The z-Transform

✧ Introduction

- Why do we study them?
  - A generalization of DTFT.
    
    Some sequences that do not converge for DTFT have valid z-transforms.
  - Better notation (compared to FT) in analytical problems (complex variable theory)
  - Solving difference equation. → algebraic equation.

✦ Fourier Transform, Laplace Transform, DTFT, & z-Transform

Fourier Transform

\[
\mathcal{Z}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} \, dt
\]

To encompass a broader class of signals:

\[
\int_{-\infty}^{\infty} (x(t)e^{-\alpha t})e^{-j\Omega t} \, dt = \int_{-\infty}^{\infty} x(t)e^{-s t} \, dt \equiv L\{x(t)\}
\]

Laplace Transform

\[
\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-s t} \, dt
\]

Region of Convergence

\[ j\Omega \]

S-domain

\[ \sigma \]

Region of Convergence
The z-Transform

$$x(t) = \sum_{k=-\infty}^{\infty} x[k] \delta(t - kT)$$

Similarly,

$$L\{x(t)\} = L\left\{ \sum_{k=-\infty}^{\infty} x[k] \delta(t - kT) \right\} = \int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta(t - kT) \right\} e^{-st} dt = \sum_{k=-\infty}^{\infty} x[k] \int_{-\infty}^{\infty} \delta(t - kT) e^{-st} dt$$

$$= \sum_{k=-\infty}^{\infty} x[k] e^{-skT} = \sum_{k=-\infty}^{\infty} x[k] z^{-k} = Z\{x[n]\} = X(z)$$

**z-Transform**

- Eigenfunctions of discrete-time LTI systems

$$z^n \quad \text{Discrete-Time LTI} \quad H(z)z^n$$

If \(x[n] = z_0^n\), \(z_0^n\): some complex constant

$$y[n] = x[n] \ast h[n] = \sum_{k=-\infty}^{\infty} x[n - k] h[k] = \sum_{k=-\infty}^{\infty} z_0^{-k} h[k] = \left\{ \sum_{k=-\infty}^{\infty} h[k] z_0^{-k} \right\} z_0^n = H(z_0)z_0^n$$

**Remark:**

$$X(z) \bigg|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-jnw}$$

DTFT can be viewed as a special case: \(z = e^{j\omega}\)
**z-Transform**

- **(Two-sided) z-Transform** (bilateral z-Transform)
  
  **Forward:** \( Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \equiv X(z) \)

  From DTFT viewpoint: \( Z\{x[n]\} = F\{r^n x[n]\}\big|_{r=1} \)

  (Or, DTFT is a special case of z-T when \( z = e^{j\omega}, \) unit circle.)

  **Inverse:** \( x[n] = \frac{1}{2\pi j} \oint_{\Gamma} X(z)z^{n-1}dz \equiv Z^{-1}[X(z)] \)

  *Note:* The integration is evaluated along a counterclockwise circle on the complex \( z \) plane with a radius \( r \). (A proof of this formula requires the complex variable theory.)

- **Single-sided z-Transform** (unilateral) – for causal sequences
  
  \( X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \)

- **Region of Convergence (ROC)**

  The set of values of \( z \) for which the z-transform converges.

  - **Uniform convergence**
    
    If \( z = re^{j\omega} \) (polar form), the z-transform converges uniformly if \( x[n]r^{-n} \) is *absolutely summable*; that is,
    
    \[ \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \]

    - In general, if some value of \( z \), say \( z = z_1 \), is in the ROC, then all values of \( z \) on the circle defined by \( |z| = |z_1| \) are also in the ROC. \( \Rightarrow \) ROC is a “ring”.
    
    - If ROC contains the unit circle, \( |z| = 1 \), then the FT of this sequence converges.

    - By its definition, \( X(z) \) is a Laurent series (complex variable)
      
      \( \Rightarrow \) \( X(z) \) is an *analytic function* in its ROC
      
      \( \Rightarrow \) *All* its derivatives are continuous (in \( z \)) within its ROC.
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The z-Transform

- **DTFT v.s. z-Transform**
  
  \[ x_1[n] = \frac{\sin \omega_0 n}{\pi n}, \quad -\infty < n < \infty \]

  Not absolutely summable; but square summable

  \[ \Rightarrow \text{z-transform does not exist; DTFT (in m.s. sense) exists.} \]

  \[ x_2[n] = \cos \omega_0 n, \quad -\infty < n < \infty \]

  Not absolutely summable; not square summable

  \[ \Rightarrow \text{z-transform does not exist; \textquotedblleft useful\textquotedblright{} DTFT (impulses) exists.} \]

  \[ x_3[n] = a^n u[n], \quad |a| > 1, \quad -\infty < n < \infty \]

  \[ \Rightarrow \text{z-transform exist (a certain ROC); DTFT does not exists.} \]

- **Some Common Z-T Pairs**

  - \( \delta[n] \leftrightarrow 1 \), \( \delta[n - m] \leftrightarrow z^{-m}, \quad m > 0, |z| > 0 \)

  \( \delta[n + m] \leftrightarrow z^m, \quad m > 0, |z| < \infty \)

  - \( u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1 \), \( -u[-n - 1] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| < 1 \)

  - \( a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a| \)

  \[ -a^n u[-n - 1] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| < |a| \]

  - \( r^n \cos \Theta n u[n] \leftrightarrow \frac{1 - [r \cos \Theta] z^{-1}}{1 - [2 r \cos \Theta] z^{-1} + r^2 z^{-2}}, \quad |z| > r \)

  \( r^n \sin \Theta n u[n] \leftrightarrow \frac{1 - [r \sin \Theta] z^{-1}}{1 - [2 r \sin \Theta] z^{-1} + r^2 z^{-2}}, \quad |z| > r \)
Properties of ROC for z-Transform

- Rational functions

\[ X(z) = \frac{P(z)}{Q(z)} \]

**Poles** – Roots of the denominator; the \( z \) such that \( X(z) \to \infty \)

**Zeros** – Roots of the numerator; the \( z \) such that \( X(z) = 0 \)

Properties of ROC

1. The ROC is a ring or disk in the \( z \)-plane centered at the origin.
2. The F.T. of \( x[n] \) converges absolutely \( \iff \) its ROC includes the unit circle.
3. The ROC cannot contain any poles.
4. If \( x[n] \) is **finite-duration**, then the ROC is the entire \( z \)-plane except possibly \( z = 0 \) or \( z = \infty \).
5. If \( x[n] \) is **right-sided**, the ROC, if exists, must be of the form \( |z| > r_{\text{max}} \) except possibly \( z = \infty \), where \( r_{\text{max}} \) is the magnitude of the largest pole.
6. If \( x[n] \) is **left-sided**, the ROC, if exists, must be of the form \( |z| < r_{\text{min}} \) except possibly \( z = 0 \), where \( r_{\text{min}} \) is the magnitude of the smallest pole.
7. If \( x[n] \) is **two-sided**, the ROC must be of the form \( r_1 < |z| < r_2 \) if exists, where \( r_1 \) and \( r_2 \) are the magnitudes of the interior and exterior poles.
8. The ROC must be a connected region.

In general, if \( X(z) \) is rational, its inverse has the following form (assuming \( N \) poles: \( \{d_k\} \))

\[ x[n] = \sum_{k=1}^{N} A_k (d_k)^n. \]

For a right-sided sequence, it means \( n \geq N_1 \), where \( N_1 \) is the first nonzero sample.

The \( n \)th term in the \( z \)-transform is \( x[n] r^{-n} = \sum_{k=1}^{N} A_k (d_k r^{-1})^n. \)

This sequence converges if \( \sum_{n=N_1}^{\infty} |d_k r^{-1}|^n < \infty \) for every pole \( k = 1, \ldots, N \). In order to be so, \( |r| > |d_k|, k = 1, \ldots, N. \)
Pole Location and Time-Domain Behavior for Causal Signals

Reference: Digital Signal Processing by Proakis & Manolakis

Figure 3.11 Time-domain behavior of a single-real pole causal signal as a function of the location of the pole with respect to the unit circle.

Figure 3.12 Time-domain behavior of causal signals corresponding to a double \((m = 2)\) real pole, as a function of the pole location.
Figure 3.13  A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

Figure 3.14  Causal signal corresponding to a double pair of complex-conjugate poles on the unit circle.
The Inverse z-Transform

Inverse formula: \[ x[n] = \frac{1}{2\pi j} \oint_{C} X(z)z^{-n-1}dz \]

This formula can be proved using Cauchy integral theorem (complex variable theory).

- **Methods of evaluating the inverse z-transform**
  1. Table lookup or inspection
  2. Partial fraction expansion
  3. Power series expansion

**Inspection** (transform pairs in the table) – memorized them

**Partial Fraction Expansion**

\[ X(z) = \frac{b_0 + b_1z^{-1} + \cdots + b_Mz^{-M}}{a_0 + a_1z^{-1} + \cdots + a_Nz^{-N}} \]

\[ \Rightarrow X(z) = \frac{z^N(b_0z^{-M} + \cdots + b_M)}{z^M(a_0z^N + \cdots + a_N)} \]

Hence, it has \( M \) zeros (roots of \( \sum b_k z^{M-k} \)), \( N \) poles (roots of \( \sum a_k z^{N-k} \)), and \((M-N)\) poles at zero if \( M>N \) (or \((N-M)\) zeros at zero if \( N>M \)).

\[ \Rightarrow X(z) = \frac{b_0(1-c_1z^{-1}) \cdots (1-c_Mz^{-1})}{a_0(1-d_1z^{-1}) \cdots (1-d_Nz^{-1})} ; \quad c_k, \text{ non-zero zeros; } d_k, \text{ non-zero poles.} \]

**Case 1**: \( M < N \), strictly proper

*Simple (single) poles:*

\[ X(z) = \frac{A_1}{(1-d_1z^{-1})} + \frac{A_2}{(1-d_2z^{-1})} + \cdots + \frac{A_N}{(1-d_Nz^{-1})} \]

where \( A_k = (1-d_kz^{-1})X(z) \big|_{z=d_k} \)

*Multiple poles: Assume \( d_i \) is the \( s \)th order pole. (Repeated \( s \) times)*

\[ X(z) = \sum_{k=1}^{N} \frac{A_k}{(1-d_kz^{-1})} + \sum_{s=1}^{C_1} \frac{C_1}{(1-d_1z^{-1})^s} + \sum_{s=1}^{C_2} \frac{C_2}{(1-d_2z^{-1})^s} + \cdots + \sum_{s=1}^{C_N} \frac{C_N}{(1-d_Nz^{-1})^s} \]

*single-pole terms  \quad multiple-pole terms*

where \( C_m = \frac{1}{(s-m)!(-d_m)^{-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1-d_m)w^s X(w^{-1})] \right\}_{w=d_m} \)

**Case 2**: \( M \geq N \)

\[ X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{(1-d_kz^{-1})} + \sum_{m=1}^{C_m} \frac{C_m}{(1-d_mz^{-1})^m} \]

*impulses  \quad single-poles  \quad multiple-pole*
• **Power Series Expansion**

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

- **Case 1:** Right-sided sequence, ROC: \(|z| > r_{\text{max}}\)

  It is expanded in powers of \(z^{-1}\).

  \[ X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a| \]

- **Case 2:** Left-sided sequence, ROC: \(|z| < r_{\text{min}}\)

  It is expanded in powers of \(z\).

  \[ X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a| \]

- **Case 3:** Two-sided sequence, ROC: \(r_1 < |z| < r_2\)

  \[ X(z) = X_+(z) + X_-(z) \]

  converges for \(|z| > r_1\) \quad converges for \(|z| < r_2\)

  \[ x[n] = x_+[n] + x_-[n] \]

  causal sequence \quad anti-causal sequence
z-Transform Properties

If \( x[n] \leftrightarrow X[z] \) and \( y[n] \leftrightarrow Y[z] \), ROC: \( R_X, R_Y \)

- **Linearity:** \( ax[n] + by[n] \leftrightarrow aX(z) + bY(z) \)
  ROC: \( R' \supset R_X \cap R_Y \) -- At least as large as their intersection; larger if pole/zero cancellation occurs

- **Time Shifting:** \( x[n - n_0] \leftrightarrow z^{-n_0} X(z) \)
  ROC: \( R' = R_X \pm \{0 \text{ or } \infty\} \)

- **Multiplication by an exponential sequence:**
  \( a^n x[n] \leftrightarrow X(z/a) \)
  ROC: \( R' = \{0\} R_X \) -- expands or contracts

- **Differentiation of \( X(z) \):**
  \( nx[n] \leftrightarrow -z \frac{dX(z)}{dz}, \quad \text{ROC: } R' = R_X \)

- **Conjugation of a complex sequence:** \( x^*[n] \leftrightarrow X^*[z^*], \quad \text{ROC: } R' = R_X \)

- **Time reversal:** \( x^*[-n] \leftrightarrow X^*(1/z^*) \),
  ROC: \( R' = 1/R_X \) (Meaning: If \( R_X : r_R < |z| < r_L \), then \( R': 1/r_L < |z| < 1/r_R \). Corollary: \( x[-n] \leftrightarrow X(1/z) \)

- **Convolution:** \( x[n] * y[n] \leftrightarrow X(z)Y(z) \)
  ROC: \( R' \supset R_X \cap R_Y \) (=, if no pole/zero cancellation)

- **Initial Value Theorem:**
  \( \text{If } x[n]=0, \text{ then } x[0] = \lim_{z \to \infty} X(z) \)
■ Final Value Theorem:

*If*  
(1) \( x[n]=0, \ n<0, \) and  

(2) all singularities of \((1 - z^{-1})X(z)\) are inside the unit circle,  

*then*  
\[ x[\infty] = \lim_{z \to 1}(1 - z^{-1})X(z) \]

Remarks:  
(1) If all poles of \(X(z)\) are inside unit circle, \(x[n] \to 0\) as \(n \to \infty\)  
(2) If there are multiple poles at “1”, \(x[n] \to \infty\) as \(n \to \infty\)  
(3) If poles are on the unit circle but not at “1”, \(x[n] \approx \cos \omega_0 n\)

<Supplementary>

**z-Transform Solutions of Linear Difference Equations**

Use single-sided z-transform:  
\[
Z\{y[n-1]\} = z^{-1}Y(z) + y[-1] \\
Z\{y[n-2]\} = z^{-2}Y(z) + z^{-1}y[-1] + y[-2] \\
Z\{y[n-3]\} = z^{-3}Y(z) + z^{-2}y[-1] + z^{-1}y[-2] + y[-3]
\]

For causal signals, their single-sided z-transforms are identical to their two-sided z-transforms.

**Ex.** Find \(y[n]\) of the difference eqn.  
\[ y[n] - 0.5y[n-1] = x[n] \quad \text{with} \quad x[n] = 1, \ n \geq 0, \ \text{and} \ y[-1] = 1 \]

*(Sol)*  
Take the single-sided z-transform of the above eqn.  
\[
\Rightarrow \ Y(z) - 0.5\{z^{-1}Y(z) + y[-1]\} = X(z) = \frac{1}{1 - z^{-1}}
\]

\[
\Rightarrow \ Y(z) = \left(\frac{1}{1 - 0.5z^{-1}}\right)\left(0.5 + \frac{1}{1 - z^{-1}}\right)
\]

\[
\Rightarrow \ Y(z) = \frac{0.5}{1 - 0.5z^{-1}} + \frac{1}{(1 - 0.5z^{-1})(1 - z^{-1})}
\]

\[
\Rightarrow \ Y(z) = \frac{2}{1 - z^{-1}} - \frac{0.5}{1 - 0.5z^{-1}}
\]

Take the inverse z-transform  
\[
\Rightarrow \ y[n] = 2 - 0.5(0.5)^n, \ n \geq 0
\]