Chapter 7: The z-Transform

Chih-Wei Liu

Outline

- Introduction
- The z-Transform
- Properties of the Region of Convergence
- Properties of the z-Transform
- Inversion of the z-Transform
- The Transfer Function
- Causality and Stability
- *Determining Frequency Response from Poles & Zeros*
- *Computational Structures for DT-LTI Systems*
- *The Unilateral z-Transform*

Introduction

- The *z-transform* provides a broader characterization of discrete-time LTI systems and their interaction with signals than is possible with DTFT
- Signal that is not absolutely summable → z-transform
  - ~~→~~ DTFT
- Two varieties of z-transform:
  - Unilateral or one-sided
  - Bilateral or two-sided
  - The unilateral z-transform is for solving difference equations with initial conditions.
  - The bilateral z-transform offers insight into the nature of system characteristics such as stability, causality, and frequency response.

A General Complex Exponential z^n

- Complex exponential $z = re^{j\Omega}$ with magnitude r and angle Ω

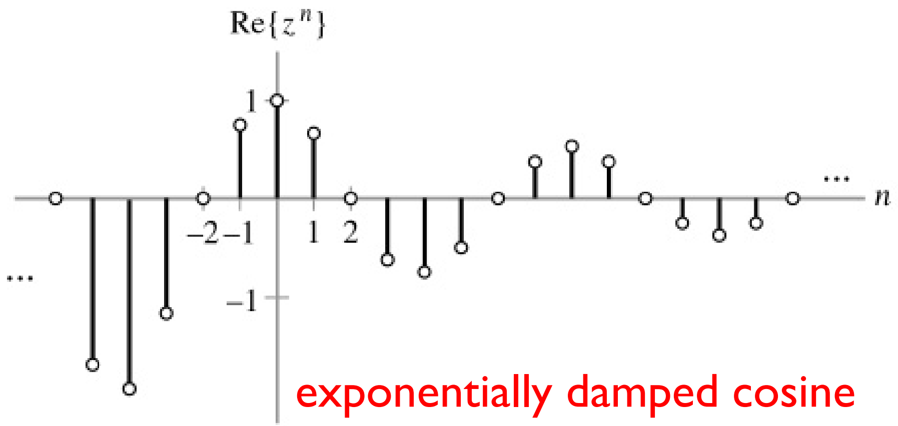
$$z^n = r^n \cos(\Omega n) + jr^n \sin(\Omega n)$$

Re{z^n}: exponential damped cosine

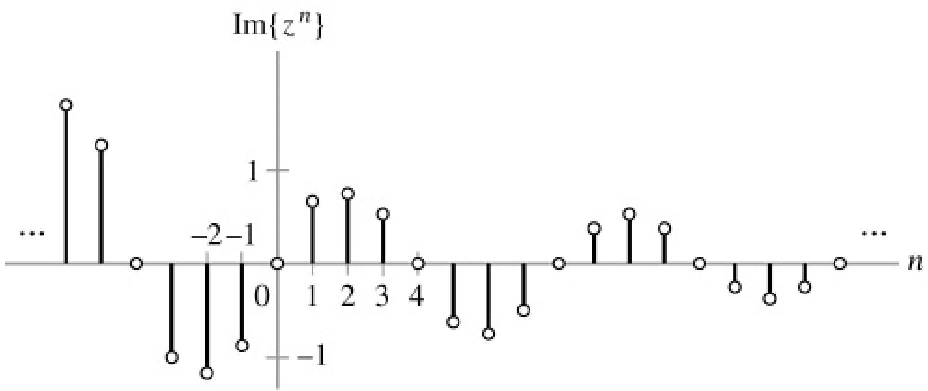
Im{z^n}: exponential damped sine

r: damping factor

Ω: sinusoidal frequency



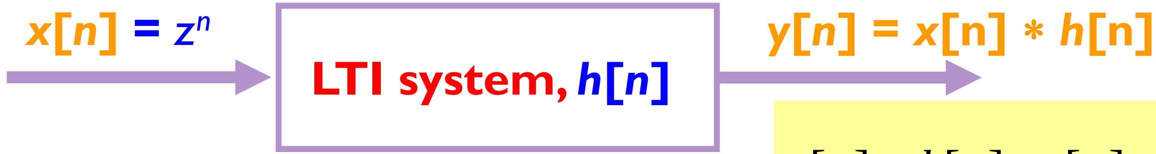
exponentially damped cosine



exponentially damped sine

- z^n is an eigenfunction of the LTI system

Eigenfunction Property of z^n



$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$
$$= \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$
$$= z^n\left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)$$
$$= z^n H(z)$$

- Transfer function $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$
- $H(z)$ is the eigenvalue of the eigenfunction z^n
- Polar form of $H(z)$: $H(z) = |H(z)|\, e^{j\phi(z)}$

$|H(z)| \equiv$ amplitude of $H(z)$; $\phi(z) \equiv$ phase of $H(z)$

Then $y[n] = |H(z)|e^{j\phi(z)}z^n.$ Let $z = re^{j\Omega}$

$$\Rightarrow y[n] = |H(re^{j\Omega})|r^n \cos(\Omega n + \phi(re^{j\Omega})) + j|H(re^{j\phi})|r^n \sin(\Omega n + \phi(re^{j\Omega})).$$

The LTI system changes the amplitude of the input by $|H(re^{j\Omega})|$ and shifts the phase of the sinusoidal components by $\phi(re^{j\Omega})$.

DEPT. OF ELECTRONICS ENGINEERING & INST. OF ELECTRONICS NCTU

The z-Transform

$$x[n] \xleftrightarrow{z} X(z)$$

- Definition: The z-transform of $x[n]$: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$,
- Definition: The inverse z-transform of $X(z)$: $x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz.$

- A representation of arbitrary signals as a weighted superposition of eigenfunctions z^n with $z = re^{j\Omega}$. We obtain

$$H(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n](re^{j\Omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(h[n]r^{-n}\right)e^{-j\Omega n}$$

$$\Rightarrow \quad h[n]r^{-n} \xleftrightarrow{DTFT} H(re^{j\Omega})$$

z-transform is the DTFT of $h[n]r^{-n}$

Hence

$$h[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(re^{j\Omega}\right)e^{j\Omega n} d\Omega$$

$$\Rightarrow \quad h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\Omega})(re^{j\Omega})^n d\Omega$$

$$= \frac{1}{2\pi j}\oint H(z) z^{n-1} dz$$

$z = re^{j\Omega}$

$dz = jre^{j\Omega} d\Omega$

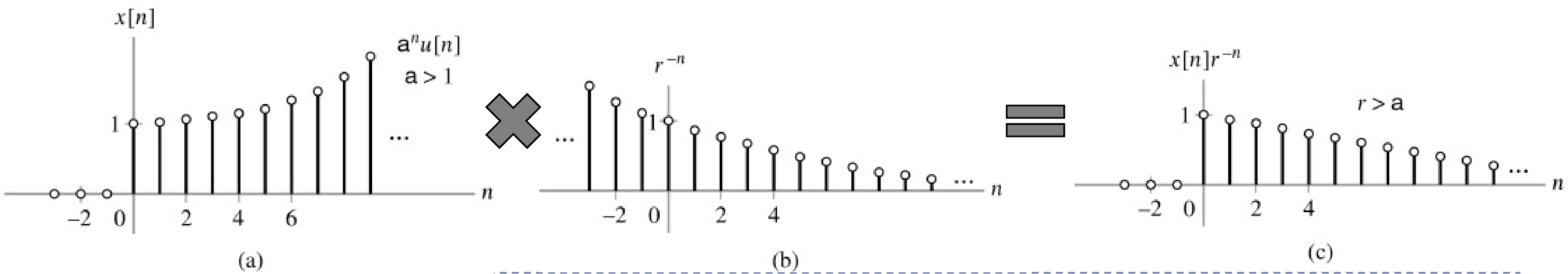
$d\Omega = (1/j)z^{-1} dz$

Convergence of Laplace Transform

- z-transform is the DTFT of $x[n]\,r^{-n}$ ➔ A necessary condition for convergence of the z-transform is the absolute summability of $x[n]\,r^{-n}$:

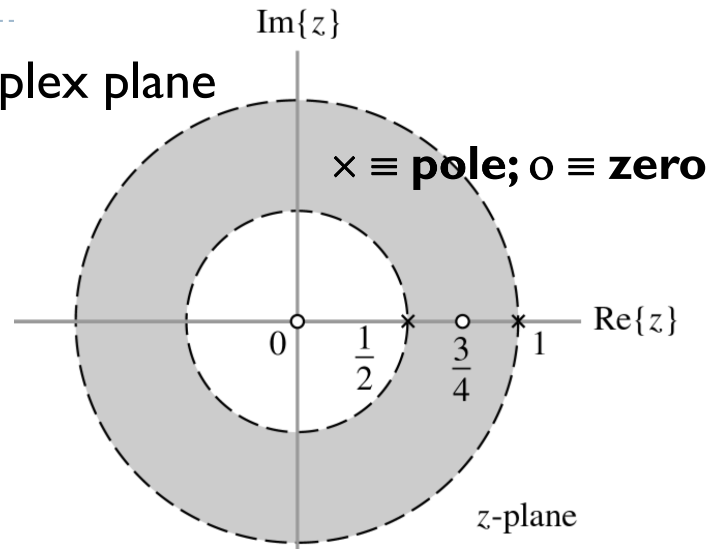
$$\sum_{n=-\infty}^{\infty}\left|x[n]r^{-n}\right| < \infty.$$

- The range of r for which the z-transform converges is termed the *region of convergence* (ROC).
- Convergence example:
  1. DTFT of $x[n]=a^n\,u[n]$, $a>1$, does not exist, since $x[n]$ is not absolutely summable.
  2. But $x[n]r^{-n}$ is absolutely summable, if $r>a$, i.e. ROC, so the z-transform of $x[n]$, which is the DTFT of $x[n]r^{-n}$, does exist.



The z-Plane, Poles, and Zeros

- To represent $z = re^{j\Omega}$ graphically in terms of complex plane
- Horizontal axis of z-plane = real part of z;
- vertical axis of z-plane = imaginary part of z.



- Relation between DTFT and z-transform:

$$X\left(e^{j\Omega}\right) = X(z)\big|_{z=e^{j\Omega}}.$$

the DTFT is given by the z-transform evaluated on the unit circle

The frequency Ω in the DTFT corresponds to the point on the unit circle at an angle Ω with respect to the positive real axis

- z-transform $X(z)$:

$$X(z) = \frac{b_0 + b_1 z_{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots a_N z^{-N}}. \quad \Rightarrow \quad X(z) = \frac{\tilde{b}\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}.$$

c_k = zeros of $X(z)$; d_k = poles of $X(z)$

$$\tilde{b} = b_0 / a_0 \equiv \text{gain} \quad \text{factor}$$

Example 7.2 Right-Sided Signal

Determine the z-transform of the signal $x[n] = \alpha^n u[n]$.

Depict the ROC and the location of poles and zeros of X(z) in the z-plane.

<Sol.>

By definition, we have $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n.$

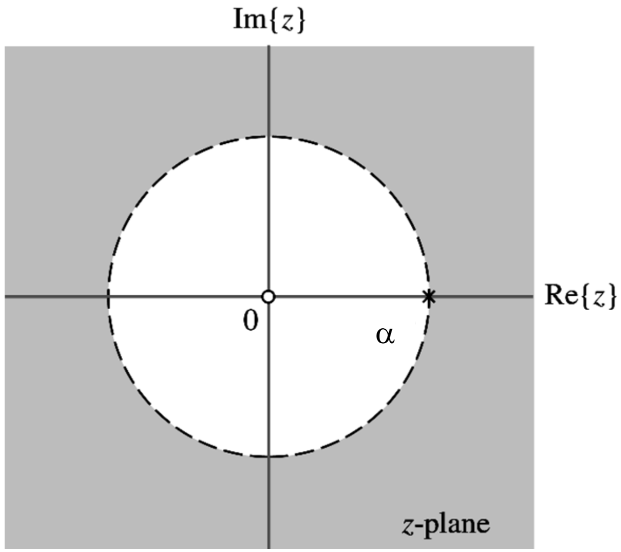
This is a geometric series of infinite length in the ratio α/z;

➡ X(z) converges if |α/z| < 1, or the ROC is |z| > |α|. And,

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

$$= \frac{z}{z-\alpha}, \quad |z| > |\alpha|.$$

There is a pole at z = α and a zero at z = 0



Right-sided signal ➔ the ROC is |z| > |α|.

Example 7.3 Left-Sided Signal

Determine the z-transform of the signal $y[n] = -\alpha^n u[-n-1]$.

Depict the ROC and the location of poles and zeros of *Y*(z) in the z-plane.

<Sol.>

By definition, we have $Y(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n$

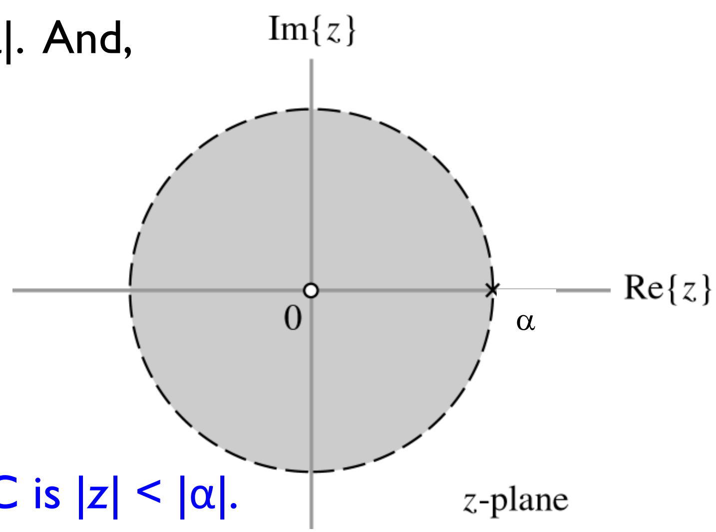
$$= -\sum_{k=1}^{\infty} \left(\frac{z}{\alpha}\right)^k = 1 - \sum_{k=0}^{\infty} \left(\frac{z}{\alpha}\right)^k .$$

➡ *Y*(z) converges if |z/α| < 1, or the ROC is |z| < |α|. And,

$$Y(z) = 1 - \frac{1}{1 - z\alpha^{-1}}, \quad |z| < |\alpha|,$$

$$= \frac{z}{z-\alpha}, \quad |z| < |\alpha|$$

There is a pole at z = α and a zero at z = 0



Left-sided signal ➔ the ROC is |z| < |α|.

Examples 7.2 & 7.3 reveal that the same z-transform but different ROC.
This ambiguity occurs in general with signals that are one sided

Properties of the ROC

- 1. The ROC cannot contain any poles

  If d is a pole, then $|X(d)| = \infty$, and the z-transform does not converge at the pole

- 2. The ROC for a finite-duration $x[n]$ includes the entire z-plane, except possibly z=0 or |z|=∞

  For finite-duration $x[n]$, we might suppose that $X(z) = \sum_{n=n_1}^{n_2} x[n] z^{-n}$.

  $X(z)$ will converge, if each term of $x[n]$ is finite.

  1) If a signal has any nonzero causal components, then the expression for $X(z)$ will have a term involving z^{-1} for $n_2 > 0$, and thus the ROC cannot include $z = 0$.

  2) If a signal has any nonzero noncausal components, then the expression for $X(z)$ will have a term involving z for $n_1 < 0$, and thus the ROC cannot include $|z| = \infty$.

- 3. $x[n]=c\delta[n]$ is the only signal whose ROC is the entire z-plane

  Consider $X(z) = \sum_{n=n_1}^{n_2} x[n] z^{-n}$.

  If $n_2 \leq 0$, then the ROC will include $z = 0$.

  If a signal has no nonzero noncausal components ($n_1 \geq 0$), then the ROC will include $|z| = \infty$.

Properties of the ROC

- 4. For the infinite-duration signals, then

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n] z^{-n}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n] z^{-n}\right| = \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n}.$$

The condition for convergence is $|X(z)| < \infty$. We may write $|X(z)| \le I_{+}(z) + I_{-}(z)$

That is, we split the infinite sum into negative- and positive-term portions:

$$I_{-}(z) = \sum_{n=-\infty}^{-1} |x[n]||z|^{-n} \quad \text{and} \quad I_{+}(z) = \sum_{n=0}^{\infty} |x[n]||z|^{-n}.$$

Note that If $I_{-}(z)$ and $I_{+}(z)$ are finite, then $|X(z)|$ is guaranteed to be finite, too.

A signal that satisfies these two bounds grows no faster than $(r_{+})^n$ for positive n and $(r_{-})^n$ for negative n.

That is,

$$|x[n]| \le A_{-}(r_{-})^n, \quad n < 0 \qquad (7.9)$$

$$|x[n]| \le A_{+}(r_{+})^n, \quad n \ge 0 \qquad (7.10)$$

If the bound given in Eq. (7.9) is satisfied, then

$k = -n$

$$I_-(z) \le A_- \sum_{n=-\infty}^{-1} (r_-)^n |z|^{-n} = A_- \sum_{n=-\infty}^{-1} \left(\frac{r_-}{|z|}\right)^n = A_- \sum_{k=1}^{\infty} \left(\frac{|z|}{r_-}\right)^k$$

$I_-(z)$ converges if and only if $|z| < r_-$.

If the bound given in Eq. (7.10) is satisfied, then

$$I_+(z) \le A_+ \sum_{n=0}^{\infty} (r_+)^n |z|^{-n} = A_+ \sum_{n=0}^{\infty} \left(\frac{r_+}{|z|}\right)^n$$

$I_+(z)$ converges if and only if $|z| > r_+$.

Hence, if $r_+ < |z| \le |< r_-$, then both $I_+(z)$ and $I_-(z)$ converge and $|X(z)|$ also converges.

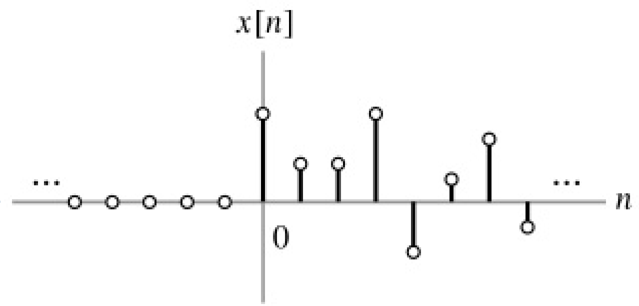
Note that if $r_+ > r_-$, then the ROC = $\varnothing$

For signals $x[n]$ satisfy the exponential bounds of Eqs. (7.9) and (7.10) , we have

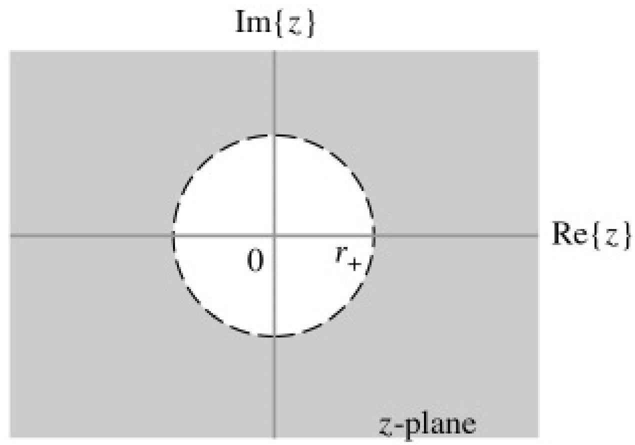
(1). The ROC of a right-sided signal is of the form $|z| > r_+$.

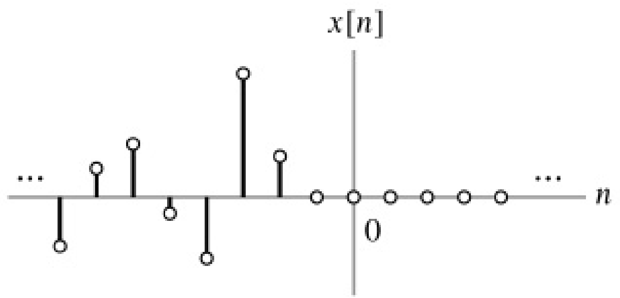
(2). The ROC of a left-sided signal is of the form $|z| < r_-$.

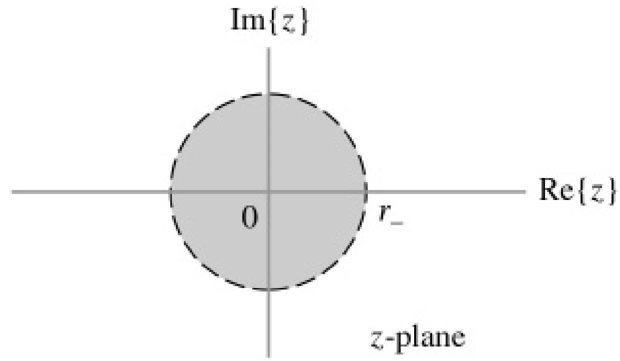
(3). The ROC of a two-sided signal is of the form $r_+ < |z| \le |< r_-$.



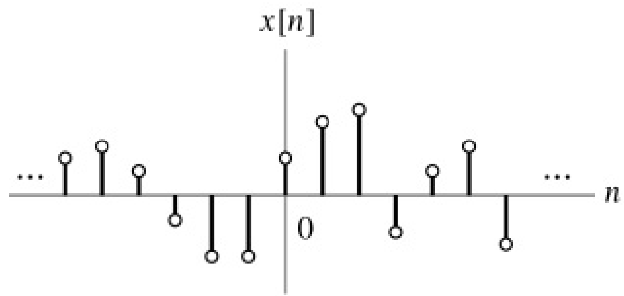
A right-sided signal has an ROC of the form $|z| > r_+$.

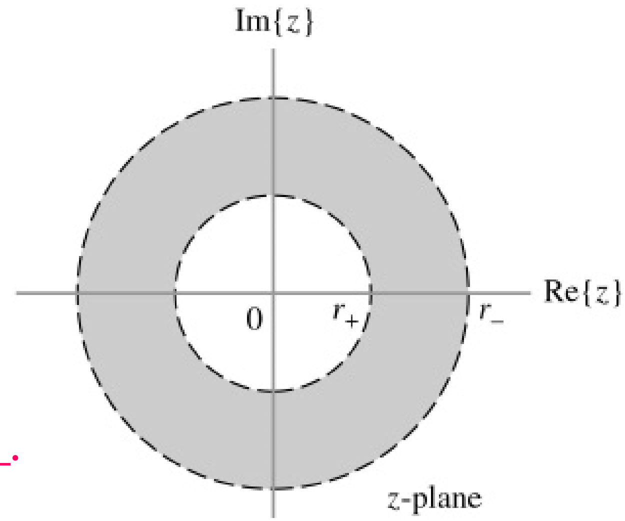






A left-sided signal has an ROC of the form $|z| < r_-$. (b)





A two-sided signal has an ROC of the form $r_+ < |z| < r_-$.

Example 7.5

Identify the ROC associated with z-transform for each of the following signal:

$$x[n]=(-1/2)^n u[-n]+2(1/4)^n u[n];\quad y[n]=(-1/2)^n u[n]+2(1/4)^n u[n];$$

$$w[n]=(-1/2)^n u[-n]+2(1/4)^n u[-n].$$

<Sol.>

1. $$X(z)=\sum_{n=-\infty}^{0}\left(\frac{-1}{2z}\right)^n+2\sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^n=\sum_{k=0}^{\infty}(-2z)^k+2\sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^n.$$

The first series converge for $|z|<1/2$, while the second converge for $|z|>1/4$. So, the ROC is $1/4 < |z| < 1/2$. Hence,

$$X(z)=\frac{1}{1+2z}+\frac{2z}{z-\frac{1}{4}},$$

Poles at $z = -1/2$ and $z = 1/4$

2. $$Y(z)=\sum_{n=0}^{\infty}\left(\frac{-1}{2z}\right)^n+2\sum_{n=9}^{\infty}\left(\frac{1}{4z}\right)^n.$$

The first series converge for $|z| > 1/2$, while the second converge for $|z| > 1/4$. Hence, the ROC is $|z| > 1/2$, and

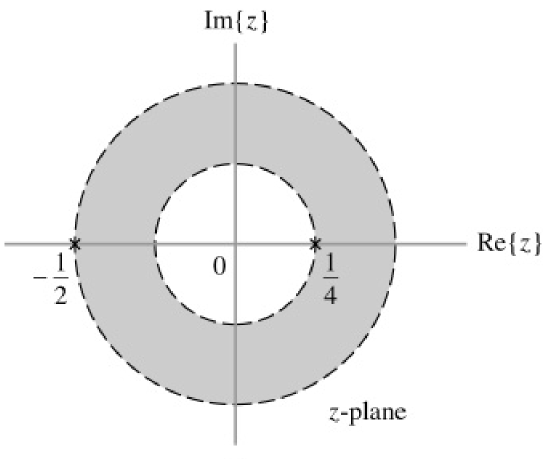
$$Y(z)=\frac{z}{z+\frac{1}{2}}+\frac{2z}{z-\frac{1}{4}},$$

Poles at $z = -1/2$ and $z = 1/4$

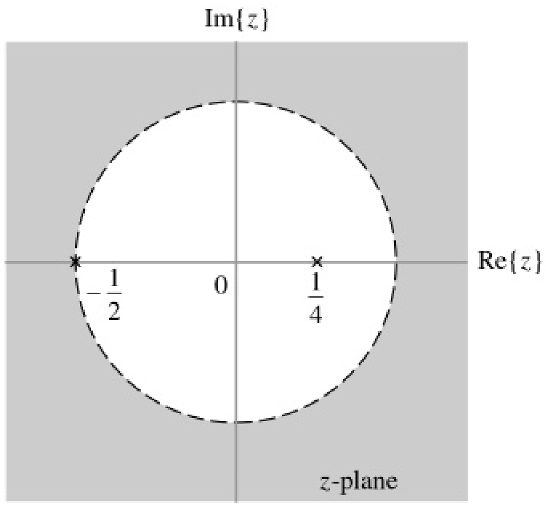
3. $$W(z)=\sum_{n=-\infty}^{0}\left(\frac{-1}{2z}\right)^{n}+2\sum_{n=-\infty}^{0}\left(\frac{1}{4z}\right)^{n}=\sum_{k=0}^{\infty}(-2z)^{k}+2\sum_{k=0}^{\infty}(4z)^{k},$$

The first series converge for $|z| < 1/2$, while the second converge for $|z| < 1/4$.
So, the ROC is $|z| < 1/4$, and

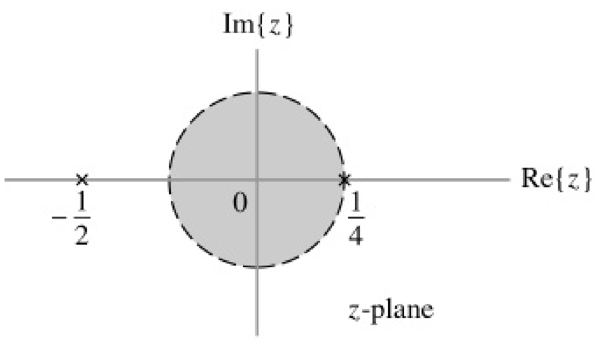
$$W(z)=\frac{1}{1+2z}+\frac{2}{1-4z},$$ Poles at $z = -1/2$ and $z = 1/4$



(a)



(b)



(c)

♣ This example illustrates that the ROC of a two-side signal (a) is a ring, i.e. in between the poles, the ROC of a right sided signal (b) is the exterior of a circle, and the ROC of a left-sided signal (c) is the interior of a circle. In each case the poles define the boundaries of the ROC.

Outline

- Introduction
- The z-Transform
- Properties of the Region of Convergence
- Properties of the z-Transform
- Inversion of the z-Transform
- The Transfer Function
- Causality and Stability
- *Determining Frequency Response from Poles & Zeros*
- *Computational Structures for DT-LTI Systems*
- *The Unilateral z-Transform*

Properties of the z-Transform

- Most properties of the z-transform are analogous to those of the DTFT.
- Assume that $x[n] \overset{z}{\longleftrightarrow} X(z), \quad \text{with} \quad \text{ROC} \quad R_x$

  $y[n] \overset{z}{\longleftrightarrow} Y(z), \quad \text{with} \quad \text{ROC} \quad R_y$
- ***Linearity:***

  $$ax[n] + by[n] \overset{z}{\longleftrightarrow} aX(z) + bY(z), \quad \text{with} \quad \text{ROC} \quad \text{at} \quad \text{least} \quad R_x \cap R_y$$

  The ROC can be larger than the intersection if one or more terms in $x[n]$ or $y[n]$ cancel each other in the sum.
- ***Time Reversal:*** $x[-n] \overset{z}{\longleftrightarrow} X(1/z), \quad \text{with ROC } 1/R_x$

  Time reversal, or reflection, corresponds to replacing z by z^{-1}. Hence, if R_x is of the form $a < |z| < b$, the ROC of the reflected signal is $a < 1/|z| < b$, or $1/b < |z| < 1/a$.
- **Time Shift:**

  $$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0}X(z), \text{with} \quad \text{ROC } R_x, \text{except} \quad \text{possibly} \quad z = 0 \text{ or} \quad |z| = \infty$$

  1. Multiplication by z^{-n_o} introduces a pole of order n_o at $z = 0$ if $n_o > 0$.
  2. If $n_o < 0$, then multiplication by z^{-n_o} introduces n_o poles at ∞. If they are not canceled by zeros at infinity in $X(z)$, then the ROC of $z^{-n_o}X(z)$ cannot include $|z| < \infty$.

Properties of the *z*-Transform

- ***Multiplication by an Exponential Sequence***:

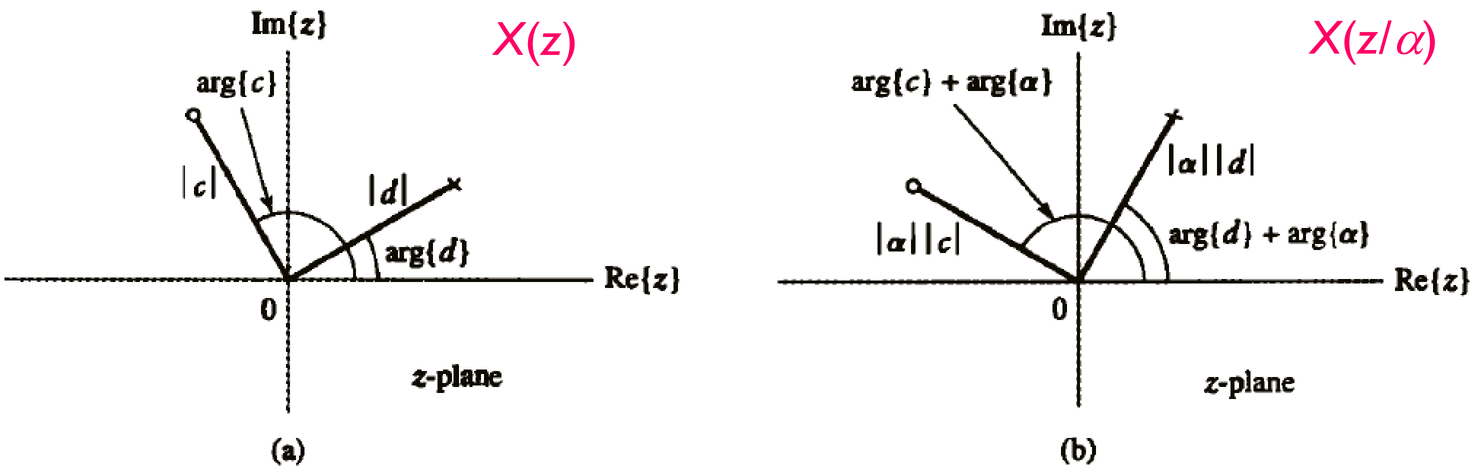
$$\alpha^n x[n] \stackrel{z}{\longleftrightarrow} X\left(\frac{z}{\alpha}\right), \text{with} \quad \text{ROC} \quad |\alpha| R_x$$

α Is a complex number

1.If $X(z)$ contains a factor $(1-dz^{-1})$ in the denominator, so that d is pole, then $X(z/\alpha)$ has a factor $(1-\alpha dz^{-1})$ in the denominator and thus has a pole at αd.

2. If c is a zero of $X(z)$, then $X(z/\alpha)$ has a zero at αc.

➔The poles and zeros of $X(z)$ have their radii changed by $|\alpha|$ in $X(z/\alpha)$, as well as their angles are changed by $\arg\{\alpha\}$ in $X(z/\alpha)$



Properties of the z-Transform

- ***Convolution***: $x[n] * y[n] \xleftrightarrow{z} X(z)Y(z), \quad \text{with} \quad \text{ROC} \quad \text{at least } R_x \cap R_y$

  Convolution of time-domain signals corresponds to multiplication of z-transforms.
  The ROC may be larger than the intersection of R_x and R_y if a pole-zero cancellation occurs in the product $X(z)Y(z)$.

- ***Differentiation in the z-Domain:***

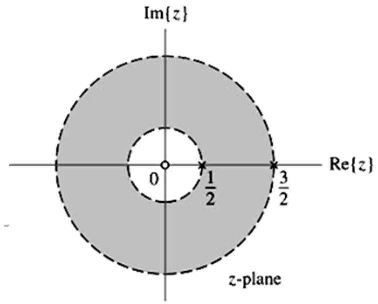
$$nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z), \quad \text{with} \quad \text{ROC } R_x$$

Multiplication by n in the time domain corresponds to differentiation with respect to z and multiplication of the result by – z in the z-domain.
This operation does not change the ROC.

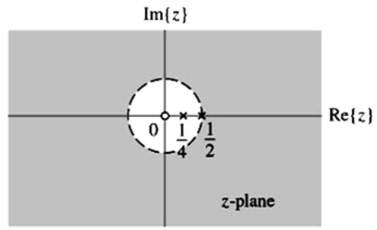
Example 7.6

Suppose

$$x[n]=\left(\frac{1}{2}\right)^n u[n]-\left(\frac{3}{2}\right)^n u[-n-1] \quad \xleftrightarrow{\;z\;} \quad X(z)=\frac{-z}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)}, \text{ and}$$



(a)



(b)

$$y[n]=\left(\frac{1}{4}\right)^n u[n]-\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{\;z\;} Y(z)=\frac{-\frac{1}{4}z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)},$$

Evaluate the z-transform of *ax*[*n*] + *by*[*n*].

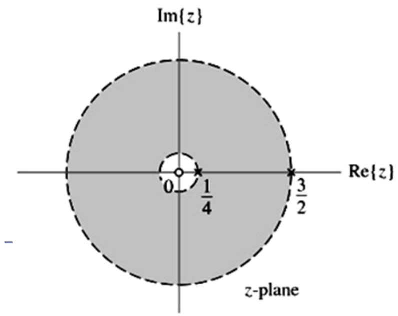
<Sol.>

$$ax[n]+by[n] \quad \xleftrightarrow{\;z\;} \quad a\frac{-z}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)}+b\frac{-\frac{1}{4}z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}.$$

In general, the ROC is the intersection of individual ROCs

However, when a = b: We see that the term $(1/2)^n$ *u*[*n*] has be canceled in *ax*[*n*] + *by*[*n*].

$$aX(z)+aY(z)=a\frac{-\frac{5}{4}z}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{2}\right)}.$$



(c)

➔ The pole at z=1/2 is canceled !! ➔ the ROC enlarges

Example 7.7

Find the z-transform of the signal $x[n]=\left(n\left(\frac{-1}{2}\right)^{n}u[n]\right)*\left(\frac{1}{4}\right)^{-n}u[-n],$

<Sol.>

First, we know that $\left(\frac{-1}{2}\right)^{n}u[n]\xleftrightarrow{z}\frac{z}{z+\frac{1}{2}}$, with ROC $|z|>\frac{1}{2}$

Apply the z-domain differentiation property, we have

$$w[n]=n\left(\frac{-1}{2}\right)^{n}u[n]\xleftrightarrow{z}W(z)=-z\frac{d}{dz}\left(\frac{z}{z+\frac{1}{2}}\right),\text{with ROC } |z|>\frac{1}{2}$$

Next, we know that $\left(\frac{1}{4}\right)^{n}u[n]\xleftrightarrow{z}\frac{z}{z-1/4}$, with ROC $|z|>\frac{1}{4}$

Apply the time-reversal property, we have

$$y[n]\xleftrightarrow{z}Y(z)=\frac{\frac{1}{z}}{\frac{1}{z}-\frac{1}{4}},\quad \text{with ROC } \frac{1}{|z|}>\frac{1}{4},$$

Last, we apply the convolution property to obtain X(z), i.e.

$$x[n]=w[n]*y[n]\xleftrightarrow{z}X(z)=W(z)Y(z),\quad \text{with ROC } R_w\cap R_y$$

$$X(z)=\frac{2z}{(z-4)\left(z+\frac{1}{2}\right)^{2}},\quad \text{with ROC } \frac{1}{2}<|z|<4$$

Example 7.8

Find the z-transform of $x[n] = a^n \cos(\Omega_0 n) u[n]$, where a is real and positive.

<Sol.>

Let $y[n] = \alpha^n u[n]$. Then we have the z-transform $Y(z) = \dfrac{1}{1 - az^{-1}}$, with ROC $|z| > a$.

Now we rewrite $x[n]$ as the sum

$$x[n] = \frac{1}{2} e^{j\Omega_0 n} y[n] + \frac{1}{2} e^{-j\Omega_0 n} y[n]$$

Then, apply the property of multiplication by a complex exponential, we have

$$\mathrm{X}(z) = \frac{1}{2} Y\left(e^{-j\Omega_0} z\right) + \frac{1}{2} Y\left(e^{j\Omega_0} z\right), \quad \text{with ROC } |z| > a$$

$$= \frac{1}{2} \frac{1}{1 - ae^{j\Omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - ae^{-j\Omega_0} z^{-1}}$$

$$= \frac{1}{2} \left(\frac{1 - ae^{-j\Omega_0} z^{-1} + 1 - ae^{j\Omega_0} z^{-1}}{\left(1 - ae^{j\Omega_0} z^{-1}\right)\left(1 - ae^{-j\Omega_0} z^{-1}\right)} \right)$$

$$= \frac{1 - a\cos(\Omega_0) z^{-1}}{1 - 2a\cos(\Omega_0) z^{-1} + a^2 z^{-2}}, \quad \text{with ROC } |z| > a.$$

Inversion of the *z*-Transform

- Direct evaluation of the inversion integral for inversion of the z-transform requires the complex variable theory
- We apply the method of partial-fraction expression, based on z-transform pairs and z-transform properties, to inverse transform.
- The inverse transform can be obtained by expressing *X*(z) as a sum of terms for which we already known the time function, which relies on the property of the ROC
- A **right-/left-** sided time signal has an ROC that lies **outside/inside** the pole radius
- Suppose that $$X(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots+a_Nz^{-N}}$$ a rational function of z^{-1}

  M < *N*

If $M \ge N$, we may use long division to express *X*(z) as $$X(z)=\sum_{k=0}^{M-N} f_k z^{-k}+\frac{\tilde{B}(z)}{A(z)}$$

1. Using the time-shift property and the pair $1 \overset{z}{\longleftrightarrow} \delta[n]$

   We obtain $$\sum_{k=0}^{M-N} f_k \delta[n-k] \overset{z}{\longleftrightarrow} \sum_{k=0}^{M-N} f_k z^{-k}$$

Inversion by Partial-Fraction Expansion

2. Factor the denominator polynomial as a product of pole factors to obtain

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 \prod_{k=1}^{N} \left(1 - d_k z^{-1}\right)}, \quad M < N$$

- Case I, If all poles d_k are distinct:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \Rightarrow A_k (d_k)^n u[n] \overset{z}{\longleftrightarrow} \frac{A_k}{1 - d_k z^{-1}}, \quad \text{with ROC } |z| > d_k$$

or

$$-A_k (d_k)^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{A_k}{1 - d_k z^{-1}}, \quad \text{with ROC } |z| < d_k$$

- Case II: If a pole d_i is repeated r times:

$$\Rightarrow \frac{A_{i_1}}{1 - d_1 z^{-1}}, \frac{A_{i_2}}{\left(1 - d_i z^{-1}\right)^2}, \ldots, \frac{A_{i_r}}{\left(1 - d_i z^{-1}\right)^r}$$

1. If the ROC is of the form $|z| > d_i$, then the right-sided inverse z-transform is chosen:

$$A \frac{(n+1)\cdots(n+m-1)}{(m-1)!} (d_i)^n u[n] \overset{z}{\longleftrightarrow} \frac{A}{\left(1 - d_i z^{-1}\right)^m}, \quad \text{with ROC } |z| > d_i$$

Appendix B !!

Inversion by Partial-Fraction Expansion

- Case II: If a pole d_i is repeated r times:

  2. If the ROC is of the form $|z| < d_i$, then the left-sided inverse z-transform is chosen:

$$-A\frac{(n+1)\cdots(n+m-1)}{(m-1)!}(d_i)^n u[-n-1] \xleftrightarrow{z} \frac{A}{\left(1-d_i z^{-1}\right)^m}, \quad \text{with} \quad \text{ROC} \quad |z| < d_i$$

- Example 7.9 Find the inverse z-transform of

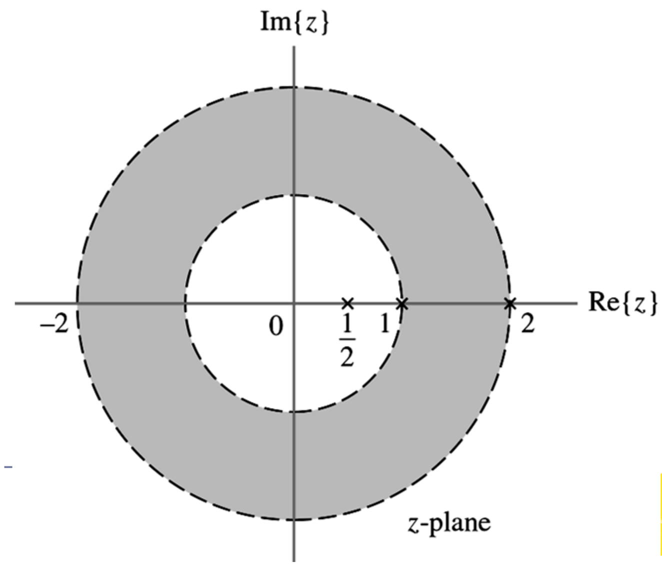
$$X(z) = \frac{1 - z^{-1} + z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\left(1-z^{-1}\right)}, \quad \text{with} \quad \text{ROC} \quad 1 < |z| < 2$$

<Sol.>

By partial fraction expansion, we obtain

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-2z^{-1}} + \frac{2}{1-z^{-1}}.$$

right-sided left-sided

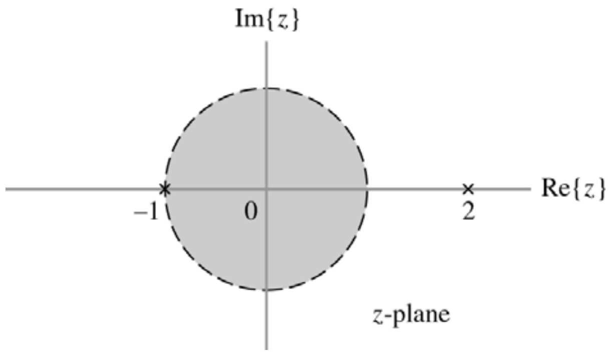


$$\Rightarrow \quad x[n] = \left(\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1] - 2u[n].$$

Example 7.10

Find the inverse z-transform of $X(z)=\dfrac{z^3-10z^2-4z+4}{2z^2-2z-4}$, with ROC $|z|<1$

<Sol.>



First, convert $X(z)$ into a ratio of polynomials in z^{-1}

$$X(z)=\frac{1}{2}z\left(\frac{1-10z^{-1}-4z^{-2}+4z^{-3}}{1-z^{-1}-2z^{-2}}\right)=\frac{1}{2}zW(z)$$

Since 3>2,

$$\frac{1-10z^{-1}-4z^{-2}+4z^{-3}}{1-z^{-1}-2z^{-2}}=-2z^{-1}+3+\frac{-5z^{-1}-2}{1-z^{-1}-2z^{-2}}=-2z^{-1}+3+\frac{-5z^{-1}-2}{(1-z^{-1})(1-2z^{-1})}.$$

$$W(z)=-2z^{-1}+3+\frac{1}{1+z^{-1}}-\frac{3}{1-2z^{-1}},\quad \text{with ROC } |z|<1$$

left-sided

$$w[n]=-2\delta[n-1]+3\delta[n]-(-1)^n u[-n-1]+3(2)^n u[-n-1].$$

Finally, apply the time-shift property to obtain $x[n]=\frac{1}{2}w[n+1]$

$$x[n]=-\delta[n]+\frac{3}{2}\delta[n+1]-\frac{1}{2}(-1)^{n+1}u[-n-2]+3(2)^n u[-n-2].$$

Remarks

- Causality, stability, or the existence of the DTFT is sufficient to determine the inverse transform

- **Causality:**
- If a signal is known to be causal, then the right-sided inverse transforms are chosen
- **Stability:**
- If a signal is stable, then it is absolutely summable and has a DTFT. Hence, stability and the existence of the DTFT are equivalent conditions. In both cases, the ROC includes the unit circle in the z-plane, $|z| = 1$. The inverse z-transform is determined by comparing the locations of the poles with the unit circle.
- If a pole is inside the unit circle, then the right-sided inverse z-transform is chosen; if a pole is outside the unit circle, then the left-sided inverse z-transform is chosen

Inversion by Power Series Expansion

- **Only one-sided signal is applicable!**
- Express X(z) as a power series in z^{-1} or in z.

  1. If the ROC is $|z| < a$, then we express $X(z)$ as power series in z^{-1}, so that we obtain a right-sided signal.

  2. If the ROC is $|z| > a$, then we express $X(z)$ as power series in z, so that we obtain a left-sided signal.

- Ex 7.11 Find the inverse z-transform of $X(z)=\dfrac{2+z^{-1}}{1-\frac{1}{2}z^{-1}}$, with ROC $|z|>\dfrac{1}{2}$

$$
\begin{array}{r l}
 & 2+2z^{-1}+z^{-2}+\tfrac{1}{2}z^{-3}+\ldots \\
\hline
1-\tfrac{1}{2}z^{-1} \,\big) & 2+z^{-1} \\
 & 2-z^{-1} \\
\hline
 & 2z^{-1} \\
 & 2z^{-1}-z^{-2} \\
\hline
 & z^{-2} \\
 & z^{-2}-\tfrac{1}{2}z^{-3} \\
\hline
 & \tfrac{1}{2}z^{-3}
\end{array}
$$

$$\Rightarrow X(z)=2+2z^{-1}+z^{-2}+\frac{1}{2}z^{-3}+\ldots$$

$$\Rightarrow x[n]=2\delta[n]+2\delta[n-1]+\delta[n-2]+\tfrac{1}{2}\delta[n-3]+\ldots$$

This may not lead to a closed-form expression !!

Example 7.12

An advantage of the power series approach is the ability to find inverse z-transforms for signals that are **not a ratio of polynomials in z**.

Find the inverse z-transform of $X(z) = e^{z^2}, \quad \text{with ROC all z except } |z| = \infty$

<Sol.>

Using the power series representation for e^a: $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$

$$\Rightarrow X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!} = 1 + \frac{z^2}{1!} + \frac{(z^2)^2}{2!} + \cdots$$

Thus $x[n] = \delta[n] + \frac{\delta[n+2]}{1!} + \frac{\delta[n+4]}{2!} + \cdots$

$$\Rightarrow x[n] = \begin{cases} 0, & n > 0 \text{ or } n \text{ odd} \\ \frac{1}{(\frac{-n}{2})!}, & \text{otherwise} \end{cases}$$

Outline

- Introduction
- The z-Transform
- Properties of the Region of Convergence
- Properties of the z-Transform
- Inversion of the z-Transform
- The Transfer Function
- Causality and Stability
- *Determining Frequency Response from Poles & Zeros*
- *Computational Structures for DT-LTI Systems*
- *The Unilateral z-Transform*

The Transfer Function

$x[n] = z^n$ → **LTI system, $h[n]$** → $y[n] = x[n] * h[n]$

$z = re^{j\Omega}$

- Recall that the transfer function $H(z)$ of an LTI system is $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$
- If we take the z-transform of both sides of y[n], then

$$Y[z] = H[z]X[z] \quad \Rightarrow \quad H[z] = \frac{Y[z]}{X[z]}$$

This definition applies only at values of z for which $X[z] \neq 0$

- From difference equation: $\sum_{K=0}^{\infty} a_k y[n-k] = \sum_{K=0}^{M} b_k x[n-k]$

After Substituting z^n for $x[n]$ and $z^n H(z)$ for $y[n]$, we obtain rational transfer function

$$z^n \sum_{K=0}^{N} a_k z^{-k} H(z) = z^n \sum_{K=0}^{M} b_k z^{-k} \quad \Rightarrow \quad H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

⇒ Knowledge of the poles d_k, zeros c_k, and factor $\tilde{b} = b_0 / a_0$ completely determine the system

Example 7.13

Find the transfer function and impulse response of a causal LTI system if the input to the system is $x[n] = (-1/3)^n u[n]$ and the output is $y[n] = 3(-1)^n u[n] + (1/3)^n u[n]$

<Sol.>

The *z*-transforms of the input and output are respectively given by

$$X(z) = \frac{1}{1+(1/3)z^{-1}}, \ \text{ROC} \ |z| > \tfrac{1}{3} \quad \text{and} \quad Y(z) = \frac{3}{1+z^{-1}} + \frac{1}{1-(1/3)z^{-1}}, \ \text{ROC} \ |z| > 1$$

Hence, the transfer function is

$$H(z) = \frac{4(1+(1/3)z^{-1})}{(1+z^{-1})(1+(-1/3)z^{-1})}, \ \text{ROC} \ |z| > 1$$

The impulse response of the system is obtain by finding the inverse *z*-transform of *H*(*z*). Applying a partial fraction expansion to *H*(z) yields

$$H(z) = \frac{2}{1+z^{-1}} + \frac{2}{1-(1/3)z^{-1}}, \quad \text{with} \quad \text{ROC} \quad |z| > 1$$

$$h[n] = 2(-1)^n u[n] + 2(1/3)^n u[n]$$

Examples 7.14 & 7.15

Determine the transfer function and the impulse response for the causal LTI system described by $y[n]-(1/4)y[n-1]-(3/8)y[n-1]=-x[n]+2x[n-1]$

<Sol.>

We first obtain the transfer function by taking the z-transform:

$$H(z)=\frac{-1+2z^{-1}}{1-(1/4)z^{-1}-(3/8)z^{-2}} \Rightarrow H(z)=\frac{-2}{1+(1/2)z^{-1}}+\frac{1}{1-(3/4)z^{-1}}$$

The system is causal, so we choose the right-side inverse z-transform for each term to obtain the following impulse response: $h[n]-2(-1/2)^n u[n]+(3/4)^n u[n]$

Find the difference-equation description of an LTI system with transfer function

$$H(z)=\frac{5z+2}{z^2+3z+2}$$

<Sol.>

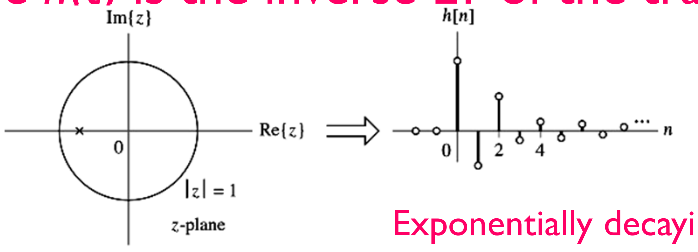
We rewrite $H(z)$ as a ratio of polynomials in z^{-1}: $H(z)=\dfrac{5z^{-1}+2z^{-2}}{(1+3z^{-1}+2z^{-2})}$

$$\Rightarrow y[n]+3y[n-1]+2y[n-2]=5x[n-1]+2x[n-2]$$

Causality and Stability

- The impulse response $h(t)$ is the inverse LT of the transfer function $H(z)$
- Causality

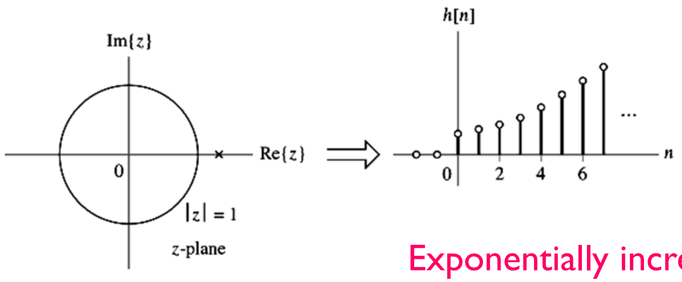
right-sided
inverse z-transform



(a)

Exponentially decaying term

Pole d_k inside the unit circle, i.e., $|d_k| < 1$

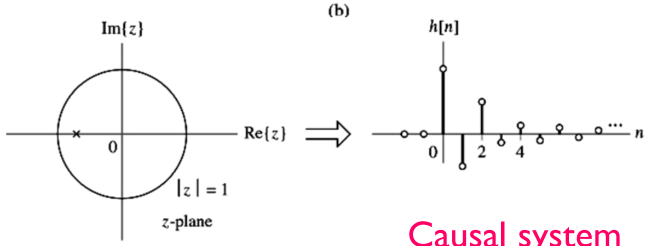


(b)

Exponentially increasing term

- Stability

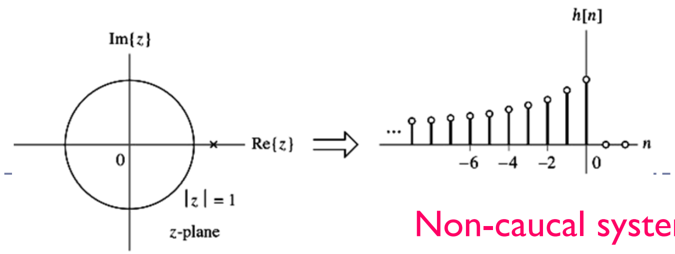
the ROC includes
unit circle in z-plane



(a)

Causal system

Pole d_k inside the unit circle, i.e., $|d_k| < 1$



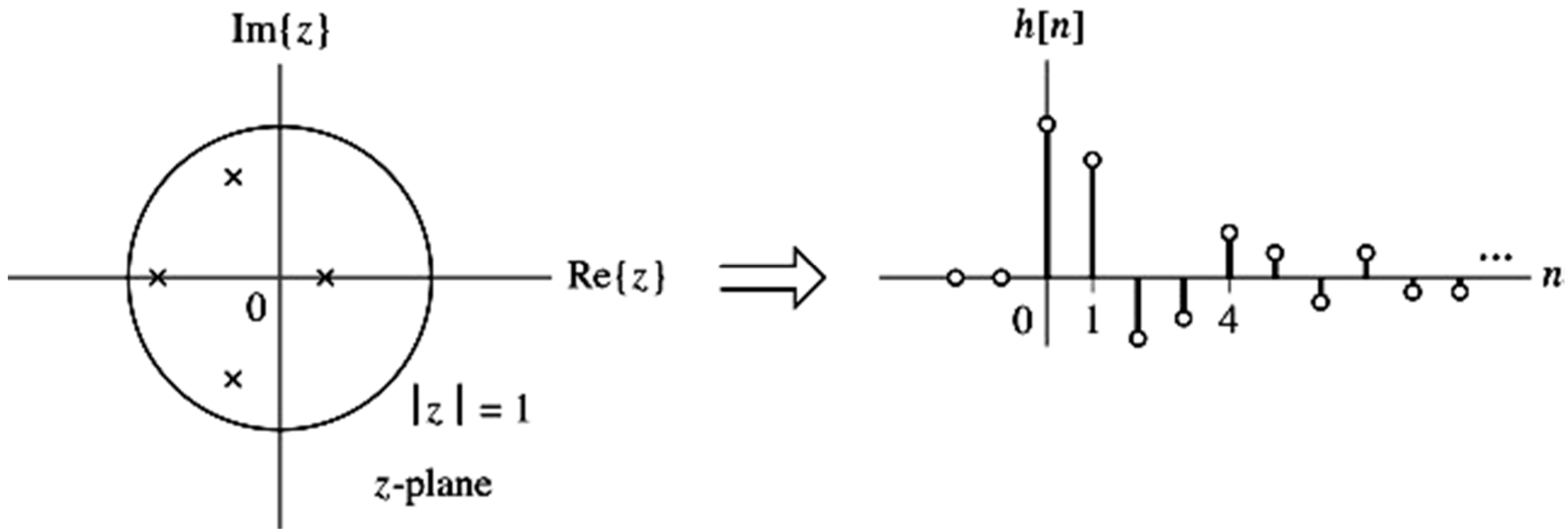
(b)

Non-caucal system

Pole d_k outside the unit circle, i.e., $|d_k| > 1$

Causal and Stable LTI System

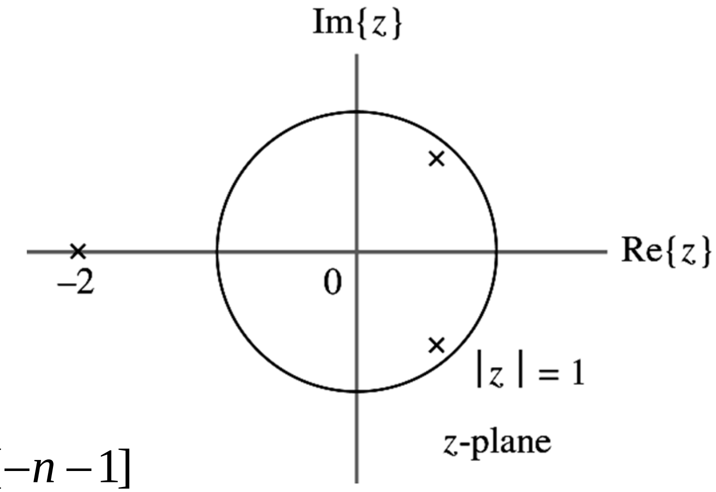
- To obtain a unique inverse transform of $H(z)$, we must know the ROC or have other knowledge(s) of the impulse response
- The relationships between the poles, zeros, and system characteristics can provide some additional knowledges
- **Systems that are stable and causal must have all their poles inside the unit circle of the z-plane**:



Example 7.16

An LTI system has the transfer function $H(z) = \dfrac{2}{1-0.9e^{j\frac{\pi}{4}}z^{-1}} + \dfrac{2}{1-0.9e^{-j\frac{\pi}{4}}z^{-1}} + \dfrac{3}{1+2z^{-1}}$

Find the impulse response, assuming that the system is (a) stable, or (b) causal. (c) Can this system both stable and causal?



<Sol.>

a. If the system is stable, then the ROC includes the unit circle. The two conjugate poles inside the unit circle contribute the right-sided term to the impulse response, while the pole outside the unit circle contributes a left-sided term.

$$h(n) = 2(0.9e^{j\frac{\pi}{4}})^n u[n] + 2(0.9e^{-j\frac{\pi}{4}})^n u[n] - 3(-2)^n u[-n-1]$$

$$= 4(0.9)^n \cos(\frac{\pi}{4}n)u[n] - 3(-2)^n u[-n-1]$$

b. If the system is causal, then all poles contribute right-sided terms to the impulse response.

$$h(n) = 2(0.9e^{j\frac{\pi}{4}})^n u[n] + 2(0.9e^{-j\frac{\pi}{4}})^n u[n] + 3(-2)^n u[n]$$

$$= 4(0.9)^n \cos[\frac{\pi}{4}n]u[n] + 3(-2)^n u[n]$$

c. the LTI system cannot be both stable and causal, since there is a pole outside the unit circle.

Example 7.17

The first-order recursive equation $y[n]-\rho y[n-1]=x[n]$
may be used to describe the value $y[n]$ of an investment by setting $\rho=1+r/100$, where r is the interest rate per period, expressed in percent. Find the transfer function of this system and determine whether it can be both stable and causal.

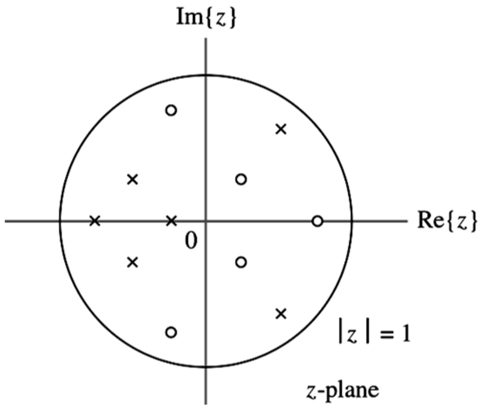
<Sol.>

The transfer function is determined by using z-transform: $$H(z)=\frac{1}{1-\rho z^{-1}}$$

This LTI system cannot be both stable and causal, because the pole at $z=\rho>1$

Inverse System

- Given an LTI system with impulse response $h[n]$, the impulse response of the inverse system, $h^{inv}[n]$, satisfies the condition $h^{inv}[n] * h[n] = \delta[n]$
  - $H^{inv}(z)H(z) = 1$ or $H^{inv}(z) = \dfrac{1}{H(z)}$
  - the poles of the inverse system $H^{inv}(z)$ are the zeros of $H(z)$, and vice versa
  - a stable and causal inverse system exists only if all of the zeros of $H(z)$ are inside the unit circle in the z-plane.
  - A (stable and causal) $H(z)$ has all of its poles and zeros inside the unit circle
  - ***H*(z) is *minimum phase*.**



A *nonminimum-phase system* cannot have a stable and causal inverse system.

Example 7.18

An LTI system is described by the difference equation

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=x[n]+\frac{1}{4}x[n-1]+\frac{1}{8}x[n-2]$$

Find the transfer function of the inverse system. Does a stable and causal LTI inverse system exist?

<Sol.>

The transfer function of the given system is $H(z)=\dfrac{1+\frac{1}{2}z^{-1}-\frac{1}{8}z^{-2}}{1-z^{-1}+\frac{1}{4}z^{-2}}$

$$=\frac{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})^2}$$

Hence, the inverse system then has the transfer function

$$H^{inv}(z)=\frac{(1-\frac{1}{2}z^{-1})^2}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$

Both of the poles of the inverse system, i.e. $z=1/4$ and $z=-1/2$, are inside the unit circle. The inverse system can be both stable and causal. Note that this system is also minimum phase, since all zeros and poles of the system are inside the unit circle.

Example 7.19

Recall a two-path communication channel is described by $y[n] = x[n] + ax[n-1]$

Find the transfer function and difference-equation description of the inverse system. What must the parameter *a* satisfy for the inverse system to be stable and causal?

<Sol.>

First find the transfer function of the two-path multiple channel system:

$$H(z) = 1 + az^{-1}$$

Then, the transfer function of the inverse system is $H^{inv}(z) = \dfrac{1}{H(z)} = \dfrac{1}{1+az^{-1}}$

➔ The corresponding difference-equation representation is $y[n] + ay[n-1] = x[n]$

The inverse system is both stable and causal when |*a*| < 1.

Outline

- Introduction
- The z-Transform
- Properties of the Region of Convergence
- Properties of the z-Transform
- Inversion of the z-Transform
- The Transfer Function
- Causality and Stability
- *Determining Frequency Response from Poles & Zeros*
- *Computational Structures for DT-LTI Systems*
- *The Unilateral z-Transform*