Sampling of Continuous-time Signals

- Advantages of digital signal processing, e.g., audio/video CD.
- Things to look at:
  - Continuous-to-discrete (C/D)
  - Discrete-to-continuous (D/C) – perfect reconstruction
  - Frequency-domain analysis of sampling process
  - Sampling rate conversion

Periodic Sampling

- Ideal continuous-to-discrete-time (C/D) converter

\[
x_c(t) \rightarrow \text{C/D} \rightarrow x[n]
\]

Continuous-time signal: \( x_c(t) \)
Discrete-time signal: \( x[n] = x_c(nT), -\infty < n < \infty, T: \) sampling period

In theory, we break the C/D operation into two steps:

1. Ideal sampling using “analog delta function (impulse)”
2. Conversion from impulse train to discrete-time sequence

Step (1) can be modeled by mathematical equation.
Step (2) is a “concept”, no mathematical model.

In reality, the electronic analog-to-digital (A/D) circuits can approximate the ideal C/D operation. This circuitry is one piece; it cannot be split into two steps.
● Ideal sampling

\[
x_c(t) \xrightarrow{\text{Sampling}} x_s(t)
\]

**Ideal sampling signal:** impulse train (an analog signal)

\[
s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad T: \text{sampling period}
\]

**Analog (continuous-time) signal:** \(x_c(t)\)

**Sampled (continuous-time) signal:** \(x_s(t)\)

\[
x_s(t) = x_c(t)s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

\[
= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)
\]

✧ **Frequency-domain Representation of Sampling**

\[
s(t) \leftrightarrow S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s), \text{ where } \Omega_s = 2\pi / T
\]

**Remark:** \(\Omega\): analog frequency (radians/s)

\(\omega\): discrete (normalized) frequency (radians/sample)

\[
\Omega = \frac{\omega}{T}; \quad -\pi < \omega \leq \pi, \quad -\frac{\pi}{T} < \Omega \leq \frac{\pi}{T}
\]

**Step 1: Ideal Sampling** (all in analog domain)

\[
X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{1}{T} X_c(j\Omega) * \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)
\]

\[
= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega) * \delta(\Omega - k\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s(j(\Omega - k\Omega_s))
\]

The sampled signal spectrum is the sum of shifted copies of the original.

**Remark:** In analog domain,

\[
x(t)y(t) \leftrightarrow X(f) * Y(f)
\]

\[
= \frac{1}{2\pi} X(j\Omega) * Y(j\Omega)
\]
Step 2: Analog Impulses to Sequence (analog to discrete-time)

No mathematical model. The spectrum of \( x_s(t) \), \( X_s(j\Omega) \), is the same as the spectrum of \( x[n] \), \( X(e^{j\Omega T}) \). (See the Appendix at the end.)

Now, \( X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \)

Thus, \( X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) \)

Remark: In time domain, \( x_s(t) \) and \( x[n] \) are two very different signals but they have the “same” spectra in frequency domain.

Two Cases:

(1) no aliasing: \( \Omega_s > 2\Omega_N \), and

(2) aliasing: \( \Omega_s < 2\Omega_N \), where \( \Omega_N \) is the highest nonzero frequency component of \( X_c(j\Omega) \).

After sampling, the replicas of \( X_c(j\Omega) \) overlap (in frequency domain). That is, the higher frequency components of \( X_c(j\Omega) \) overlap with the lower frequency components of \( X_c(j(\Omega - \Omega_s)) \).
Nyquist Sampling Theorem:
Let \( x(t) \) be a bandlimited signal with \( X_c(j\Omega) = 0 \) for \( |\Omega| \geq \Omega_N \) (i.e., no components at frequencies greater than \( \Omega_N \)). Then \( x_c(t) \) is uniquely determined by its samples \( x[n] = x_c(nT), n = 0, \pm 1, \pm 2, \ldots \), if \( \Omega_s = \frac{2\pi}{T} \geq 2\Omega_N \). (Nyquist, Shannon)

- Nyquist frequency = \( \Omega_N \), the bandwidth of signal.
- Nyquist rate = \( 2\Omega_N \), the minimum sampling rate without distortion. (In some books, Nyquist frequency = Nyquist rate.)
- Undersampling: \( \Omega_s < 2\Omega_N \)
- Overdamping: \( \Omega_s > 2\Omega_N \)

Fourier Series, Fourier Transform, Discrete-Time

Fourier Series & Discrete-Time Fourier Transform

- Fourier Series

\( x(t) \): periodic continuous-time signal with period \( T_0 \)

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k t/T_0} = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_k t}
\]

\[
X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\Omega_k t} dt,
\]

\( \Omega_0 = \frac{2\pi}{T_0} \)

Power:
\[
P_s = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2
\]
Fourier Transform

\( x(t) \): continuous-time signal

\[
\begin{align*}
  x(t) & = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \\
  X(j\Omega) & = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt
\end{align*}
\]

Energy: \( P_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega \)

Remark: (1) Other Notations

\[
\begin{align*}
  x(t) & = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jwt} dw \\
  X(w) & = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt
\end{align*}
\]

(2) Relationships between F.S. & F.T.

\[
\begin{align*}
  x(t) & = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\
  X(f) & = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt
\end{align*}
\]

\[
X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\Omega k t} dt, \quad X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt
\]

\[
T_0 \frac{X_k}{X(j\Omega) \rightarrow T_0 \rightarrow \infty} \rightarrow X(j\Omega) = \lim_{T_0 \rightarrow \infty} (T_0 X_k)
\]
(3) Periodic Signal

- Discrete-Time Fourier Series
  \(x[n]\): periodic discrete-time signal with period \(N\).

\[
\begin{align*}
  x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j \frac{2\pi nk}{N}} \\
  X_k &= \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi nk}{N}}
\end{align*}
\]

Power: \(P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X_k|^2\)
• Discrete-Time Fourier Transform

\[ x[n]: \text{discrete-time signal} \]

\[
\begin{align*}
    x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega \\
    X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}
\end{align*}
\]

Energy: \[ E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega \]

Remark: \( X(e^{j\omega}) \) v.s. \( X(j\Omega) \)

\( X(e^{j\omega}) \) is a frequency-scaled version of \( X(j\Omega) \)

\[ X(j\Omega) = X(e^{j\omega}) \bigg|_{\omega = \Omega T} \]
Reconstruction of a Band-limited Signal from Its Samples

-- **Perfect reconstruction**: recover the original continuous-time signal without distortion, e.g., ideal lowpass (bandpass) filter

Based on the frequency-domain analysis, if we can “clip” one copy of the original spectrum, $X_c(j\Omega)$, without distortion, we can achieve the perfect reconstruction. For example, we use the ideal low-pass filter as the reconstruction filter.

**Remark:** Note that $x_s(t)$ is an analog signal (impulses).

$x_c(t) \rightarrow sampling \rightarrow x_s(t) = \sum x[nT]\delta(t-nT) \rightarrow seq.\,-\,convr. \rightarrow x[n]$

$x[n] \rightarrow impulse\,-\,convr. \rightarrow x_s(t) = \sum x[n]\delta(t-nT) \rightarrow recon. \rightarrow x_r(t)$

$$x_r(t) = x_s(t) * h_r(t) = \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} x[n]\delta(\lambda-nT)h_r(t-\lambda) \right\} d\lambda$$

$$= \sum_{n=-\infty}^{\infty} \left\{ x[n]\right\}_{\lambda=nT}\delta(\lambda-nT)h_r(t-\lambda)d\lambda = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]H_r(j\Omega)e^{-j\Omega n} = H_r(j\Omega)\left\{ \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \right\}$$

$$= H_r(j\Omega)X(e^{j\Omega}) \bigg|_{\omega=\Omega T} = H_r(j\Omega)X(e^{j\Omega T}) = H_r(j\Omega)X(j\Omega)$$

**Ideal low-pass** reconstruction filter:

$$H_r(j\Omega) = \begin{cases} T & -\pi/T < \Omega \leq \pi/T \\ 0 & \text{otherwise} \end{cases} \quad h_r(t) = \frac{\sin(\pi/T)}{(\pi T)}$$
Sampling of Continuous-time Signals

\[ x[n] \quad n \]

\[ x_s(t) \quad t \]

\[ x_r(t) \quad t \]
Discrete-time Processing of Continuous-time Signals

If this is an LTI system,

1. \( x[n] \rightarrow y[n] \): \( Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \)

2. \( x_c(t) \rightarrow x[n] \): \( X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j \left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right) \)

3. \( y[n] \rightarrow y_r(t) \): \( Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}) \)

Note that “\( T \)” is included in the expression of \( Y(e^{j\omega}) \), \( \omega \leftarrow \Omega T \). This means “physical” frequency (not normalized).

4. \( x_c(t) \rightarrow \cdots \rightarrow y_r(t) \):

\[
Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T})
= H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j\Omega - j \frac{2\pi k}{T} \right)
\]

If \( H_r(j\Omega) \) is an ideal low-pass reconstruction filter, then

\[
Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi / T \\ 0, & \text{otherwise} \end{cases}
\]

In other words, if \( x_c(t) \) is band-limited and is ideally sampled at a rate above the Nyquist rate, and the reconstruction filter is the ideal low-pass filter, then the equivalent analog filter has the same spectrum shape of the discrete-time filter.

\[
H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & \text{otherwise} \end{cases}
\]
Remark: In order to have the above equivalent relation between $H(e^{j\omega})$ and $H_{\text{eff}}(j\Omega)$, we need

(i) The system is LTI;

(ii) The input is bandlimited;

(iii) The input is sampled without aliasing and the ideal impulse train is used in sampling;

(iv) The ideal reconstruction filter is used to produce the analog output.

In practice, the above conditions are only approximately valid at best. However, there are methods in designing the sampling and the reconstruction processes to make the approximation better.

\[ H_{\text{eff}}(j\Omega) = H_c(j\Omega) \]

\[ H(e^{j\omega}) = H_c(j\frac{\omega}{T}) \quad |\omega| < \pi \]

$T$ is chosen s.t. $H_c(j\Omega) = 0$, for $|\Omega| \geq \frac{\pi}{T}$

\[ \Rightarrow h[n] = Th_c(nT) \]

The impulse response of the discrete-time system is a scaled, sampled version of $h_c(t)$. 
Continuous-time Processing of Discrete-time Signals

\[ X_c(j\Omega) = TX(e^{j\Omega T}), \quad |\Omega| < \frac{\pi}{T} \]
\[ Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega), \quad |\Omega| < \frac{\pi}{T} \]
\[ Y(e^{j\omega}) = \frac{1}{T}Y_c(j\frac{\omega}{T}), \quad |\omega| < \pi \]

\[ \Rightarrow Y(e^{j\omega}) = \frac{1}{T}H_c(j\frac{\omega}{T})X_c(j\frac{\omega}{T}) = H_c(j\frac{\omega}{T})X(e^{j\omega}) \quad |\omega| < \pi \]

\[ \Rightarrow H(e^{j\omega}) = H_c(j\frac{\omega}{T}) \quad |\omega| < \pi \]

or, equivalently,
\[ H(e^{j\Omega T}) = H_c(j\Omega) \quad |\Omega| < \frac{\pi}{T} \]

Example: Noninteger Delay

\[ H(e^{j\omega}) = e^{-j\omega \Delta} \quad |\omega| < \pi \]
Change the Sampling Rate Using Discrete-time Processing

\[ x_c(t) \begin{cases} \rightarrow T & \rightarrow x[n] = x_c(nT) \\ \rightarrow T' & \rightarrow x'[n] = x_c(nT') \end{cases} \]

Original sampling period: \( T \)
New sampling period: \( T' \), \( T \neq T' \)

- Sampling rate reduction by an integer factor
  - Sampling rate compressor:
    \[ T' = MT \], where \( M \) is an integer
    \[ x_d[n] = x[nM] = x_c(nMT) \]

![Diagram showing the process of changing the sampling rate using discrete-time processing.](image)

Compressor

Aliasing: If the original signal BW is not small enough to meet the Nyquist rate requirement,
prefiltering is needed.

The Original

\[ X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left( f\left( \frac{\omega}{T} - \frac{2\pi k}{T} \right) \right) \]

The Downsampled

\[ X_d(e^{j\omega}) = \frac{1}{T'} \sum_{r=-\infty}^{\infty} X_c\left( f\left( \frac{\omega}{T'} - \frac{2\pi r}{T'} \right) \right) \]

Old and new index: \( r = i + kM \)

\[ r, k = -\infty, \ldots, -2, -1, 0, 1, \ldots (M-1), M, (M+1), \ldots \]
\[ i = 0, 1, 2, \ldots, M-1 \]

\[
\begin{align*}
X_d(e^{j\omega}) &= \frac{1}{T'} \sum_{r=-\infty}^{\infty} X_c\left( f\left( \frac{\omega}{T'} - \frac{2\pi r}{T'} \right) \right) \\
&= \frac{1}{MT'} \sum_{r=-\infty}^{\infty} \sum_{k=-\infty}^{M-1} X_c\left( f\left( \frac{\omega}{MT'} - \frac{2\pi kM}{MT'} - \frac{2\pi r}{MT'} \right) \right) \\
X_d(e^{j\omega}) &= \frac{1}{M} \sum_{i=-\infty}^{\infty} \frac{1}{T} \sum_{k=-\infty}^{M-1} X_c\left( f\left( \frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{MT} \right) \right) \\
&= \frac{1}{M} \sum_{i=-\infty}^{\infty} X_c\left( f\left( \frac{\omega - 2\pi i}{MT} \right) \right)
\end{align*}
\]

The down-sampled spectrum = sum of shifted replica of the original

---

**Figure 3.9**: Frequency-domain illustration of downsampling.
Downsampling with aliasing

To avoid aliasing

$$w_N M < \pi$$

$$x[n] \xrightarrow{\text{Lowpass filter \ Cutoff} = \pi/M} \tilde{x}[n] \xrightarrow{\downarrow M} x_d[n] = \tilde{x}[nM]$$

General System for Sampling Rate Reduction by M
- Increasing sampling rate by an integer factor

\[ T' = \frac{T}{L}, \text{ where } L \text{ is an integer} \]

\[ x[n] = x_e \left( n \frac{T}{L} \right) \]

(1) shape is compressed; (2) replicas are removed
(1) Increase samples

<Time-domain>
\[
x_c[n] = \begin{cases} 
  x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\
  0, & \text{otherwise}
\end{cases}
\]
\[= \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL]\]

<Frequency-domain>
\[
X_c(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x[k] \delta[n - kL] \right) e^{-j\omega n}
\]
\[= \sum_{k=-\infty}^{\infty} x[k] \left( \sum_{n=-\infty}^{\infty} \delta[n - kL] e^{-j\omega n} \right) = X(e^{j\omega})
\]

Note that \( \sum_{n=-\infty}^{\infty} \delta[n - kL] e^{-j\omega n} = e^{-j\omega k} \)

Remark: Essentially, the horizontal frequency axis is compressed.

The shape of the spectrum is not changed.

old \( \omega = \Omega T \), new \( \omega = \Omega T' = \Omega T/L \), old \( \omega = \) new \( \omega \cdot L \)

Remark: At this point, we only insert zeros into the original signal. In time domain, this signal doesn’t look like the original.

(2) Ideal lowpass filtering

<Frequency-domain>
\[
H_i(j\Omega) = \begin{cases} 
  1 & \pi/(TL) < \Omega \leq \pi/(TL) \\
  0 & \text{otherwise}
\end{cases}
\]

<Time-domain>
\[
h_i[n] = \frac{\sin(\pi n/L)}{(\pi n/L)}, \text{ an interpolator!}
\]
\[
x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n - kL)/L]}{\pi(n - kL)/L}
\]
Linear interpolation

\[ h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \leq L \\ 0, & \text{otherwise} \end{cases} \]

\[ H_{\text{lin}}(e^{j\omega}) = \frac{1}{L} \left[ \sin(\omega L/2)^2 \right] \]

\[ x_{\text{lin}}[n] = \sum_{k=-\infty}^{\infty} x[k] h_{\text{lin}}[n - kL] \]

Figure 4.26 Impulse response for linear interpolation.

Figure 4.27 (a) Illustration of linear interpolation by filtering, (b) Frequency response of linear interpolator compared with ideal lowpass interpolation filter.
- Changing sampling rate by a rational factor

**Idea:** Sampling period $T$ \( \frac{1}{L} \) interpolation \( T \) decimation \( \frac{M}{L} T \)

![Diagram showing the process of changing sampling rate by a rational factor](image)

**Remark:** In general, if the factor is not rational, go back to the continuous signals.
In summary:

-- Sampling

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<th>Frequency -domain</th>
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<td>Limit bandwidth $\Omega_s &gt; 2\Omega_N$</td>
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<tr>
<td>Analog sampling (impulse train)</td>
<td>Duplicate and shift ($\Omega$)</td>
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<td>Analog to discrete $\delta(t) \rightarrow \delta[n]$</td>
<td>$\Omega \rightarrow \omega$</td>
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-- Reconstruction

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<td>$\omega \rightarrow \Omega$</td>
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<tr>
<td>Interpolation</td>
<td>Remove extra copies ($\Omega$)</td>
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</table>

-- Down-sampling

<table>
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<tr>
<th>Time-domain</th>
<th>Frequency -domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prefiltering</td>
<td>Limit bandwidth</td>
</tr>
<tr>
<td>Drop samples (rearrange index)</td>
<td>Expand (by a factor of M) and duplicate (insert (M-1) copies)</td>
</tr>
</tbody>
</table>

-- Up-sampling

<table>
<thead>
<tr>
<th>Time-domain</th>
<th>Frequency -domain</th>
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<tbody>
<tr>
<td>Insert zeros</td>
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DSP (Spring, 2007)  
Sampling of Continuous-time Signals

普通话 Processing of Analog Signals

Ideal C/D converter \(\rightarrow\) (approximation) analog-to-digital (A/D) converter

Ideal D/C converter \(\rightarrow\) (approximation) digital-to-analog (D/A) converter

![Diagram](image)

**Prefiltering to Avoid Aliasing**

Ideal antialiasing filter: Ideal low-pass filter (difficult to implement sharp-cutoff analog filters).

\[ \leftarrow \] A solution: simple prefilter and oversampling followed by sharp antialiasing filters in discrete-time domain.

Remark: Sharp cutoff analog filters are expensive and difficult to implement.

A/D conversion \(\Rightarrow\) the input continuous-time signal is sampled at a very high sampling rate.

![Diagram](image)
A/D Conversion

Digital: discrete in time and discrete in amplitude

\[ x_a(t) \xrightarrow{\text{Sample and hold}} x_0(t) \xrightarrow{\text{A/D converter}} \hat{x}_a[n] \]

**Ideal sample-and-hold:** Sample the (input) analog signal and hold its value for \( T \) seconds.

\[
x_0(t) = \sum_{n=-\infty}^{\infty} x[n] h_0(t - nT)
\]

\[
h_0(t) = \begin{cases} 
1, & 0 < t < T \\
0, & \text{otherwise}
\end{cases}
\]

\[
x_0(t) = \sum_{n=-\infty}^{\infty} x_a(nT) h_0(t - nT) = \left\{ \sum_{n=-\infty}^{\infty} x_a(nT) \delta(t - nT) \right\} * h_0(t)
\]
Quantization: Transform the input sample $x[n]$ into one of a finite set of prescribed values.

$\hat{x}[n] = Q(x[n])$, $\hat{x}[n]$ is the quantized sample

Note: Quantization is a non-linear operation.

(i) Uniform quantizer – uniformly spaced quantization levels; very popular (also called linear quantizer)

(ii) Nonuniform quantizer – may be more efficient for certain applications

- Parameters in a quantizer
  
  (1) Decision levels – partition the dynamic range of input signal
  
  (2) Quantization (representation) levels – the output values of a quantizer; a quantization level represents all samples between two nearby decision levels
  
  (3) Full-scale level – the quantizer input dynamic range

Note: Typically, when the decision levels are first chosen, then the best quantization levels are decided (for a given input probability distribution). On the other hand, when the quantization levels are chosen, the best decision levels are decided.
Quantization error analysis

For a uniform quantizer, there are two key parameters:

(i) step size $\Delta$, and (ii) full-scale level ($\pm X_m$)

Assume $(B+1)$ bits are used to represent the quantized values.

$$\Delta = \frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B}$$

Quantization error: $e[n] = \hat{x}[n] - x[n] =$ quantized value – true value

It is clear that $-\frac{\Delta}{2} < e[n] < \frac{\Delta}{2}$.

Statistical characteristics of $e[n]$:

(1) $e[n]$ is stationary (probability distribution unchanged)

(2) $e[n]$ is uncorrelated with $x[n]$

(3) $e[n]$, $e[n+1]$, … are uncorrelated (white)

(4) $e[n]$ has a uniform distribution

The preceding assumptions are (approximately) valid if the signal is sufficiently complex and the quantization steps are sufficiently small, …
Mean square error (MSE) of $e[n]$ (= variance if zero mean)

$$(\sigma_e^2)^2 = E(e(n)^2) = \int_{-\Delta/2}^{\Delta/2} e^2 \frac{1}{\Delta} de = \frac{\Delta^2}{12}$$

-- Expressed in terms of $2^B$ and $X_m$

$$\sigma_e^2 = \frac{2^{-2B}X_m^2}{12}$$

-- SNR (signal-to-noise ratio) due to quantization

$$\text{SNR} = 10 \log_{10} \frac{\sigma_x^2}{\sigma_e^2} = 10 \log_{10} \frac{12 \cdot 2^{-2B} \sigma_x^2}{X_m^2} = 10.8 + 6.02B - 20 \log_{10} \frac{X_m}{\sigma_x}$$

Remarks:

(1) One bit buys a 6dB SNR improvement.

(2) If the input is Gaussian, a small percentage of the input samples would have an amplitude greater than $4\sigma_x$.

If we choose $X_m = 4\sigma_x$, $\text{SNR} \approx 6B - 1.25dB$

For example, a 96dB SNR requires a 16-bit quantizer.
**D/A Conversion**

The ideal lowpass filter is replaced by a “practical” filter.

Examples of practical filters: zero-order hold and first-order hold.

Mathematical model:

\[
x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_0(t-nT)
\]

= quantized input * impulse response of “practical” interpolation filter

\[
x_{DA}(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT) + \sum_{n=-\infty}^{\infty} e[n]h_0(t-nT)
\]

= \(x_0(t) + e_0(t)\)

**Purpose:** Find a compensation filter \(\tilde{h}_r(t)\) to compensate for the distortion caused by the non-ideal \(h_0(t)\) so that its output \(\hat{x}_r(t)\) is close to the analog original \(x_a(t)\).

In frequency domain:

\[
X_0(j\Omega) = F \left\{ \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT) \right\} = \sum_{n=-\infty}^{\infty} x[n]H_0(j\Omega)e^{-j\Omega nT}
\]

= \(\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\right)H_0(j\Omega) = X(e^{j\Omega T})H_0(j\Omega)\)
Because
\[
X(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - k \frac{2\pi}{T}\right)\right),
\]
\[
X_0(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - k \frac{2\pi}{T}\right)\right)H_0(j\Omega)
\]

[The interpolation filter $H_0(j\Omega)$ is used to remove the replicas.]

If $H_0(j\Omega)$ is not an ideal lowpass filter, we design a compensated reconstruction filter,
\[
\tilde{H}_r(j\Omega) = \frac{H_r(j\Omega)}{H_0(j\Omega)}
\]

(1) **Zero-order hold**

\[
h_0(t) = \begin{cases} 
1, & 0 < t < T \\
0, & \text{otherwise}
\end{cases}
\]

or
\[
H_0(j\Omega) = \frac{2\sin\left(\Omega T / 2\right)}{\Omega} e^{-\Omega^2 T / 2}
\]

Thus, the compensated reconstruction filter is
\[
\tilde{H}_r(j\Omega) = \begin{cases} 
\frac{\Omega T / 2}{\sin(\Omega T / 2)} e^{\Omega^2 T / 2}, & |\Omega| < \pi / T \\
0, & |\Omega| > \pi / T
\end{cases}
\]

**Remark:** A “practical” filter cannot achieve this approximation.
Overall system:

\[ X_a(j\Omega) \rightarrow H_{aa}(j\Omega) \rightarrow H(e^{j\Omega T}) \rightarrow H_0(j\Omega) \rightarrow \tilde{H}_r(j\Omega) \rightarrow Y_a(j\Omega) \]

Anti-aliasing Processing  Zero-order-hold Compensated reconstruction.

\[ H_{\text{eff}}(j\Omega) = \tilde{H}_r(j\Omega) \cdot H_0(j\Omega) \cdot H(e^{j\Omega T}) \cdot H_{aa}(j\Omega) \]

\[ P_e(j\Omega) = \left| \tilde{H}_r(j\Omega) \cdot H_0(j\Omega) \cdot H(e^{j\Omega T}) \right|^2 \sigma_e^2 \quad \text{where} \quad \sigma_e^2 = \frac{\Delta^2}{12} \]