Chapter 6:
The Laplace Transform

Chih-Wei Liu

Outline

- Introduction
- The Laplace Transform
- The Unilateral Laplace Transform
- Properties of the Unilateral Laplace Transform
- Inversion of the Unilateral Laplace Transform
- Solving Differential Equations with Initial Conditions
- Laplace Transform Methods in Circuit Analysis
- Properties of the Bilateral Laplace Transform
- Properties of the Region of convergence
- Inversion of the bilateral Laplace Transform

Outline

- The Transfer Function
- Causality and Stability
- Determining Frequency Response from Poles & Zeros

Introduction

- The *Laplace transform* (LT) provides a broader characterization of continuous-time LTI systems and their interaction with signals than is possible with Fourier transform
- Signal that is not absolutely integral → Laplace transform; ✗→ Fourier transform
- Two varieties of LT:
  - Unilateral or one-sided
  - Bilateral or two-sided
  - The unilateral Laplace transform (ULT) is for solving differential equations with initial conditions.
  - The bilateral Laplace transform (BLT) offers insight into the nature of system characteristics such as stability, causality, and frequency response.

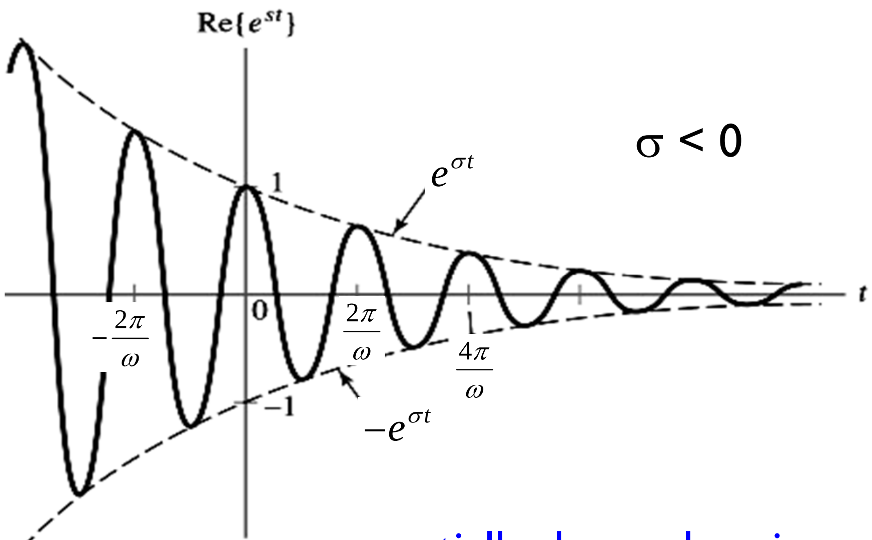
A General Complex Exponential e^{st}

- Complex exponential e^{st} with complex frequency $s=\sigma+j\omega$

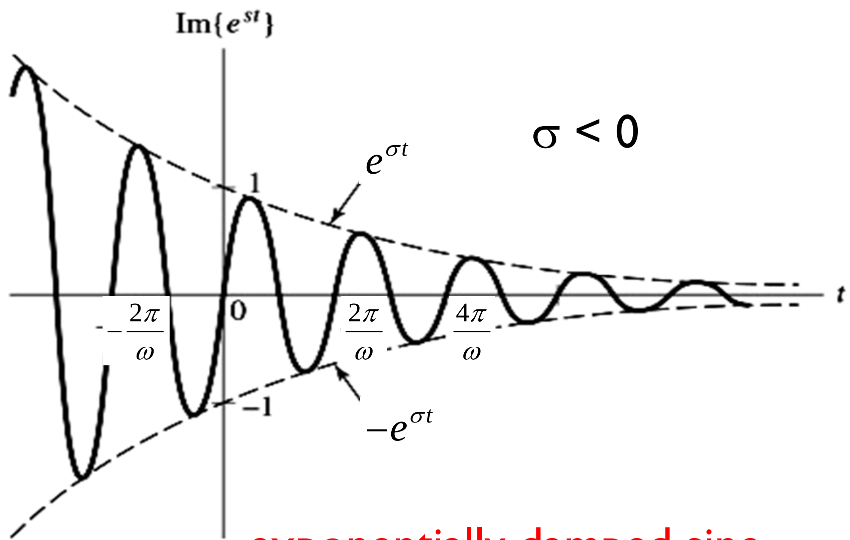
$$e^{st} = e^{\sigma t}\cos(\omega t) + j e^{\sigma t}\sin(\omega t).$$

Re{s}=σ: exponential damping factor

Im{s}=ω: frequency of the cosine and sine factor



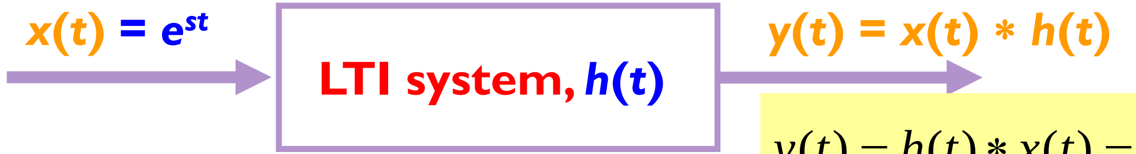
exponentially damped cosine



exponentially damped sine

- e^{st} is an eigenfunction of the LTI system

Eigenfunction Property of e^{st}



$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$
$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$
$$= e^{st}H(s)$$

- Transfer function $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$
- $H(s)$ is the eigenvalue of the eigenfunction e^{st}
- Polar form of $H(s)$: $H(s) = |H(s)|e^{j\phi(s)}$

$|H(s)| \equiv$ amplitude of $H(s)$; $\phi(s) \equiv$ phase of $H(s)$

Then $y(t) = |H(s)|e^{j\phi(s)}e^{st}$ Let $s = \sigma + j\omega$

$$y(t) = |H(\sigma + j\omega)|e^{\sigma t}e^{j\omega t + \phi(\sigma + j\omega)}$$
$$= |H(\sigma + j\omega)|e^{\sigma t}\cos(\omega t + \phi(\sigma + j\omega)) + j|H(\sigma + j\omega)|e^{\sigma t}\sin(\omega t + \phi(\sigma + j\omega))$$

The LTI system changes the amplitude of the input by $|H(\sigma + j\omega)|$ and shifts the phase of the sinusoidal components by $\phi(\sigma + j\omega)$.

The Laplace Transform

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$$

- Definition: The Laplace transform of $x(t)$: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$
- Definition: The inverse Laplace transform of $X(s)$: $x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$

- A representation of arbitrary signals as a weighted superposition of eigenfunctions e^{st} with $s=\sigma+j\omega$. We obtain

$$H(\sigma + j\omega) = \int_{-\infty}^{\infty} h(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} \left[h(t) e^{-\sigma t} \right] e^{-j\omega t} dt$$

$$\Rightarrow h(t)e^{-\sigma t} \overset{FT}{\longleftrightarrow} H(\sigma + j\omega)$$

***Laplace transform* is the FT of $h(t)\,e^{-\sigma t}$**

Hence

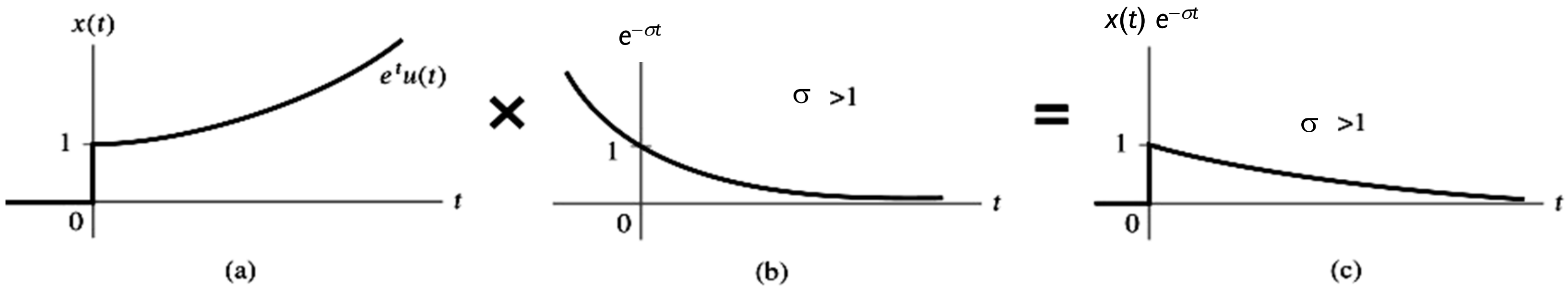
$$h(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega) e^{j\omega t} d\omega \Rightarrow h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega$$

$s = \sigma + j\omega$, $ds = jd\omega$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + \infty} H(s) e^{st} ds$$

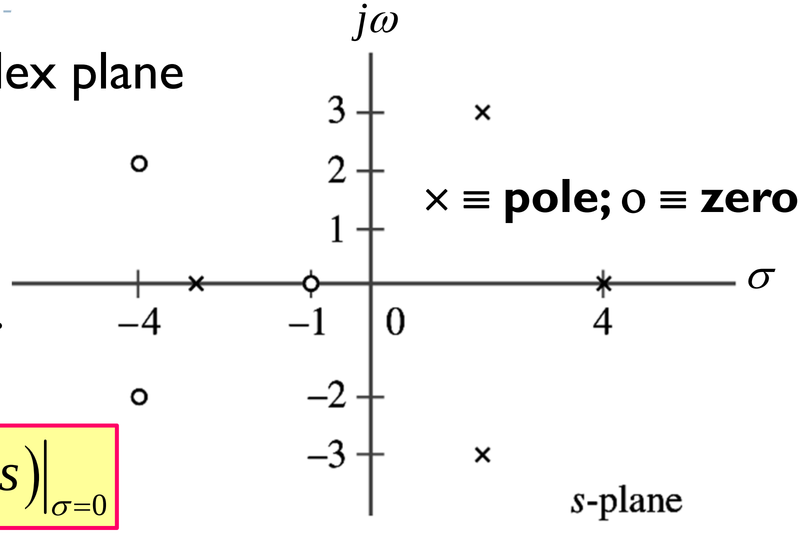
Convergence of Laplace Transform

- LT is the FT of $x(t)\,e^{-\sigma t}$ ➔ A necessary condition for convergence of the LT is the absolute integrability of $x(t)\,e^{-\sigma t}$: $\int_{-\infty}^{\infty} \left| x(t) e^{-\sigma t} \right| dt < \infty$
- The range of σ for which the Laplace transform converges is termed the *region of convergence* (ROC).
- Convergence example:
  1. Fourier transform of $x(t)=e^{t}u(t)$ does not exist, since $x(t)$ is not absolutely integrable.
  2. But $x(t)\,e^{-\sigma t} = e^{(1-\sigma)t}u(t)$ is absolutely integrable, if $\sigma>1$, i.e. ROC, so the Laplace transform of $x(t)$, which is the Fourier transform of $x(t)e^{-\sigma t}$, does exist.



The *s*-Plane, Poles, and Zeros

- To represent *s* graphically in terms of complex plane
- $s = \sigma + j\omega$
- Horizontal axis of *s*-plane = real part of *s*;
- vertical axis of *s*-plane = imaginary part of *s*.



- Relation between FT and LT: $X(j\omega) = X(s)\big|_{\sigma=0}$

the Fourier transform is given by the LT evaluated along the imaginary axis

- The $j\omega$-axis divides the *s*-plane in half: *left-half* and *right-half* *s*-plane
- Laplace transform $X(s)$:

$$X(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0} \Rightarrow X(s) = \frac{b_M \prod_{k=1}^{M}(s - c_k)}{\prod_{k=1}^{N}(s - d_k)}$$

c_k = zeros of $X(s)$; d_k = poles of $X(s)$

Example 6.1

Determine the Laplace transform of $x(t) = e^{at}u(t)$, and depict the ROC and locations of poles and zeros in the s-plane. Assume that a is real.

<Sol.>

1. First find the LT of $x(t)$: $X(s) = \int_{-\infty}^{\infty} e^{at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s-a)t}dt = \frac{-1}{s-a}e^{-(s-a)t}\Big|_0^{\infty}$

2. To evaluate the limit value, we use $s = \sigma + j\omega$ to re-write $X(s)$:

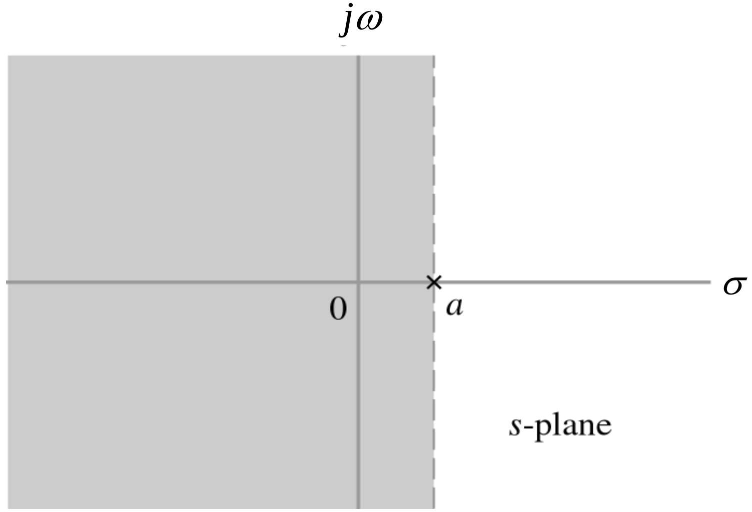
$$X(s) = \frac{-1}{\sigma + j\omega - a}e^{-(\sigma-a)t}e^{j\omega t}\Big|_0^{\infty}$$

⇒ if $\sigma > a$, then $e^{-(\sigma-a)t}$ goes to zero at $t \to \infty$

3. ROC: $\sigma > a$ or $\text{Re}(s) > a$, and

$$X(s) = \frac{-1}{\sigma + j\omega - a}(0-1)$$

$$= \frac{1}{\sigma + j\omega - a} = \frac{1}{s-a}, \quad \text{Re}(s) > a$$



Example 6.2

Determine the Laplace transform of $y(t) = -e^{at}u(-t)$, and depict the ROC and locations of poles and zeros in the s-plane. Assume that a is real.

<Sol.>

1. First find the LT of $y(t)$: $Y(s) = \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st}dt = \int_{-\infty}^{0} -e^{-(s-a)t}dt = \frac{1}{s-a}e^{-(s-a)}\Big|_{-\infty}^{0}$

2. To evaluate the limit value, we use $s = \sigma + j\omega$ to re-write $Y(s)$:

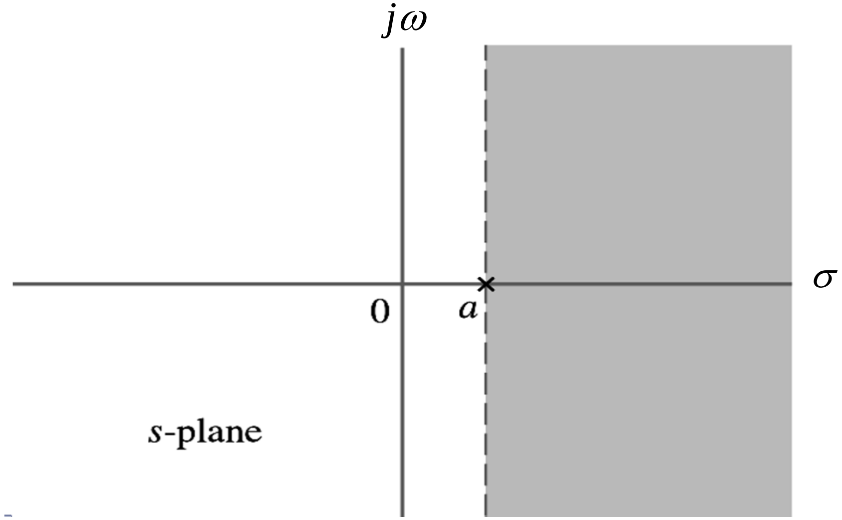
$$Y(s) = \frac{1}{\sigma + j\omega - a}e^{-(\sigma-a)t}e^{j\omega t}\Big|_{-\infty}^{0}$$

➡ if $\sigma < a$, then $e^{-(\sigma-a)t}$ goes to zero at $t \to -\infty$

3. ROC: $\sigma < a$ or $\mathrm{Re}(s) < a$, and

$$Y(s) = \frac{1}{\sigma + j\omega - a}(1-0)$$

$$= \frac{1}{\sigma + j\omega - a} = \frac{1}{s-a}, \quad \mathrm{Re}(s) < a$$



Concluding Remarks

- Examples 6.1 & 6.2 reveal that the same Laplace transform but different ROCs for the different signals $x(t)$ and $y(t)$
- This ambiguity occurs in general with signals that are one sided. To see why, let $x(t)=g(t)u(t)$ and $y(t)=-g(t)u(-t)$. We may thus write

$$X(s)=\int_0^{\infty} g(t)e^{-st}dt = G(s,\infty)-G(s,0)$$

where $G(s,t)=\int g(t)e^{-st}dt$

And, $Y(s)=-\int_{-\infty}^{0} g(t)e^{-st}dt=\int_0^{-\infty} g(t)e^{-st}dt=G(s,-\infty)-G(s,0)$

We conclude that $X(s) = Y(s)$ whenever $G(s,\infty) = G(s,-\infty)$.

- The value of s for which the integral represented by $G(s,\infty)$ converges differ from those for which the integral represented by $G(s,-\infty)$ converges, and thus the ROCs are different

The ROC must be specified for the Laplace transform to be unique !!!

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The Unilateral (or One-Sided) LT

- We may assume that the signals involved are causal, that is, zero for $t < 0$.
- The **unilateral Laplace transform (ULT)** of a signal $x(t)$ is defined by

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt \qquad x(t) \overset{\mathscr{L}_u}{\longleftrightarrow} X(s)$$

The lower limit of 0^- implies that we do include discontinuities and impulses

- Note that ULT and LT are equivalent for signals that are causal.

$$e^{at}u(t) \xleftrightarrow{ULT} \frac{1}{s-a} \quad \text{equivalent to} \Leftrightarrow \quad e^{at}u(t) \xleftrightarrow{LT} \frac{1}{s-a} \text{ with ROC } \mathrm{Re}\{s\} > a$$

Since one-sided, do not specify ROC

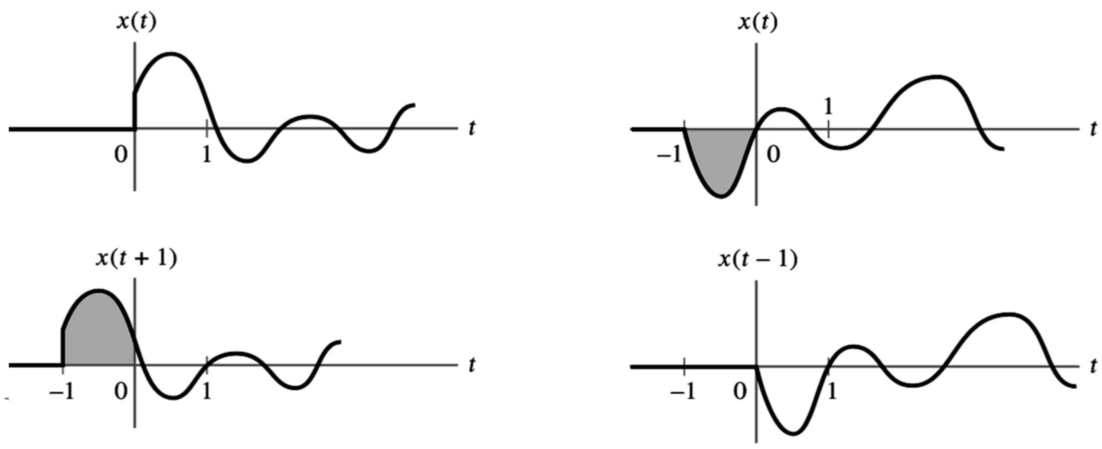
- The ambiguity of LT is removed in ULT

Properties of ULT

- Properties of ULT are similar to those of the FT
- Here we assume that $x(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} X(s)$ and $y(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} Y(s)$
- ***Linearity:*** $ax(t)+by(t) \stackrel{\mathcal{L}_u}{\longleftrightarrow} aX(s)+bY(s)$
- ***Scaling:*** $x(at) \stackrel{\mathcal{L}_u}{\longleftrightarrow} \frac{1}{a}X\left(\frac{s}{a}\right)$ for $a>0$
- ***Time Shift:***

$$x(t-\tau) \stackrel{\mathcal{L}_u}{\longleftrightarrow} e^{-s\tau}X(s) \text{ for all } \tau \text{ such that } x(t-\tau)u(t)=x(t-\tau)u(t-\tau)$$

This property applies only if the shift does not move a nonzero $t \geq 0$ component of signal to $t < 0$, or does not move a nonzero $t < 0$ portion of the signal to $t \geq 0$:



(a) (b)

Properties of ULT

- ***s*-Domain Shift:** $e^{s_0 t}x(t) \overset{\mathcal{L}_u}{\longleftrightarrow} X(s-s_0)$

- ***Convolution:*** $x(t)*y(t) \overset{\mathcal{L}_u}{\longleftrightarrow} X(s)Y(s)$

- ***Differentiation in the s-Domain:*** $-tx(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \dfrac{d}{ds}X(s)$

- Example 6.3:
  Find the unilateral Laplace transform of $x(t)=\left(-e^{3t}u(t)\right)*\left(tu(t)\right)$

  <Sol.>

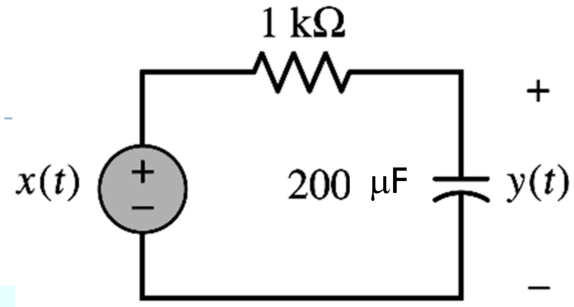
  Since $-e^{3t}u(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \dfrac{-1}{s-3}$ and $u(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \dfrac{1}{s}$

  Apply the s-domain differentiation property, we have $tu(t) \overset{\mathcal{L}_u}{\longleftrightarrow} 1/s^2$

  Now, from the convolution property, we obtain

$$x(t)=\left(e^{3t}u(t)\right)*\left(tu(t)\right) \overset{\mathcal{L}_u}{\longleftrightarrow} X(s)=\frac{-1}{(s-3)s^2}$$

Example 6.4 RC Circuit



Find the Laplace transform of the output of the RC circuit for the input $x(t) = te^{2t}u(t)$.

<Sol.>

1. The impulse response of the RC circuit is $h(t) = \frac{1}{RC} e^{-t/(RC)} u(t)$

   Then $H(s) = \frac{1}{RC} \frac{1}{s - \frac{-1}{RC}} = \frac{1}{1 + sRC}$ Using $RC = 0.2$ s ⇒ $H(s) = \frac{5}{s+5}$

2. Next, we use the s-domain differentiation property $-tx(t) \xleftrightarrow{\mathcal{L}_u} \frac{d}{ds} X(s)$

   Then $$X(s) = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$

3. We apply the convolution property to obtain the LT of the output $y(t)$:

$$Y(s) = \frac{5}{(s-2)^2 (s+5)}$$

Properties of ULT

- ***Differentiation in the Time-Domain:***

$$\frac{d}{dt}x(t) \overset{\mathcal{L}_u}{\longleftrightarrow} \int_{0^-}^{\infty}\left(\frac{d}{dt}x(t)\right)e^{-st}dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty}x(t)e^{-st}dt$$

Integration by part

$x(t)e^{-st}$ approaches zero as $t\to\infty$

$$\Rightarrow \quad \frac{d}{dt}x(t) \overset{\mathcal{L}_u}{\longleftrightarrow} sX(s) - x(0^-)$$

The **general form** for the differentiation property

$$\frac{d^n}{dt^n}x(t) \overset{\mathcal{L}_u}{\longleftrightarrow} s^nX(s) - \frac{d^{n-1}}{dt^{n-1}}x(t)\Big|_{t=0^-} - s\frac{d^{n-2}}{dt^{n-2}}x(t)\Big|_{t=0^-} - \cdots - s^{n-2}\frac{d^{n-1}}{dt^{n-1}}x(t)\Big|_{t=0^-} - s^{n-1}x(0^-)$$

Properties of ULT

- ***Integration Property:*** $\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathscr{L}_u}{\longleftrightarrow} \frac{x^{(-1)}(0^-)}{s} + \frac{X(s)}{s}$

where $x^{(-1)}(0^-) = \int_{-\infty}^{0^-} x(\tau)d\tau$ is the area under $x(t)$ from $t = -\infty$ to $t = 0^-$.

- ***Initial- and Final-Value Theorem:***

$\lim_{s\to\infty} sX(s) = x(0^+)$ $\lim_{s\to 0} sX(s) = x(\infty)$

Recall $\int_{0^-}^{\infty}\left(\frac{d}{dt}x(t)\right)e^{-st}dt = sX(s) - x(0^-)$ Then

(1) $\lim_{s\to 0} sX(s) - x(0^-) = \lim_{s\to 0}\int_{0^-}^{\infty}\left(\frac{d}{dt}x(t)\right)e^{-st}dt = \int_{0^-}^{\infty}\frac{d}{dt}x(t)dt = \int_{0^-}^{\infty} dx(t) = x(\infty) - x(0^-)$

$\Rightarrow \lim_{s\to 0} sX(s) = x(\infty)$

(2) $\lim_{s\to\infty} sX(s) - x(0^-) = \lim_{s\to\infty}\int_{0^-}^{\infty}\left(\frac{d}{dt}x(t)\right)e^{-st}dt = 0 \Rightarrow \lim_{s\to\infty} sX(s) = x(0^-)$

Remarks

- For initial-value theorem:
- ➔ This theorem does not apply to rational functions $X(s)$ in which the order of the numerator polynomial is greater than or equal to that of the denominator polynomial
- For final-value theorem
- ➔ This theorem applies only if all the poles of $X(s)$ are in the left half of the s-plane, with at most a single pole at s=0
- Example 6.6
  Determine the initial and final values of a signal $x(t)$ whose ULT is $X(s) = \frac{7s+10}{s(s+2)}$
  <Sol.>

1. Initial value:

$$x(0^+) = \lim_{s\to\infty} s\frac{7s+10}{s(s+2)} = \lim_{s\to\infty}\frac{7s+10}{s+2} = 7$$

2. Final value:

$$x(\infty) = \lim_{s\to 0} s\frac{7s+10}{s(s+2)} = \lim_{s\to 0}\frac{7s+10}{s+2} = 5$$

Note that $X(s)$ is the Laplace transform of $x(t) = 5u(t) + 2e^{-2t}u(t)$

Inversion of the ULT

- Direct inversion of the LT (ILT) requires a contour integration
- We shall determine the ILT using the one-to-one relationship between a signal and its ULT

D.1 *Basic Laplace Transforms*

Signal $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$	Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$	ROC
$u(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} > 0$
$\delta(t-\tau), \quad \tau \geq 0$	$e^{-s\tau}$	for all s
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > -a$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} > -a$
$[\cos(\omega_1 t)]u(t)$	$\frac{s}{s^2+\omega_1^2}$	$\text{Re}\{s\} > 0$
$[\sin(\omega_1 t)]u(t)$	$\frac{\omega_1}{s^2+\omega_1^2}$	$\text{Re}\{s\} > 0$
$[e^{-at}\cos(\omega_1 t)]u(t)$	$\frac{s+a}{(s+a)^2+\omega_1^2}$	$\text{Re}\{s\} > -a$
$[e^{-at}\sin(\omega_1 t)]u(t)$	$\frac{\omega_1}{(s+a)^2+\omega_1^2}$	$\text{Re}\{s\} > -a$

Inversion by Partial-Fraction Expansion

- The inverse transform can be obtained by expressing $X(s)$ as a sum of terms for which we already known the time function
- Suppose $$X(s)=\frac{B(s)}{A(s)}=\frac{b_M s^M+b_{M-1}s^{M-1}+\cdots+b_1 s+b_0}{s^N+a_{N-1}s^{N-1}+\cdots+a_1 s+a_0}$$ $M < N$

If $M \geq N$, we may use long division to express $X(s)$ as $X(s)=\sum_{k=0}^{M-N} c_k s^k+\tilde{X}(s)$

1. Using the differentiation property and the pair $\delta(t) \xleftrightarrow{\mathcal{L}_u} 1$

   We obtain $$\sum_{k=0}^{M-N} c_k \delta^{(k)}(t) \xleftrightarrow{\mathcal{L}_u} \sum_{k=0}^{M-N} c_k s^k$$

2. Factor the denominator polynomial as a product of pole factors to obtain

$$\tilde{X}(s)=\frac{b_p s^p+b_{p-1}s^{p-1}+\cdots+b_1 s+b_0}{\prod_{k=1}^{N}(s-d_k)}$$ $P < N$

Inversion by Partial-Fraction Expansion

$$\tilde{X}(s) = \frac{b_p s^p + b_{p-1} s^{p-1} + \cdots + b_1 s + b_0}{\prod_{k=1}^{N} (s - d_k)}$$

- Case I: If all poles d_k are distinct:

$$\Rightarrow \tilde{X}(s) = \sum_{k=1}^{N} \frac{A_k}{s - d_k} \quad \Rightarrow \quad A_k e^{d_k t} u(t) \xleftrightarrow{\mathcal{L}_u} \frac{A_k}{s - d_k}$$

- Case II: If a pole d_i is repeated r times:

$$\Rightarrow \frac{A_{i_1}}{s - d_i}, \quad \frac{A_{i_2}}{(s - d_i)^2}, \quad \ldots, \quad \frac{A_{i_r}}{(s - d_i)^r}$$

Apply differentiation in the *s*-domain:

$$\Rightarrow \frac{A t^{n-1}}{(n-1)!} e^{d_k t} u(t) \xleftrightarrow{\mathcal{L}_u} \frac{A}{(s - d_k)^n}$$

Example 6.7

Find the inverse Laplace transform of $X(s)=\dfrac{3s+4}{(s+1)(s+2)^2}$

<Sol.>

Use a partial-fraction expansion of $X(s)$ to write $X(s)=\dfrac{A_1}{s+1}+\dfrac{A_2}{s+2}+\dfrac{A_3}{(s+2)^2}$

Solving for A_1, A_2, and A_3, we obtain $X(s)=\dfrac{1}{s+1}-\dfrac{1}{s+2}+\dfrac{2}{(s+2)^2}$

1) The pole of the first term is at s=−1, so

$$e^{-t}u(t) \xleftrightarrow{\mathscr{L}_u} \frac{1}{s+1}$$

2) The second term has pole at s = −2; thus,

$$-e^{2t}u(t) \xleftrightarrow{\mathscr{L}_u} -\frac{1}{s+2}$$

3) The double pole in the last term is also at s = −2; hence,

$$2te^{-2t}u(t) \xleftrightarrow{\mathscr{L}_u} \frac{2}{(s+2)^2}$$

$$x(t)=e^{-t}u(t)-e^{-2t}u(t)+2te^{-2t}u(t)$$

Example 6.8

Find the inverse unilateral Laplace transform of $X(s)=\dfrac{2s^3-9s^2+4s+10}{s^2-3s-4}$

<Sol.>

$$
\begin{array}{r}
2s-3 \\
s^2-3s-4\,\overline{\big)\,2s^3-9s^2+4s+10} \\
\underline{2s^3-6s^2-8s} \\
-3s^2+12s+10 \\
\underline{-3s^2+9s+12} \\
3s-2
\end{array}
$$

$$\Rightarrow X(s)=2s-3+\frac{3s-2}{s^2-3s-4}$$

$$\Rightarrow X(s)=2s-3+\frac{1}{s+1}+\frac{2}{s-4}$$

$$\Rightarrow x(t)=2\delta^{(1)}(t)-3\delta(t)+e^{-t}u(t)+2e^{4t}u(t)$$

Conjugate Poles

- If the coefficients in the denominator are real, then all the complex poles occur in complex-conjugate pairs.
- Combine complex-conjugate poles to obtain real-valued expansion coefficients

1. Suppose that $\alpha + j\omega_0$ and $\alpha - j\omega_0$ make up a pair of complex-conjugate poles. Then

$$\frac{A_1}{s-\alpha-j\omega_0}+\frac{A_2}{s-\alpha+j\omega_0}$$

A_1 and A_2 must be complex conjugates of each other
➔ it's hard to solve A_1 and A_2 from real-valued coefficients

2. Combine conjugate poles: $\alpha + j\omega_0$ and $\alpha - j\omega_0$. Then

$$\frac{B_1 s+B_2}{(s-\alpha-j\omega_0)(s-\alpha+j\omega_0)}=\frac{B_1 s+B_2}{(s-\alpha)^2+\omega_0^{\ 2}}$$

where both B_1 and B_2 are real valued.

in term of the perfect square

$$\frac{B_1 s+B_2}{(s-\alpha)^2+\omega_0^{\ 2}}=\frac{C_1(s-\alpha)}{(s-\alpha)^2+\omega_0^{\ 2}}+\frac{C_2\omega_0}{(s-\alpha)^2+\omega_0^{\ 2}}$$

where $C_1=B_1$ and $C_2=(B_2+\alpha)/\omega_0$

$$C_1 e^{\alpha t}\cos(\omega_0 t)u(t)\xleftrightarrow{ULT}\frac{C_1(s-\alpha)}{(s-\alpha)^2+\omega_0^2}$$

$$C_2 e^{\alpha t}\sin(\omega_0 t)u(t)\xleftrightarrow{ULT}\frac{C_2\omega_0}{(s-\alpha)^2+\omega_0^2}$$

Example 6.9

Find the inverse Laplace transform of $$X(s)=\frac{4s^2+6}{s^3+s^2-2}$$

<Sol.>

$$s^3+s^2-2=(s-1)(s^2+2s+2)=(s-1)((s+1)^2+1)$$

write the quadratic s^2+2s+2 in terms of the perfect square

Then $$X(s)=\frac{A}{s-1}+\frac{B_1s+B_2}{(s+1)^2+1}$$

$$A=X(s)(s-1)\big|_{s=1}=\frac{4s^2+6}{(s+1)^2+1}\bigg|_{s=1}=2$$

$$4s^2+6=2\left((s+1)^2+1\right)+(B_1s+B_2)(s-1)$$

$$=(2+B_1)s^2+(4-B_1+B_2)s+(4-B_2)$$

$$X(s)=\frac{2}{s-1}+\frac{2s-2}{(s+1)^2+1}$$

$$=\frac{2}{s-1}+2\frac{s+1}{(s+1)^2+1}-4\frac{1}{(s+1)^2+1}$$

$$x(t)=2e^tu(t)+2e^{-t}\cos(t)u(t)-4e^{-t}\sin(t)u(t)$$

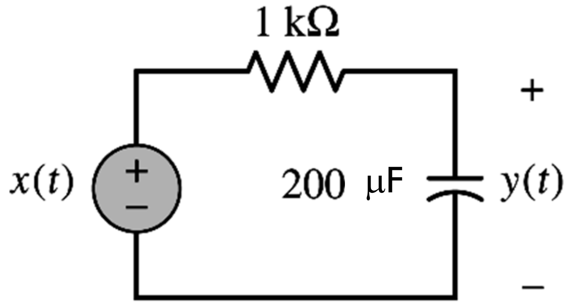
Outline

- Introduction
- The Laplace Transform
- The Unilateral Laplace Transform
- Properties of the Unilateral Laplace Transform
- Inversion of the Unilateral Laplace Transform
- Solving Differential Equations with Initial Conditions
- Laplace Transform Methods in Circuit Analysis
- Properties of the Bilateral Laplace Transform
- Properties of the Region of convergence
- Inversion of the bilateral Laplace Transform

Example 6.6: Solving Differential Equations with Initial Conditions

- Recall $\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}_u} sX(s) - x(0^-)$ The initial condition

Use the Laplace transform to find the voltage across the capacitor, $y(t)$, for the RC circuit in response to the applied voltage $x(t) = (3/5)e^{-2t}u(t)$ and initial condition $y(0^-) = -2$.



<Sol.>

Using Kirchhoff's voltage law, we have the differential equation

$$\frac{d}{dt}y(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

RC = 0.2 s

$$\Rightarrow \frac{d}{dt}y(t) + 5y(t) = 5x(t)$$

$$\Rightarrow sY(s) - y(0^-) + 5Y(s) = 5X(s) \Rightarrow Y(s) = \frac{1}{s+5}\left[5X(s) + y(0^-)\right]$$

$$x(t) \xleftrightarrow{\mathcal{L}_u} X(s) = \frac{3/5}{s+2}$$

$y(0^-) = -2$

$$\Rightarrow Y(s) = \frac{3}{(s+2)(s+5)} + \frac{-2}{s+5}$$

$$Y(s) = \frac{1}{s+2} + \frac{-1}{s+5} + \frac{-2}{s+5} = \frac{1}{s+2} - \frac{3}{s+5} \Rightarrow y(t) = e^{-2t}u(t) - 3e^{-5t}u(t)$$

Solving Differential Equations with Initial Conditions

- From the general differential equation:

$$\frac{d^N}{dt^N}y(t)+a_{N-1}\frac{d^{N-1}}{dt^{N-1}}y(t)+\cdots+a_1\frac{d}{dt}y(t)+a_0y(t)=$$

$$b_M\frac{d^M}{dt^M}x(t)+b_{M-1}\frac{d^{M-1}}{dt^{M-1}}x(t)+\cdots+b_1\frac{d}{dt}x(t)+b_0x(t)$$

- **Assume that the input is zero for *t* < 0.** Taking the unilateral Laplace transform of both sides, we obtain $A(s)Y(s)-C(s)=B(s)X(s)$ where

$$A(s)=s^N+a_{N-1}s^{N-1}+\cdots+a_1s+a_0$$

$$B(s)=b_Ms^M+b_{M-1}s^{M-1}+\cdots+b_1s+b_0$$

$$C(s)=\sum_{k=1}^{N}\sum_{l=0}^{k-1}a_ks^{k-1}\frac{d^l}{dt^l}y(t)\bigg|_{t=0^-}$$

Initial conditions

$$Y(s)=\frac{B(s)X(s)}{A(s)}+\frac{C(s)}{A(s)}$$

$$=Y^{(f)}(s)+Y^{(n)}(s)$$

Forced response due to input with zero I.C.

Natural response due to I.C. with zero input

The Laplace transform offers a clear separation between the natural response to initial conditions and the forced response with the input

Example 6.11

Use the unilateral Laplace transform to determine the output of the system described by the differential equation $\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=\frac{d}{dt}x(t)+6x(t)$

in response to the input $x(t) = u(t)$.

Assume that $y(0^-)=1$ and $\left.\frac{d}{dt}y(t)\right|_{t=0^-}=2$

Identity the forced response of the system, $y^{(f)}(t)$, and the natural response $y^{(n)}(t)$.

<Sol.>

1. Apply ULT on the both sides of the differential equations, we obtain:

$$(s^2+5s+6)Y(s)-\left.\frac{d}{dt}y(t)\right|_{t=0^-}-sy(0^-)-5y(0^-)=(s+6)X(s)$$

$$\Rightarrow Y(s)=Y^{(f)}(s)+Y^{(n)}(s)=\frac{(s+6)X(s)}{s^2+5s+6}+\frac{sy(0^-)+\left.\frac{d}{dt}y(t)\right|_{t=0^-}+5y(0^-)}{s^2+5s+6}$$

$$\Rightarrow Y^{(f)}(s)=\frac{s+6}{s(s+2)(s+3)} \quad \text{and} \quad Y^{(n)}(s)=\frac{s+7}{(s+2)(s+3)}$$

$$\Rightarrow y^{(f)}(t)=u(t)-2e^{-2t}u(t)+e^{-3t}u(t) \qquad y^{(n)}(t)=5e^{-2t}u(t)-4e^{-3t}u(t)$$

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- Properties of the Region of convergence
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Bilateral Laplace Transform (BLT)

- The BLT involves the values of the signal for both $t \geq 0$ and $t < 0$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

BLT is well suited problems involving noncausal signals and systems

- Assume that

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \quad \text{with ROC } R_x$$
$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) \quad \text{with ROC } R_y$$

ROC should be given

- **Linearity:** $ax(t) + by(t) \overset{\mathcal{L}}{\longleftrightarrow} aX(s) + bY(s) \quad \text{with ROC } R_x \cap R_y$

The ROC may be larger than $R_x \cap R_y$ if a pole and a zero cancel in $aX(s) + bY(s)$.

- Example 6.14

$$x(t) = e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) = \frac{1}{s+2} \quad \text{with ROC } \operatorname{Re}(s) > -2$$

and

$$y(t) = e^{-2t}u(t) - e^{-3t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = \frac{1}{(s+2)(s+3)} \quad \text{with ROC } \operatorname{Re}(s) > -2$$

Then

$$X(s) - Y(s) = \frac{(s+3)-1}{(s+2)(s+3)} = \frac{1}{s+3} \Rightarrow x(t) - y(t) = e^{-3t}u(t) \text{ with ROC } \operatorname{Re}(s) > -3$$

- If the intersection of ROCs is an empty set, the LT of $ax(t)+by(t)$ does not exist

Properties of BLT

- ***Time Shift***: $x(t-\tau) \overset{\mathscr{L}}{\longleftrightarrow} e^{-st}X(s)$

  Since the BLT is evaluated over both positive and negative *t*, the ROC is unchanged by a time shift.

- ***Differentiation in the Time Domain:***

$$\frac{d}{dt}x(t) \overset{\mathscr{L}}{\longleftrightarrow} sX(s)$$

with ROC at least R_x

The ROC associated with *sX*(*s*) may be larger than R_x if *X*(*s*) has a single pole at *s* = 0 on the ROC boundary

- ***Integration with respect to Time:***

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathscr{L}}{\longleftrightarrow} \frac{X(s)}{s}$$

with ROC $R_x \cap \mathrm{Re}(s) > 0$

Integration corresponds to division by *s*. Since this introduces a pole at *s* = 0 and we are integrating to the right, the ROC must lie to the right of *s* = 0

Example 6.15

Find the Laplace transform of $x(t) = \frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right)$

<Sol.>

1. We know from Ex. 6.1 that $e^{-3t}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+3}$ with ROC $\mathrm{Re}(s) > -3$

2. The time-shift property implies that

$$e^{-3(t-2)}u(t-2) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+3}e^{-2s} \quad \text{with} \quad \text{ROC} \quad \mathrm{Re}(s) > -3$$

3. Apply the time-differentiation property twice, we obtain

$$x(t) = \frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right) \stackrel{\mathscr{L}}{\longleftrightarrow} X(s) = \frac{s^2}{s+3}e^{-2s} \quad \text{with} \quad \text{ROC} \quad \mathrm{Re}(s) > -3$$

Properties of the ROC

- Recall that the BLT is not unique unless the ROC is specified.
- The ROC is related to he characteristics of a signal $x(t)$ indeed.
- (1) For rational LTs, the ROC cannot contain any poles

  Suppose d is a pole of $X(s)$, then $X(d) = \pm\infty$
- (2) The ROC consists of strips parallel to the $j\omega$-axis in the s-plane.

  Convergence of the BLT for a signal $x(t)$ implies that

  $$I(\sigma) = \int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty \quad \text{for some values of } \sigma.$$

  ➡ The set of σ with finite $I(\sigma)$ determines the ROC of BLT.

  ➡ ROC consists of strips parallel to the $j\omega$-axis in the s-plane.
- (3) The ROC for a finite-duration and bounded $x(t)$ is the entire s-plane

$$I(\sigma) \le \int_a^b A e^{-\sigma t} dt$$

$$= \begin{cases} \dfrac{-A}{\sigma}\left[e^{-\sigma t}\right|_a^b, & \sigma \ne 0 \\ A(b-a), & \sigma = 0 \end{cases}$$

➡ $I(\sigma)$ is finite for all finite values of σ

Properties of the ROC

- (4) Convergence of the BLT for a signal $x(t)$ implies that

$$I(\sigma) = \int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty \text{ for some values of } \sigma.$$

Let's separate $I(\sigma)$ into two one-sided parts, i.e. positive- and negative-time sections:

$$I(\sigma) = I_-(\sigma) - I_+(\sigma)$$

where $I_-(\sigma) = \int_{-\infty}^{0} |x(t)| e^{-\sigma t} dt$ and $I_+(\sigma) = \int_{0}^{\infty} |x(t)| e^{-\sigma t} dt$

➡ In order for $I(\sigma)$ to be finite, both $I_-(\sigma)$ and $I_+(\sigma)$ must be finite.

➡ This implies that $|x(t)|$ must be bounded in some sense.

Suppose we can bound $|x(t)|$ for both $I_-(\sigma)$ and $I_+(\sigma)$ by finding **the smallest σ_p** s.t.

$|x(t)| \le Ae^{\sigma_p t}, t > 0$ and **largest σ_n** s.t. $|x(t)| \le Ae^{\sigma_n t}, t < 0$

A signal $x(t)$ that satisfies these bounds is said to be of *exponential order*.

➡ $|x(t)|$ grows no faster than $e^{\sigma_p t}$ for positive t and $e^{\sigma_n t}$ for negative t

Properties of the ROC

Using the exponential order bounds on $|x(t)|$, we may write

$$I_+(\sigma) \le \int_0^\infty A e^{\sigma_p t} e^{-\sigma t} dt = \frac{A}{\sigma_p - \sigma} e^{(\sigma_p - \sigma)t}\Big|_0^\infty$$

$\Rightarrow$ **$I_+(\sigma)$ is finite whenever $\sigma > \sigma_p$**

$$I_-(\sigma) \le \int_{-\infty}^0 A e^{\sigma_n t} e^{-\sigma t} dt = \frac{A}{\sigma_n - \sigma} e^{(\sigma_n - \sigma)t}\Big|_{-\infty}^0$$

$\Rightarrow$ **$I_-(\sigma)$ is finite whenever $\sigma < \sigma_n$**

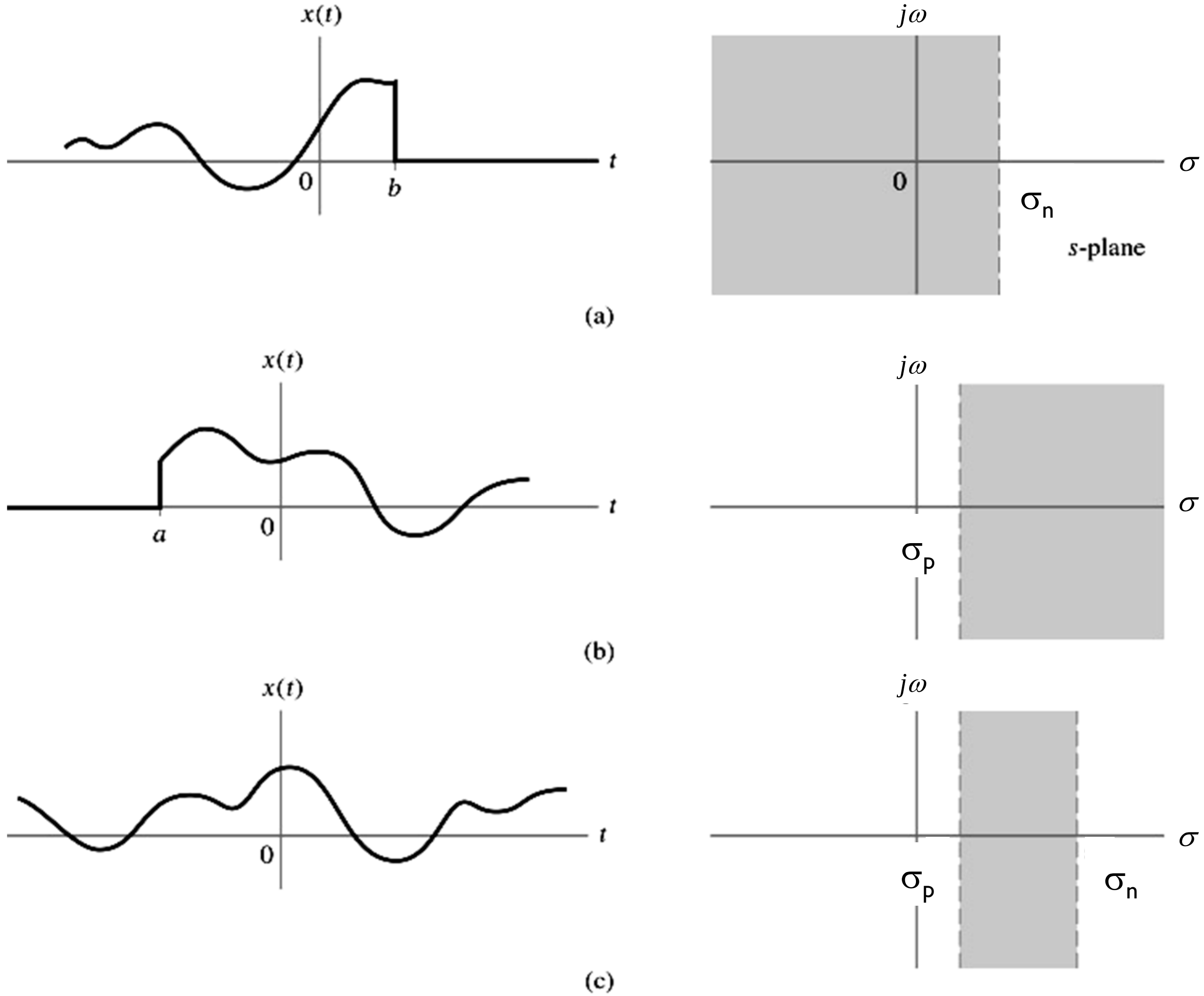
(4) If x(t) is right-sided, and if the line Re{s} = σ_p is in the ROC, then all values of s for which Re{s} > σ_p will also be in the ROC.

(5) If x(t) is left-sided, and if the line Re{s} = σ_n is in the ROC, then all values of s for which Re{s} < σ_n will also be in the ROC.

(6) If x(t) is two-sided, the ROC is of the form $\sigma_p < \sigma < \sigma_n$.

if $\sigma_p > \sigma_n$, then there are no values of σ for which the BLT converges.

Illustration Examples:



Example 6.16

Consider the two signals $x_1(t) = e^{-2t}u(t) + e^{-t}u(-t)$ and $x_2(t) = e^{-t}u(t) + e^{-2t}u(-t)$

Identify the ROC associated with the bilateral Laplace transform of each signal.

<Sol.>

1. We check the absolute integrability of $|x_1(t)|e^{-\sigma t}$ by writing

$$I_1(\sigma) = \int_{-\infty}^{\infty} |x_1(t)| e^{-\sigma t} dt$$

$$= \int_{-\infty}^{0} e^{-(1+\sigma)t} dt + \int_{0}^{\infty} e^{-(2+\sigma)t} dt$$

$$= \frac{-1}{1+\sigma}\left[e^{-(1+\sigma)t}\right|_{-\infty}^{0} + \frac{-1}{2+\sigma}\left[e^{-(2+\sigma)t}\right|_{0}^{\infty}$$

The first converges for $\sigma < -1$, while the second term converges for $\sigma > -2$. Hence, both terms converge for $-2 < \sigma < -1$. This is the intersection of the ROC for each term.

The ROC for each term and the intersection of the ROCs, which is shown as the doubly shaded region, are depicted in Fig. 6.15 (a).

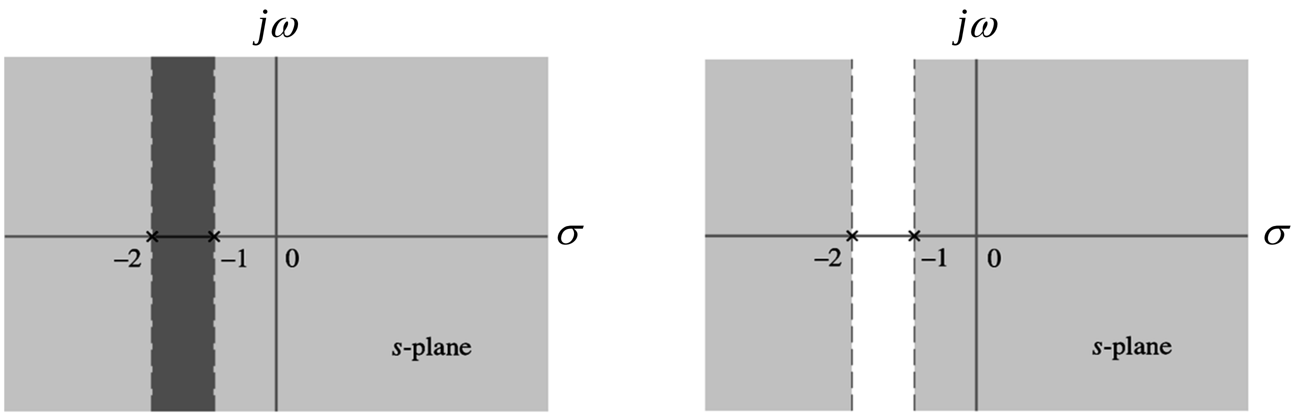
2. For the second signal, $x_2(t) = e^{-t}u(t) + e^{-2t}u(-t)$, we have

$$I_2(\sigma) = \int_{-\infty}^{\infty} \left| x_2(t) \right| e^{-\sigma t} dt$$

$$= \int_{-\infty}^{0} e^{-(2+\sigma)t} dt + \int_{0}^{\infty} e^{-1+(\sigma)t} dt$$

$$= \frac{-1}{2+\sigma} \left[e^{-(2+\sigma)t} \right|_{-\infty}^{0} + \frac{-1}{1+\sigma} \left[e^{-(1+\sigma)t} \right|_{0}^{\infty}$$

The first converges for $\sigma < -2$ and the second term converges for $\sigma > -1$.
Here, there is no value of σ for which both terms converge, so the intersection is empty.
As illustrated in Fig. 6.15(b), the bilateral Laplace transform of $x_2(t)$ does not exist.



(a) (b)

Inversion of the BLT

- Similar to the inversion of the ULT but the ROC is necessary to determine a unique inverse transform.
- Suppose $X(s) = \frac{B(s)}{A(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$ $M < N$

  If $M \geq N$, then we use long division to express $X(s) = \sum_{k=0}^{M-N} c_k s^k + \tilde{X}(s)$

$$\sum_{k=0}^{M-N} c_k \delta^{(k)}(t) \overset{\mathscr{L}_u}{\longleftrightarrow} \sum_{k=0}^{M-N} c_k s^k$$

  the LT of the impulse and its derivatives converge everywhere in the *s*-plane

- By partial-fraction expansion in terms of non-repeated poles:

$$\tilde{X}(s) = \sum_{k=1}^{N} \frac{A_k}{s - d_k}$$

Note that the ROC of $X(s)$ is the same as the ROC of $\tilde{X}(s)$

Inversion of BLT

- Two possibilities for the inverse BLT of $\tilde{X}(s)$
- 1. **Right-sided**

$$A_k e^{d_k t} u(t) \xleftrightarrow{\mathscr{L}} \frac{A_k}{s-d_k} \quad \text{with ROC} \quad \mathrm{Re}(s) > d_k$$

- 2. **Left-sided**

$$-A_k e^{d_k t} u(-t) \xleftrightarrow{\mathscr{L}} \frac{A_k}{s-d_k} \quad \text{with ROC} \quad \mathrm{Re}(s) < d_k$$

The ROC of a right-side exponential signal lies to the right of the pole, while the ROC of a left-sided exponential signal lies to the left of the pole.

- Example 6.17

Find the inverse bilateral Laplace transform of $X(s) = \dfrac{-5s-7}{(s+1)(s-1)(s+2)}$

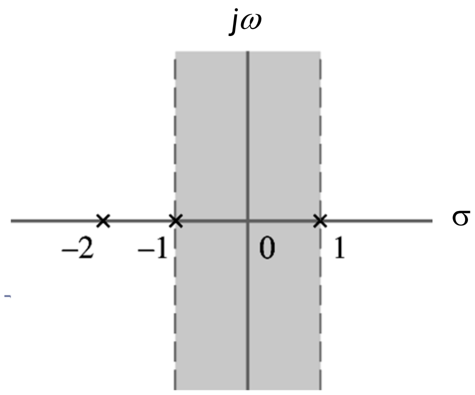
with ROC $-1 < \mathrm{Re}(s) < 1$

<Sol.>

Use the partial-fraction expansion $X(s) = \dfrac{1}{s+1} - \dfrac{2}{s-1} + \dfrac{1}{s+2}$

right-sided

left-sided



$$x(t) = e^{-t}u(t) + 2e^{t}u(-t) + e^{-2t}u(t)$$

Inversion of BLT

- By partial-fraction expansion in terms of repeated poles $\dfrac{A}{(s-d_k)^n}$
- Two possibilities for the inverse BLT of $\tilde{X}(s)$
- 1. **Right-sided** $\dfrac{At^{n-1}}{(n-1)!}e^{d_k t}u(t)$ If the ROC lies to the right of the pole d_k
- 2. **Left-sided** $\dfrac{-At^{n-1}}{(n-1)!}e^{d_k t}u(-t)$ If the ROC lies to the left of the pole d_k
- For pairs of complex-conjugate poles $\dfrac{C_1(s-\alpha)}{(s-\alpha)^2+\omega_0^2}$
- Two possibilities for the inverse BLT
- 1. **Right-sided** $C_1 e^{\alpha t}\cos(\omega_0 t)u(t)$ If the ROC lies to the right of s = $\alpha \pm j\omega_0$,
- 2. **Left-sided** $-C_1 e^{\alpha t}\cos(\omega_0 t)u(-t)$ If the ROC lies to the left of s = $\alpha \pm j\omega_0$,

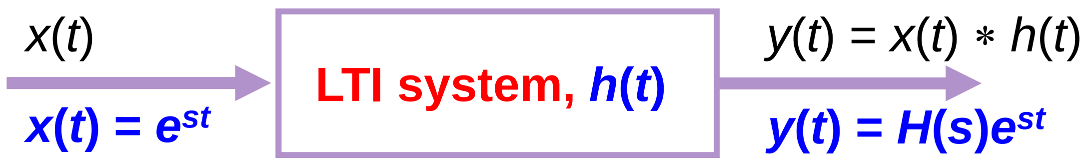
Some Remarks

- Causality
  - If the signal is known to be causal, then we choose the right-sided inverse transform of each term. (i.e. the unilateral Laplace transform)
- Stability
  - A stable signal is absolutely integrable and thus has a Fourier transform; stability and the existence of Fourier transform (i.e. Re(s)=0) are **equivalent** conditions
  - In both cases, the ROC includes the jω-axis in the s-plane, or $\mathrm{Re}(s) = 0$.
  - The inverse LT of a stable signal is obtained by comparing the poles' locations with jω-axis.
  - If a pole lies to the left/**right** of the jω-axis, then the right/**left**-sided inverse transform is chosen (the ROC should include the jω-axis)

Outline

- The Transfer Function
- Causality and Stability
- Determining Frequency Response from Poles & Zeros

The Transfer Function



- Recall that the transfer function $H(s)$ of an LTI system is $H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$
- If we take the bilateral Laplace transform of y(t), then

$$Y(s)=H(s)X(s) \Rightarrow H(s)=\frac{Y(s)}{X(s)}$$

This definition applies only at values of s for which $X(s)\neq 0$

- From differential equation: $\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t)=\sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$

After Substituting e^{st} for $x(t)$ and $e^{st}H(s)$ for $y(t)$, we obtain rational transfer function

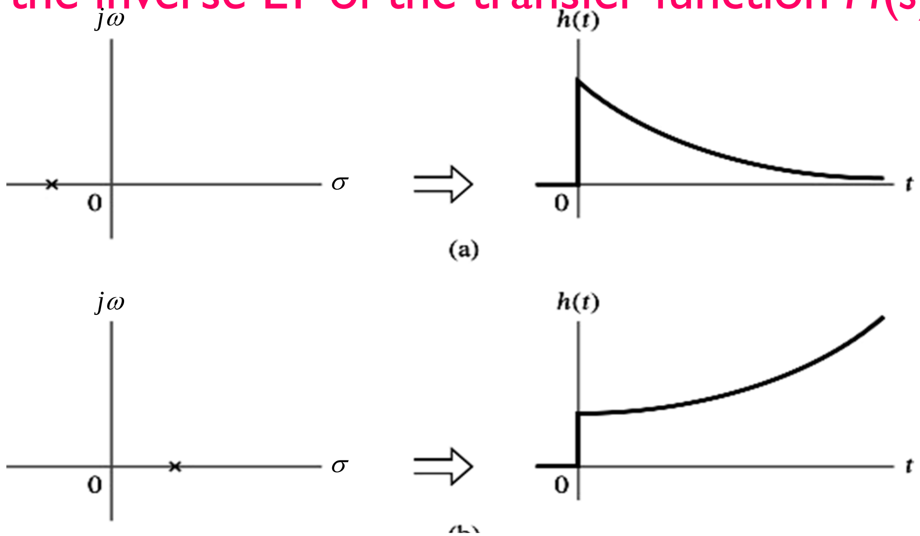
$$\left(\sum_{k=0}^{N} a_k \frac{d^k}{dt^k}\left\{e^{st}\right\}\right)H(s)=\sum_{k=0}^{M} b_k \frac{d^k}{dt^k}\left\{e^{st}\right\} \Rightarrow H(s)=\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}=\frac{\tilde{b}\prod_{k=1}^{M}(s-c_k)}{\prod_{k=1}^{N}(s-d_k)}$$

Knowledge of the poles d_k, zeros c_k, and factor $\tilde{b}\equiv b_M/a_N$ completely determine the system

Causality and Stability

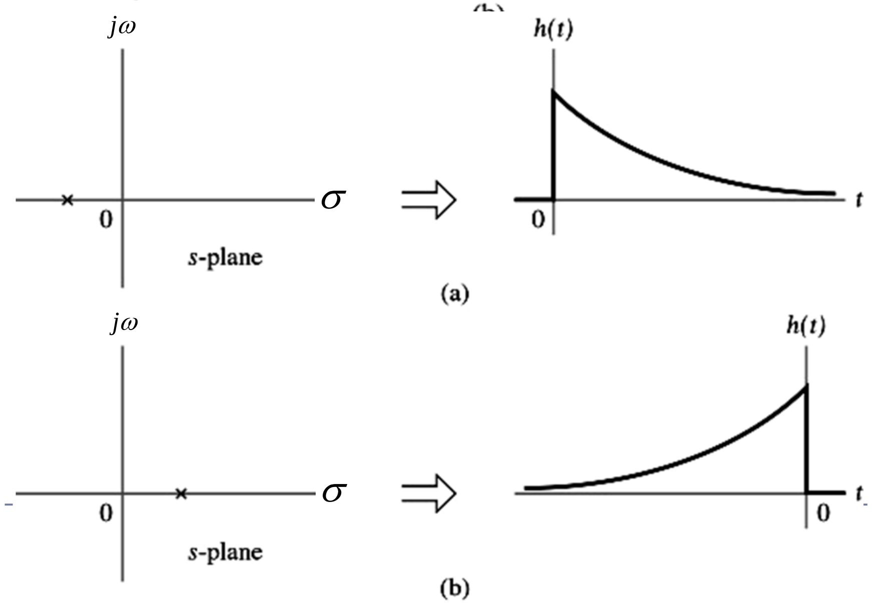
- The impulse response $h(t)$ is the inverse LT of the transfer function $H(s)$
- Causality

  right-sided inverse LT



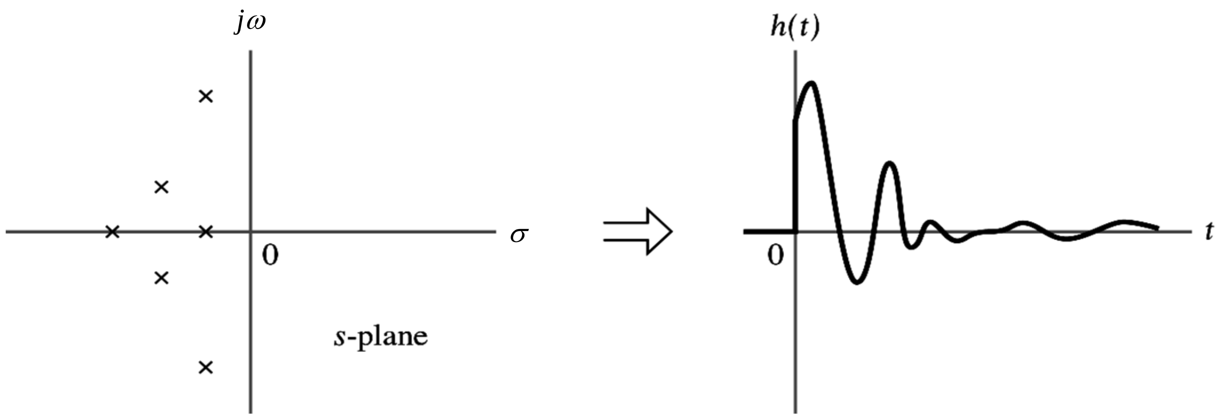
- Stability

  the ROC includes the $j\omega$-axis in the s-plane



Causal and Stable LTI System

- To obtain a unique inverse transform of $H(s)$, we must know the ROC or have other knowledge(s) of the impulse response
- The relationships between the poles, zeros, and system characteristics can provide some additional knowledges
- **Systems that are stable and causal must have all their poles in the left half of the s-plane**:



Example 6.21

A system has the transfer function $H(s) = \frac{2}{s+3} + \frac{1}{s-2}$

Find the impulse response, (a) assuming that the system is stable; (b) assuming that the system is causal; (c) can this system be both stable and causal?

<Sol.>

(a) This system has poles at $s = -3$ and $s = 2$.

Stable ➔ the ROC contains $j\omega$-axis.
the pole at $s=-3$ contributes a right-sided term;
the pole at $s=2$ contributes a left-sided term.

➡ $h(t) = 2e^{-3t}u(t) - e^{2t}u(-t)$

(b) This system has poles at $s = -3$ and $s = 2$.

Causal ➔ right-sided

➡ $h(t) = 2e^{-3t}u(t) + e^{2t}u(t)$

(c) This system has poles at $s = -3$ and $s = 2$ ➔ this system cannot be both stable and causal

Inverse System

- Given an LTI system with impulse response $h(t)$, the impulse response of the inverse system, $h^{inv}(t)$, satisfies the condition $h^{inv}(t) * h(t) = \delta(t)$
  - ➡ $H^{inv}(s)H(s) = 1$ or $H^{inv}(s) = \dfrac{1}{H(s)}$
  - ➡ the poles of the inverse system $H^{inv}(s)$ are the zeros of $H(s)$, and vice versa
  - ➡ a stable and causal inverse system exists only if all of the zeros of $H(s)$ are in the left half of the s-plane.
  - ➡ A (stable and causal) $H(s)$ has all of its poles and zeros in the left half of the s-plane
  - ➡ ***H(s)* is *minimum phase.***

A *nonminimum-phase system* cannot have a stable and causal inverse system.

Example 6.22

Consider an LTI system described by differential equation

$$\frac{d}{dt}y(t)+3y(t)=\frac{d^2}{dt^2}x(t)+\frac{d}{dt}x(t)-2x(t)$$

Find the transfer function of the inverse system. Is it a stable and causal inverse system?

<Sol.>

First find the system's transfer function $H(s)$: $H(s)=\frac{Y(s)}{X(s)}=\frac{s^2+s-2}{s+3}$

$$\Rightarrow H^{inv}(s)=\frac{1}{H(s)}=\frac{s+3}{s^2+s-2}=\frac{s+3}{(s-1)(s+2)}$$

The inverse system has pole at $s=1$ and $s=-2$. ➔ $H^{inv}(s)$ cannot be both stable and causal.

Determining Frequency Response from Poles & Zeros

- Control system

Summary

- The Laplace transform represents continuous-time signals as weighted superpositions of complex exponentials
  - The transfer function is the Laplace transform of the impulse response
  - The unilatreal Laplace transform applies to causal signals (or one-sided signals)
  - The bilateral Laplace transform applies to two-sided signals; it is not unique unless the ROC is specified.
- The Laplace transform
  - is most often used in the transient and stability analysis of system
- The Fourier transform
  - is usually employed as a signal representation tool and in solving system problems in which steady-state characteristics are of interest.