Chapter 4: Applications of Fourier Representations

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Outline

- Introduction
- Fourier Transform of Periodic Signals
- Convolution/Multiplication with (Non-)Periodic Signals
- Fourier Transform of Discrete-Time Signals
- Sampling
- Reconstruction of Continuous-Time Signals
- Discrete-Time Processing of Continuous-Time Signals
- Fourier Series of Finite-Duration Nonperiodic Signals

Introduction

- Four classes of signals in Fourier representations
  - Continuous- and discrete-time signals
  - Periodic and nonperiodc signals
- In order to use Fourier methods to analyze a general system involving a mixing of noperiodic (says, impulse response) and periodic (input) signals, we must build bridges between Fourier representations of different classes of signals
- FT/DTFT are most commonly used for analysis applications
  - We must develop FT/DTFT representations of periodic signals
- DTFS is the primary representation used for computational applications (the only one can be evaluated on a computer)
  - Use DTFS to represent the FT, FS, and DTFT

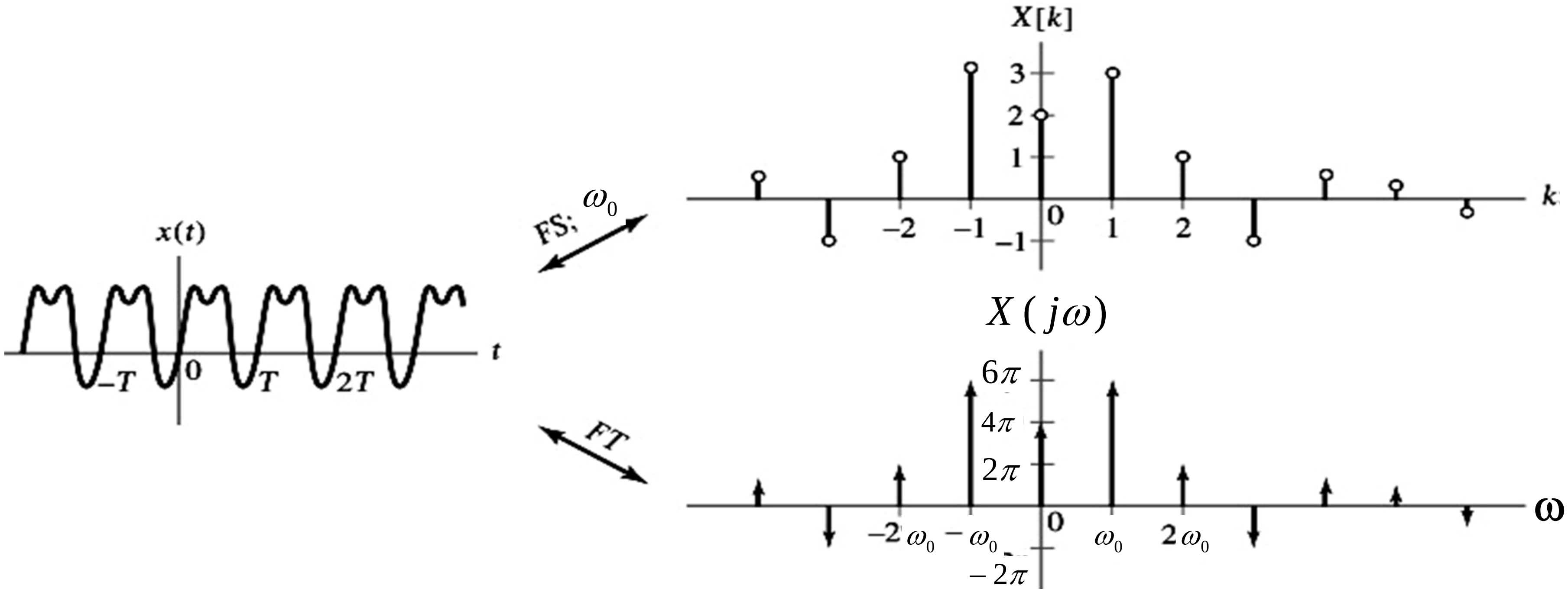
FT Representations of Periodic Signals

- Recall the FS representation of a periodic signal *x*(*t*): $x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$
- Note that $1 \xleftrightarrow{FT} 2\pi\delta(\omega)$
- Using the frequency-shift property, we have $e^{jk\omega_0 t} \xleftrightarrow{FT} 2\pi\delta\left(\omega-k\omega_0\right)$
- Let's take the FT of *x*(*t*):
- $$FT\left\{x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}\right\}=\sum_{k=-\infty}^{\infty} X[k]FT\left\{e^{jk\omega_0 t}\right\}=2\pi\sum_{k=-\infty}^{\infty} X[k]\delta(\omega-k\omega_0)$$

$$\Rightarrow \boxed{x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \xleftrightarrow{FT} X\left(j\omega\right)=2\pi\sum_{k=-\infty}^{\infty} X[k]\delta\left(\omega-k\omega_0\right)}$$

- The FT of a periodic signal is a series of impulses spaced by the fundamental frequency ω_0.
- The *k*th impulse has strength $2\pi X[k]$
- The shape of $X(j\omega)$ is identical to that of $X[k]$

FS and FT Representations of a Periodic Continuous-Time Signal



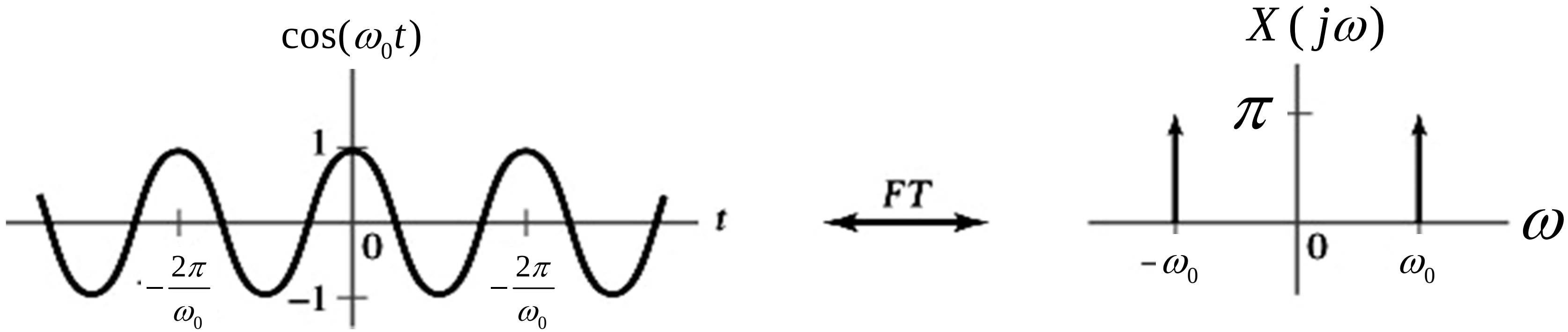
Example 4.1 FT of a Cosine

Find the FT representation of $x(t) = \cos(\omega_o t)$

<Sol.>

1. FS of $x(t)$: $$\cos(\omega_0 t) \xleftrightarrow{FS;\omega_0} X[k] = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0, & k \neq 1 \end{cases}$$

2. FT of $x(t)$: $$\cos(\omega_0 t) \xleftrightarrow{FT} X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Example 4.2 FT of a Unit Impulse Train

Find the FT of the impulse train $p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$

<Sol.>

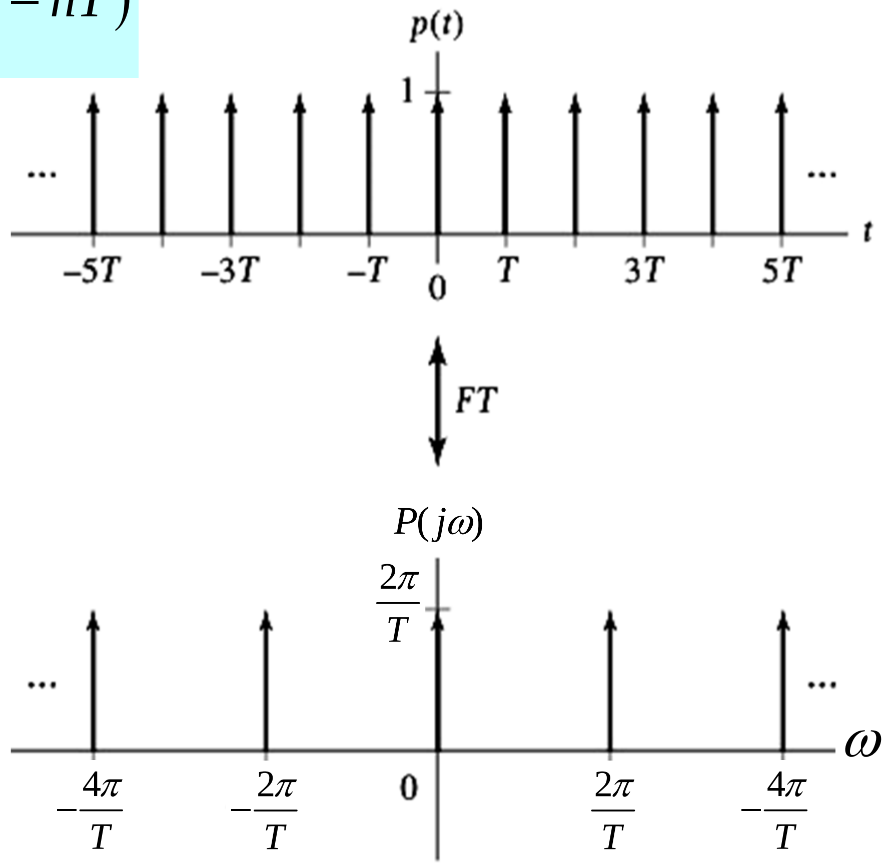
1. $p(t)$ = periodic, fundamental period = T

2. FS of $p(t)$:

$$P[k]=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\omega_0 t}dt=1/T$$

3. FT of $p(t)$:

$$P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$$



♣ The FT of an impulse train is also an impulse train.

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Relating the DTFT to the DTFS

- Recall the DTFS representation of a periodic DT signal $x[n]$:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$$

- Note that the inverse DTFT of a frequency shifted impulse, i.e. $\delta(\Omega\text{-}k\Omega)$, is a complex sinusoid. With one period of $e^{jk\Omega_0 n}$, we have

$$\frac{1}{2\pi} e^{jk\Omega_0 n} \xleftrightarrow{DTFT} \delta(\Omega - k\Omega_0) \quad -\pi < \Omega \le \pi, \ -\pi < k\Omega_0 \le \pi$$

- Let's construct an infinite series of shifted impulses separated by 2π, i.e. to obtain the following 2π-periodic function

$$\frac{1}{2\pi} e^{jk\Omega_0 n} \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - m\cdot 2\pi)$$

- Let's take the DTFT of $x[n]$:

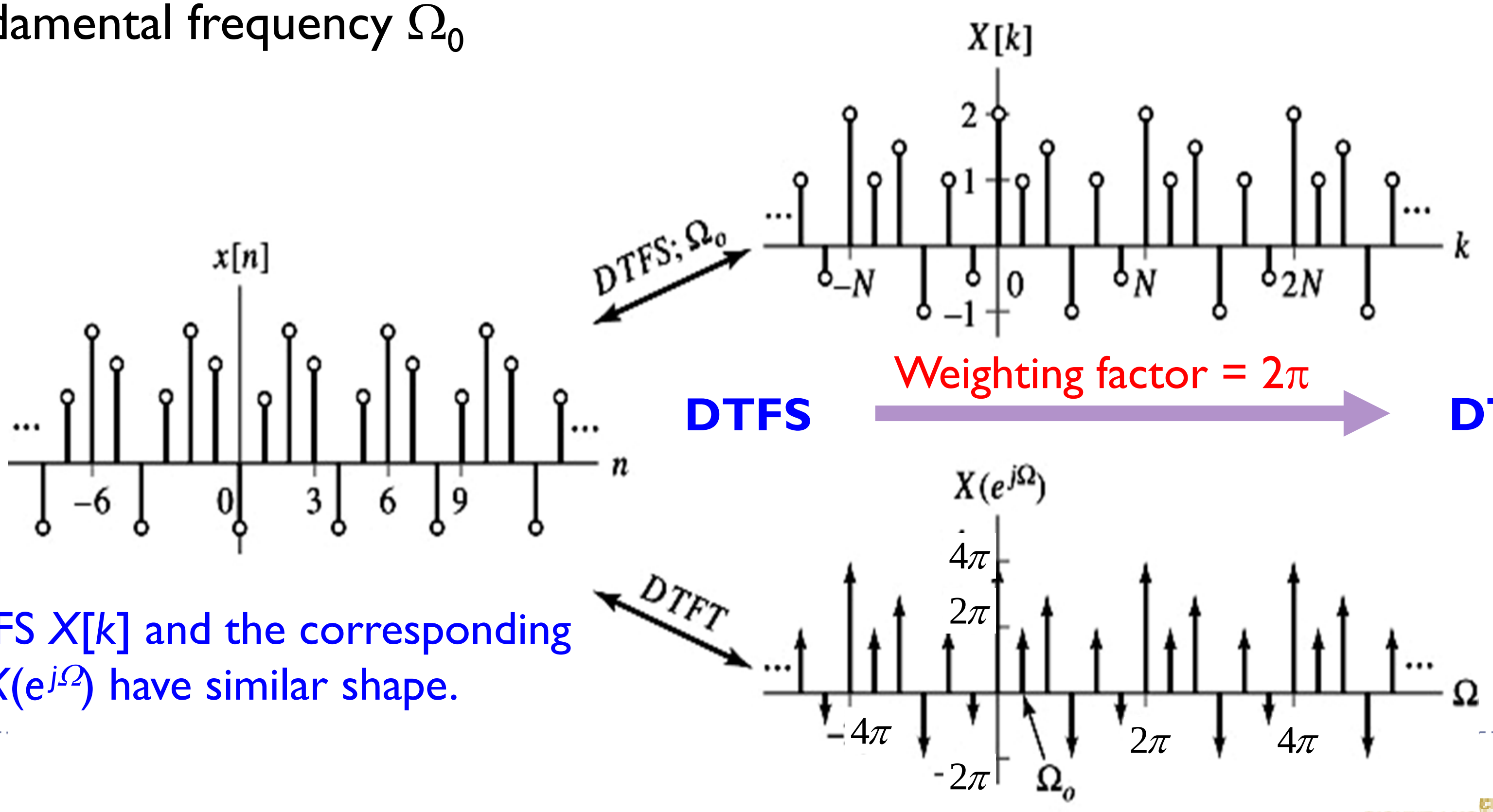
$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=0}^{N-1} X[k] \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - m\cdot 2\pi)$$

Since $X[k]$ is N periodic and $N\Omega_0 = 2\pi$, we may combine the two sums

Relating the DTFT to the DTFS

$$x[n]=\sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n} \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right)=2\pi\sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega-k\Omega_0\right)$$

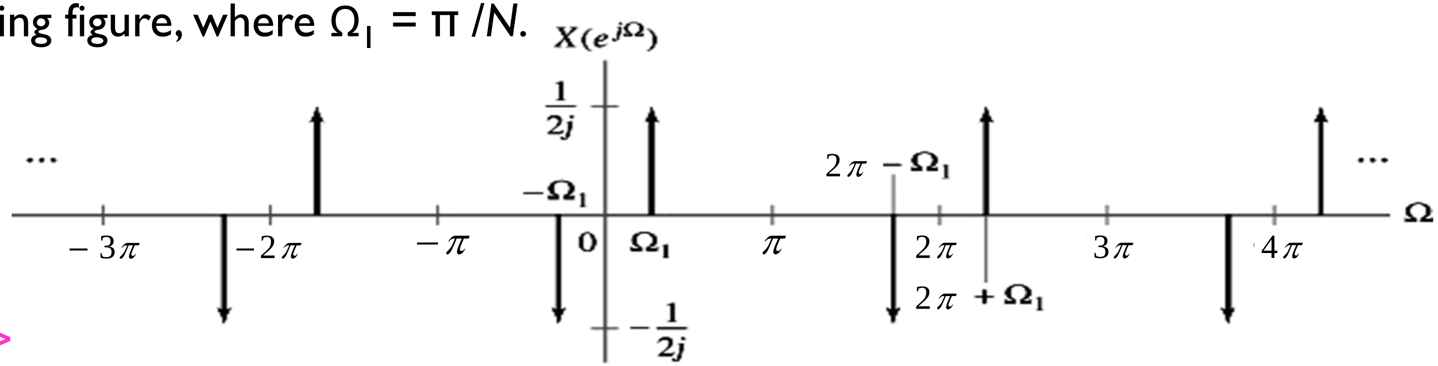
- The DTFT of a periodic signal is a series of impulses spaced by the fundamental frequency Ω_0



♣ The DTFS $X[k]$ and the corresponding DTFT $X(e^{j\Omega})$ have similar shape.

Example 4.3 DTFT of a Periodic Signal

Determine the inverse DTFT of the frequency-domain representation depicted in the following figure, where $\Omega_1 = \pi / N$.



<Sol.>

1. We express one period of $X(e^{j\Omega})$ as

$$X\left(e^{j\Omega}\right) = \frac{1}{2j}\delta\left(\Omega - \Omega_1\right) - \frac{1}{2j}\delta\left(\Omega + \Omega_1\right), \quad -\pi < \Omega \le \pi$$

from which we infer that

$$\Rightarrow X[k] = \begin{cases} 1/(4\pi j), & k = 1 \\ -1/(4\pi j), & k = -1 \\ 0, & \text{otherwise on } -1 \le k \le N-2 \end{cases}$$

2. The inverse DTFT:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$$

$$\Rightarrow x[n] = \frac{1}{2\pi}\left[\frac{1}{2j}\left(e^{j\Omega_1 n} - e^{-j\Omega_1 n}\right)\right] = \frac{1}{2\pi}\sin\left(\Omega_1 n\right)$$

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Convolution and Multiplication with Mixtures of Periodic and Nonperiodic Signals

- Example: a periodic signal fed into a stable, nonperiodic impulse response

Periodic input $x(t)$ → **Stable filter** **Nonperiodic** impulse response $h(t)$ → $y(t) = x(t) * h(t)$

- FT is applied to continuous-time (CT) case
- DTFT is applied to discrete-time (DT) case

- For CT signals: $y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$
- If $x(t)$ is CT periodic signal, then

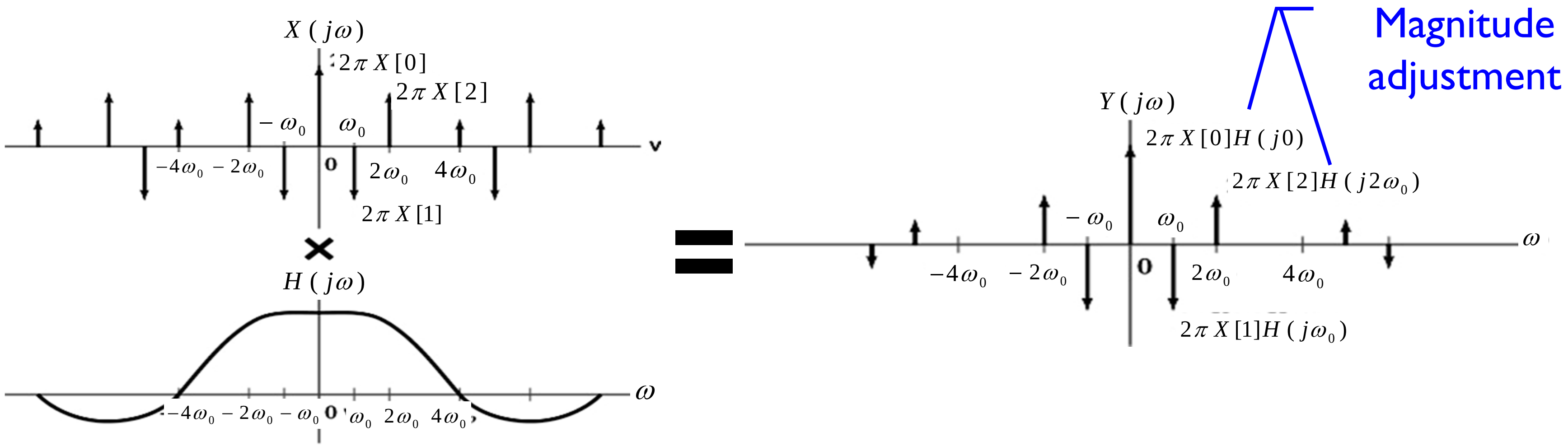
$$x(t) \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

$$\Rightarrow y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)H(j\omega)$$

$$\boxed{y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} H(jk\omega_0)X[k]\delta(\omega - k\omega_0)}$$

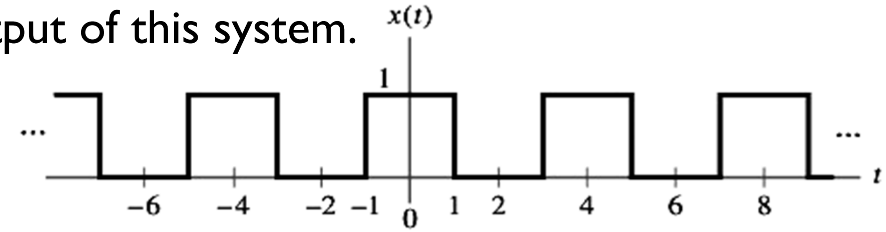
Convolution with Mixtures of Periodic and Nonperiodic Signals

$$\boxed{y(t) = x(t) * h(t) \quad \xleftrightarrow{FT} \quad Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} H(jk\omega_0)\, X[k]\, \delta(\omega - k\omega_0)}$$



Example 4.4

A periodic square wave is applied to a system with impulse response $h(t)=(1/(\pi t))\sin(\pi t$.
Use the convolution property to find the output of this system.



<Sol.>

1. The frequency response of the LTI system is by taking the FT of the impulse response $h(t)$:

$$h(t) \xleftrightarrow{FT} H(j\omega) = \begin{cases} 1, & |\omega| \le \pi \\ 0, & |\omega| > \pi \end{cases}$$

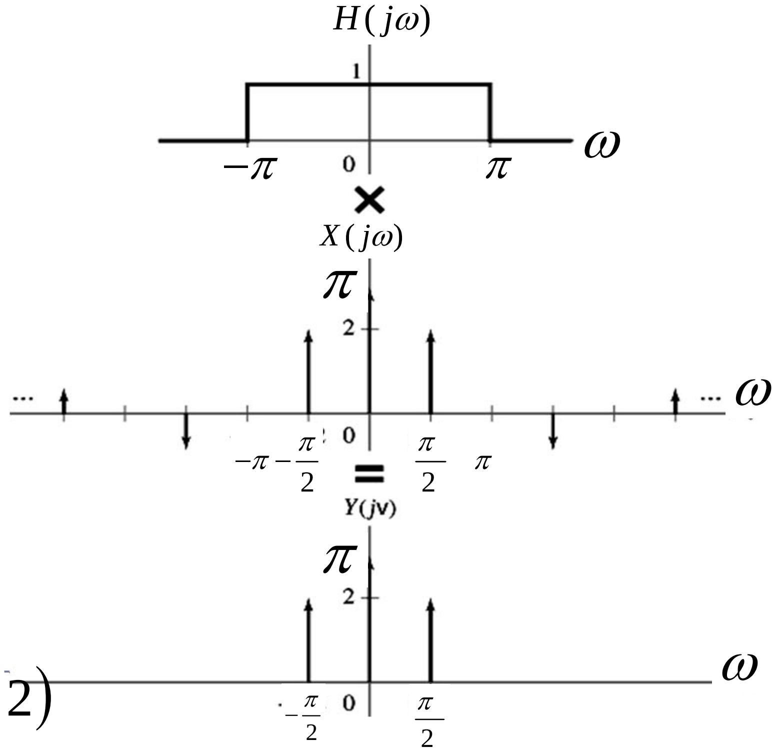
2. FT of the periodic square wave:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\pi/2)}{k}\,\delta\left(k\frac{\pi}{2}\right)$$

3. The FT of the system output is $Y(j\omega) = H(j\omega)X(j\omega)$:

$$Y(j\omega) = 2\delta\left(\omega + \frac{\pi}{2}\right) + \pi\delta(\omega) + 2\delta\left(\omega - \frac{\pi}{2}\right)$$

$H(j\omega)$ acts as a low-pass filter, passing harmonics at $-\pi/2$, 0, and $\pi/2$, while suppressing all others.



4. Inverse FT of $Y(j\omega)$: $y(t) = (1/2) + (2/\pi)\cos(t\pi/2)$

Convolution with Mixtures of Periodic and Nonperiodic Signals

Periodic input $x[n]$ → **Stable filter** / **Nonperiodic** impulse response $h[n]$ → $y[n] = x[n] * h[n]$

- For DT signals:
- If $x[n]$ is DT periodic signal, then

$$x[n]=\sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n} \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right)=2\pi\sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega-k\Omega_0\right)$$

$$\Rightarrow \boxed{y[n]=x[n]*b[n] \xleftrightarrow{DTFT} Y\left(e^{j\omega}\right)=2\pi\sum_{k=-\infty}^{\infty} H\left(e^{jk\Omega_0}\right)X[k]\delta\left(\Omega-k\Omega_0\right).}$$

♣ The form of $Y(e^{j\Omega})$ indicates that $y[n]$ is also periodic with the same period as $x[n]$.

Multiplication of Periodic and Nonperiodic Signals

- Continuous-time case:
- Original multiplication property:

$$y(t) = g(t)x(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi} G(j\omega) * X(j\omega)$$

- If $x(t)$ is periodic. The FT of $x(t)$ is

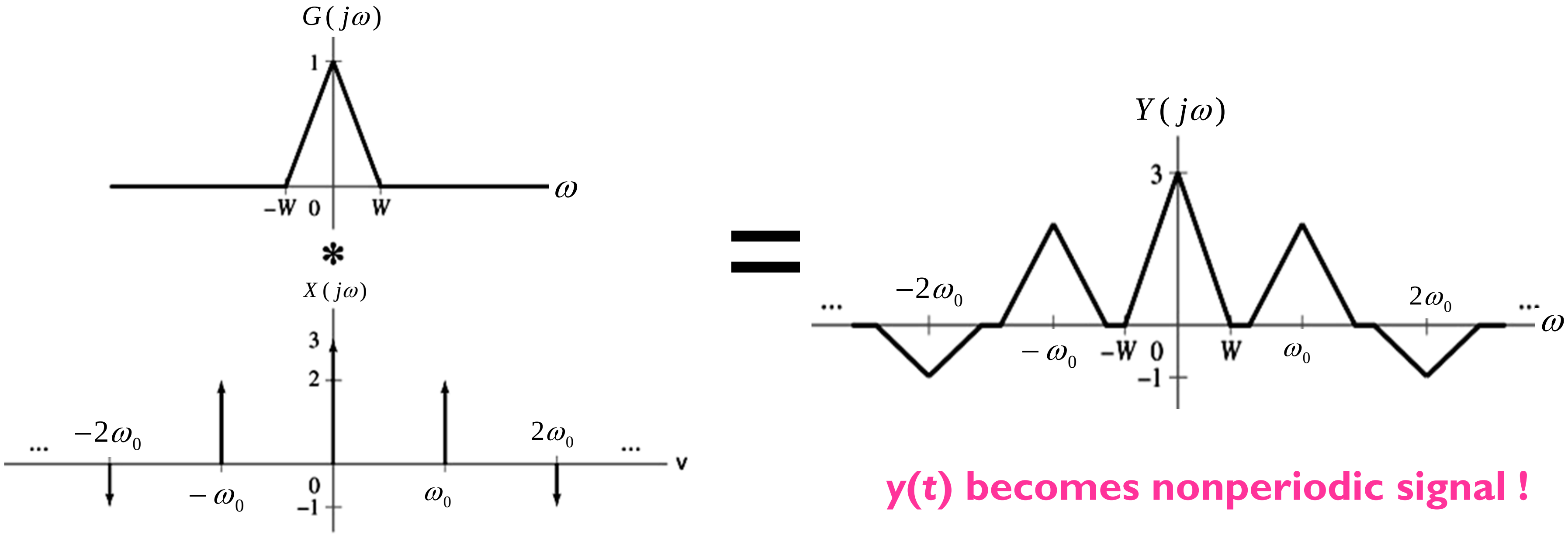
$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

$$\Rightarrow y(t) = g(t)x(t) \xleftrightarrow{FT} Y(j\omega) = G(j\omega) * \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

$$\Rightarrow \boxed{y(t) = g(t)x(t) \xleftrightarrow{FT} Y(j\omega) = \sum_{k=-\infty}^{\infty} X[k] G(j(\omega - k\omega_0))}$$

Multiplication of g(t) with the periodic function x(t) gives an FT consisting of a weighted sum of shifted versions of $G(j\omega)$

$$y(t)=g(t)x(t) \xleftrightarrow{FT} Y(j\omega)=\sum_{k=-\infty}^{\infty} X[k]G(j(\omega-k\omega_0))$$



***y(t)* becomes nonperiodic signal !**

The product of periodic and nonperiodic signals is nonperiodic

Example 4.4

Consider a system with output $y(t) = g(t)x(t)$.
Let $x(t)$ be the square wave and $g(t) = \cos(t/2)$, Find $Y(j\omega)$ in terms of $G(j\omega)$.

<Sol.>

♣ FS representation of the square wave:

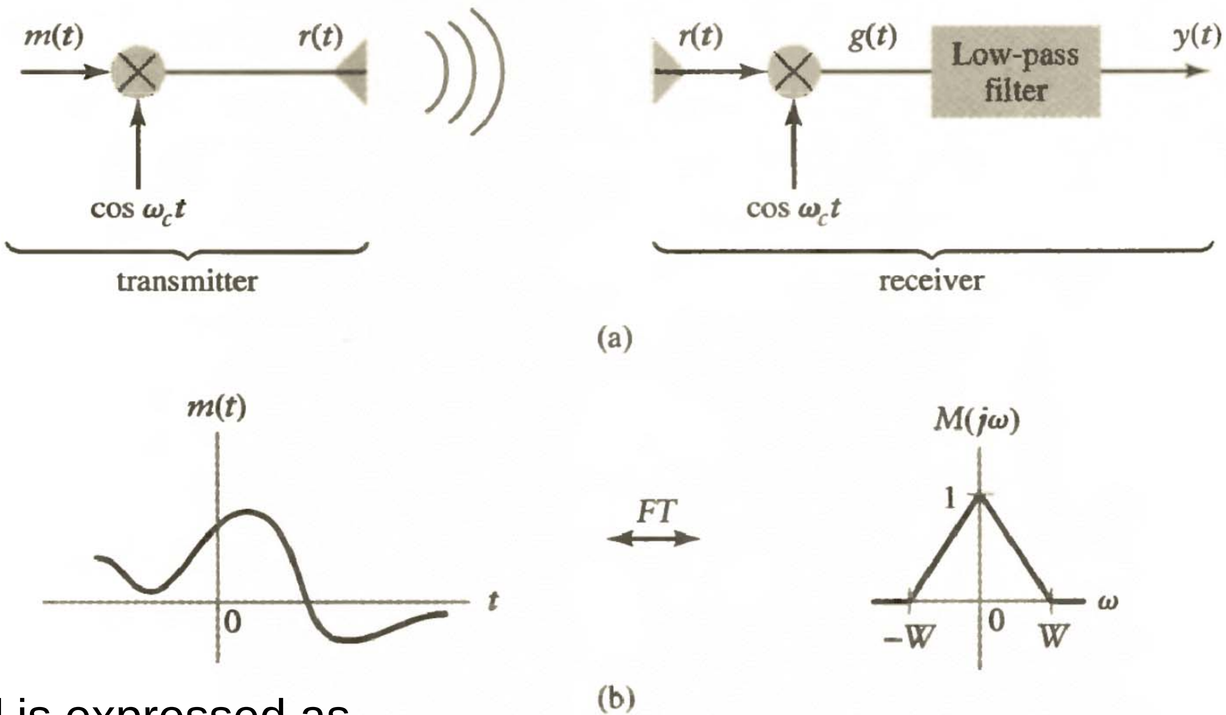
$$x(t) \xleftrightarrow{FS;\pi/2} X[k] = \frac{\sin(k\pi/2)}{\pi k}$$

And, we have $G(j\omega) = \pi\delta(\omega - 1/2) + \pi\delta(\omega + 1/2)$

$$\Rightarrow \quad Y(j\omega) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/2)}{k}\left[\delta(\omega - 1/2 - k\pi/2) + \delta(\omega + 1/2 - k\pi/2)\right]$$

Example 4.6 AM Radio

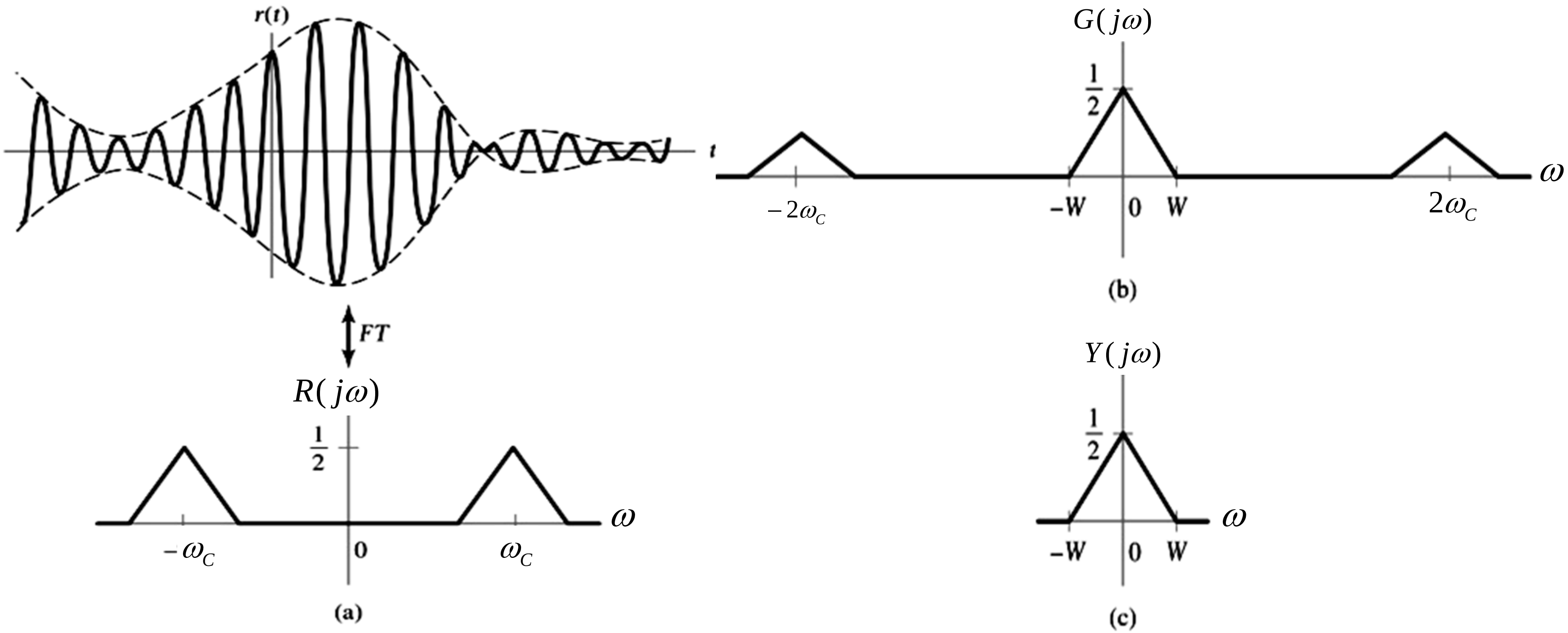
A simplified AM transmitter and receiver are depicted in Fig. 4.13(a). The effect of propagation and channel noise are ignored in this system. The signal at the receiver antenna, $r(t)$, is assumed equal to the transmitted signal. The passband of the low-pass filter in the receiver is equal to the message bandwidth, $-W < \omega < W$. Analyze this system in the frequency domain.



<Sol.>

1. The transmitted signal is expressed as

$$r(t) = m(t)\cos(\omega_c t) \xleftrightarrow{FT} R(j\omega) = (1/2)M\left(j(\omega-\omega_c)\right) + (1/2)M\left(j(\omega+\omega_c)\right)$$



$$g(t) = r(t)\cos(\omega_c t) \quad \xleftrightarrow{FT} \quad G(j\omega) = (1/2)R\big(j(\omega-\omega_c)\big) + (1/2)R\big(j(\omega+\omega_c)\big)$$

$$G(j\omega) = (1/4)M\big(j(\omega-2\omega_c)\big) + (1/2)M\big(j(\omega)\big) + (1/4)M\big(j(\omega+2\omega_c)\big)$$

♣The original message is recovered by **low-pass filtering** to remove the message replicates centered at twice the carrier frequency.

Multiplication of Periodic and Nonperiodic Signals

- Discrete-time case:
- Original multiplication property:

$$y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \otimes Z(e^{j\Omega})$$

- If $x[n]$ is periodic. The DTFS of $x[n]$ and its DTFT is

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \quad \xleftrightarrow{DTFT} \quad X\left(e^{j\Omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega - k\Omega_0\right)$$

$$\Rightarrow Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \otimes Z(e^{j\Omega}) = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_0) Z(e^{j(\Omega-\theta)}) d\theta$$

In any 2π interval of θ, there are exactly N multiples of the form $\delta(\theta - \Omega_0)$, since $\Omega_0 = 2\pi/N$.

$$\Rightarrow Y(e^{j\Omega}) = \sum_{k=0}^{N-1} X[k] \int_{-\pi}^{\pi} \delta(\Omega - k\Omega_0) Z(e^{j(\Omega-\theta)}) d\theta = \sum_{k=0}^{N-1} X[k] Z(e^{j(k\Omega_0-\theta)})$$

$$y[n] = x[n]z[n] \quad \xleftrightarrow{DTFT} \quad Y\left(e^{j\Omega}\right) = \sum_{k=0}^{N-1} X[k] Z\left(e^{j(\Omega-k\Omega_0)}\right)$$

Example 4.7 Windowing Effect

Windowing (or truncating): one common data-processing applications, which access only to a portion of a data record.

Consider the signal $x[n] = \cos\left(\frac{7\pi}{16}n\right) + \cos\left(\frac{9\pi}{16}n\right)$

Using only the 2*M*+1 values of *x*[*n*], $|n| \leq M$, evaluate the effect of computing the DTFT.

<Sol.>

1. Since $\boxed{x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_0)}$

Then $X(e^{j\Omega}) = \pi\delta\left(\Omega + \frac{9\pi}{16}\right) + \pi\delta\left(\Omega + \frac{7\pi}{16}\right) + \pi\delta\left(\Omega - \frac{7\pi}{16}\right) + \pi\delta\left(\Omega - \frac{9\pi}{16}\right)$

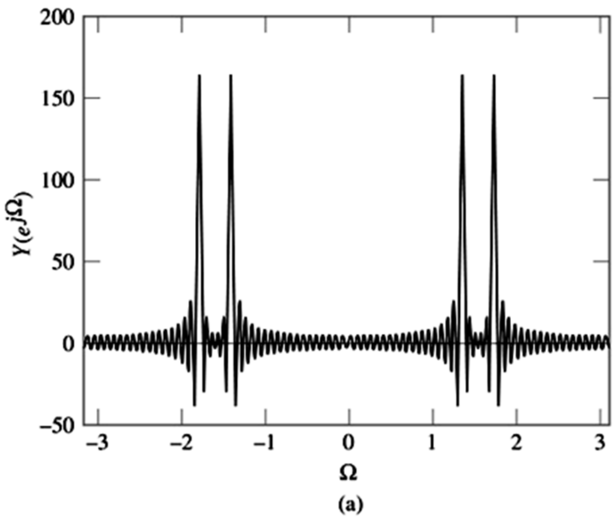
which consists of impulses at $\pm 7\pi/16$ and $\pm 9\pi/16$.

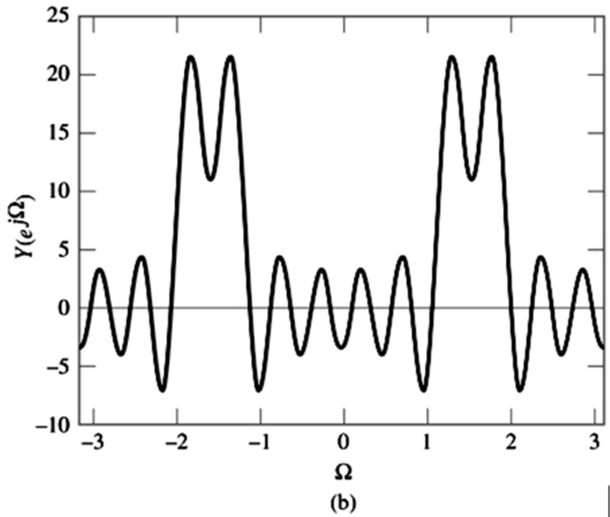
2. Define a signal *y*[*n*] = *x*[*n*]*w*[*n*], where $w[n] = \begin{cases} 1, & |n| \leq M \\ 0, & |n| > M \end{cases}$

$$y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

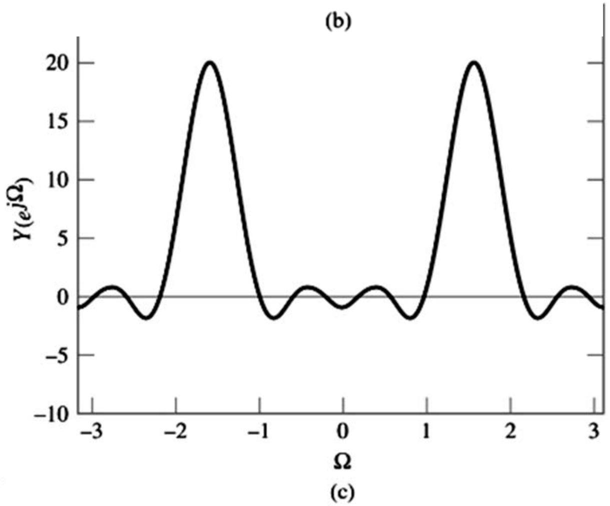
$$Y(e^{j\Omega}) = \frac{1}{2}\left\{W\left(e^{j(\Omega+9\pi/16)}\right) + W\left(e^{j(\Omega+7\pi/16)}\right) + W\left(e^{j(\Omega-7\pi/16)}\right) + W\left(e^{j(\Omega-9\pi/16)}\right)\right\}$$

3. The windowing introduces replicas of $W(e^{j\Omega})$ centered at the frequencies $7\pi/16$ and $9\pi/16$, instead of the impulses that are present in $X(e^{j\Omega})$.
4. We may view this state of affairs as a smearing of broadening of the original impulses
5. The energy in $Y(e^{j\Omega})$ is now smeared over a band centered on the frequencies of the cosines





Effect of windowing a data record. $Y(e^{j\Omega})$ for different values of M, assuming that $\Omega_1 = 7\pi/16$ and $\Omega_2 = 9\pi/16$.
(a) $M = 80$, (b) $M = 12$, (c) $M = 8$.



♣ If the number of available data points is small relative to the frequency separation, the DTFT is unable to distinguish the presence of two distinct sinusoids (e.g. M=8)

Fourier Transform of Discrete-Time Signals

- A mixtures of discrete-time and continuous-time signals
- By incorporating discrete-time impulses into the description of the signal in the appropriate manner.

Basic concepts:

Consider the following signals: $x(t) = e^{j\omega t}$ and $g[n] = e^{j\Omega n}$

Let's force $g[n]$ to be equal to the samples of $x(t)$ with sample period T_s; i.e., $g[n] = x(nT_s)$.

$$e^{j\Omega n} = e^{j\omega T_s n}$$ i.e. $\Omega = \omega T_s$

Now, consider the DTFT of a DT signal $x[n]$: $X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

Define the continuous time signal $x_\delta(t)$ with the Fourier transform $X_\delta(j\omega)$:

$$X_\delta(j\omega) = X\left(e^{j\Omega}\right)\Big|_{\Omega=\omega T_s} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T_s n}$$

Taking the inverse FT of $X_\delta(j\omega)$, using the FT pair $\delta(t - nT_s) \xleftrightarrow{FT} e^{-j\omega T_s n}$

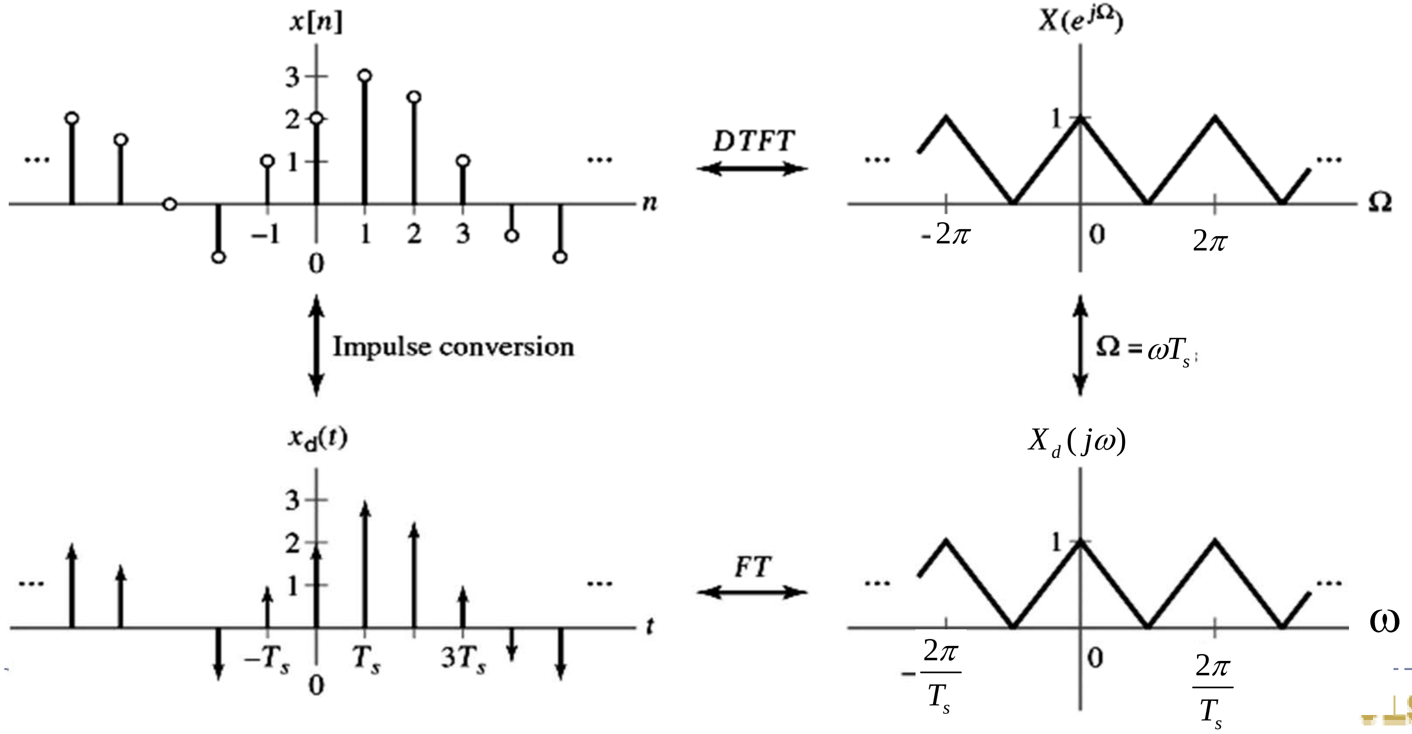
$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s).$$

Relating the FT to the DTFT

$$\boxed{x_\delta(t)=\sum_{n=-\infty}^{\infty}x[n]\delta(t-nT_s)\;\xleftrightarrow{FT}\;X_\delta(j\omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega T_s n},}$$

- $x_\delta(t)$ ≡ a CT signal corresponds to $x[n]$;
  $X_\delta(j\omega)$ ≡ Fourier transform corresponds to the CT Fourier transform $X_\delta(e^{j\Omega})$

The DTFT $X_\delta(e^{j\Omega})$ is periodic in Ω while the FT $X_\delta(j\omega)$ is $2\pi/T_s$ periodic in ω.



Example 4.8

Determine the FT pair associated with the DTFT pair

$$x[n] = a^n u[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}}$$

This pair is derived in Example 3.17 assuming that $|a| < 1$ so the DTFT converges.

<Sol.>

1. We first define the continuous-time signal $x_\delta(t) = \sum_{n=0}^{\infty} a^n \delta(t - nT_s)$
2. Using $\Omega = \omega T_s$ gives

$$x_\delta(t) \xleftrightarrow{FT} X_\delta(j\omega) = \frac{1}{1 - ae^{-j\omega T_s}}$$

Relating the FT to the DTFS

- Suppose that $x[n]$ is an N-periodic signal, then

$$x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n} \quad \xleftrightarrow{DTFT} \quad X\left(e^{j\Omega}\right)=2\pi\sum_{k=-\infty}^{\infty}X[k]\delta\left(\Omega-k\Omega_0\right)$$

  where $X[k]$ = DTFS coefficients

- Now, define $x_\delta(t) \equiv$ a CT signal corresponds to $x[n]$, then the Fourier transform $X_\delta(j\omega)$ of the continuous time signal $x_\delta(t)$ is

$$X_\delta(j\omega)=X(e^{j\Omega})\Big|_{\Omega=\omega T_s}=2\pi\sum_{k=-\infty}^{\infty}X[k]\delta(\omega T_s-k\Omega_0)$$

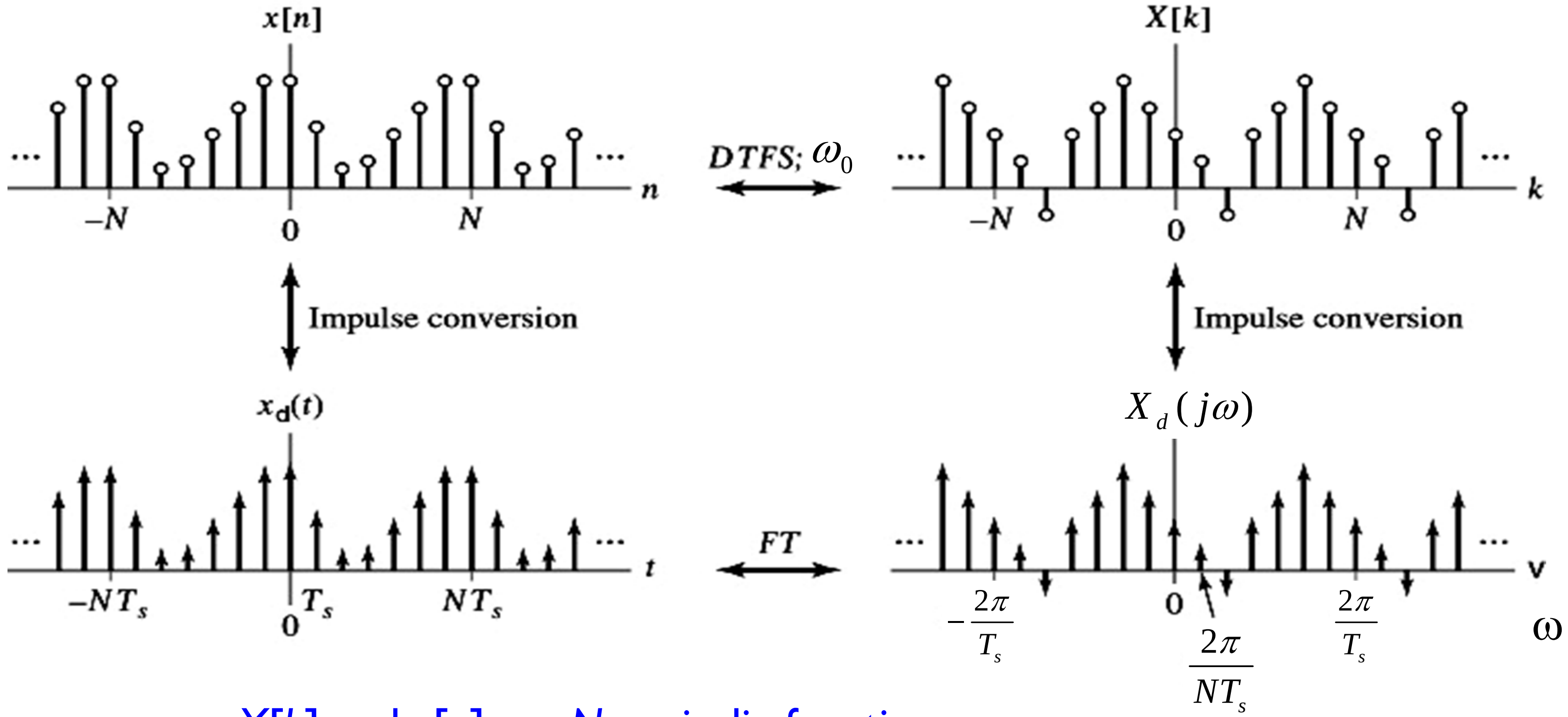
$$=2\pi\sum_{k=-\infty}^{\infty}X[k]\delta(T_s(\omega-k\frac{\Omega_0}{T_s}))$$

Use the scaling property of the impulse, $\delta(a\omega)=(1/a)\delta(\omega)$, to rewrite $X_\delta(j\omega)$ as

$$X_\delta(j\omega)=\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}X[k]\delta\left(\omega-\frac{k\Omega_0}{T_s}\right)$$

$$x_\delta(t)=\sum_{n=-\infty}^{\infty}x[n]\delta(t-nT_s)\xleftrightarrow{FT}X_\delta(j\omega)=\frac{2\pi}{T_s}\sum X[k]\delta\left(\omega-\frac{k\Omega_0}{T_s}\right)$$

1. Recall that $X[k]$ is N-periodic, which implies that $X_\delta(j\omega)$ is periodic with periodic $N\Omega_0/T_s = 2\pi/T_s$
2. Recall that $x[n]$ is N-periodic, which implies that $x_\delta(t)$ is periodic with periodic NT_s



$X[k]$ and $x[n]$ are N-periodic functions

$x_\delta(t)$ and $X_\delta(j\omega)$ are periodic impulse trains with period NT_s and $2\pi/T_s$.

Outline

- Introduction
- Fourier Transform of Periodic Signals
- Convolution/Multiplication with (Non-)Periodic Signals
- Fourier Transform of Discrete-Time Signals
- Sampling
- Reconstruction of Continuous-Time Signals
- Discrete-Time Processing of Continuous-Time Signals
- Fourier Series of Finite-Duration Nonperiodic Signals

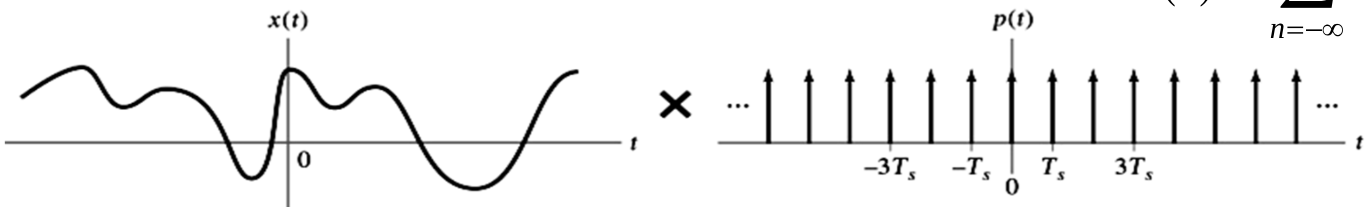
Sampling

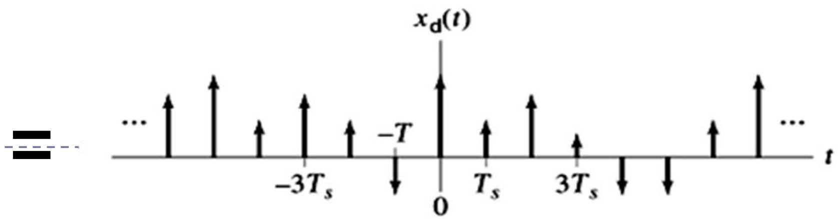
- We use FT representation of discrete-time signals to analyze the effects of uniformly sampling a signal.
- Sampling the (continuous-time) signal is often performed in order to manipulate the signal on a computer.

$$x[n] = x(nT_s)$$

- Let $x(t)$ be a CT signal and $x[n]$, a DT signal, is the "samples" of $x(t)$ at integer multiples of a sampling interval.

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s).$$





$$\boxed{x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)}$$

Sampling Process

- We note that

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = x(t)\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = x(t)\,p(t)$$

- Then

$$\boxed{x_\delta(t) = x(t)\,p(t) \xleftrightarrow{FT} X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)}$$

Example 4.2

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \qquad \omega_s = 2\pi/T_s$$

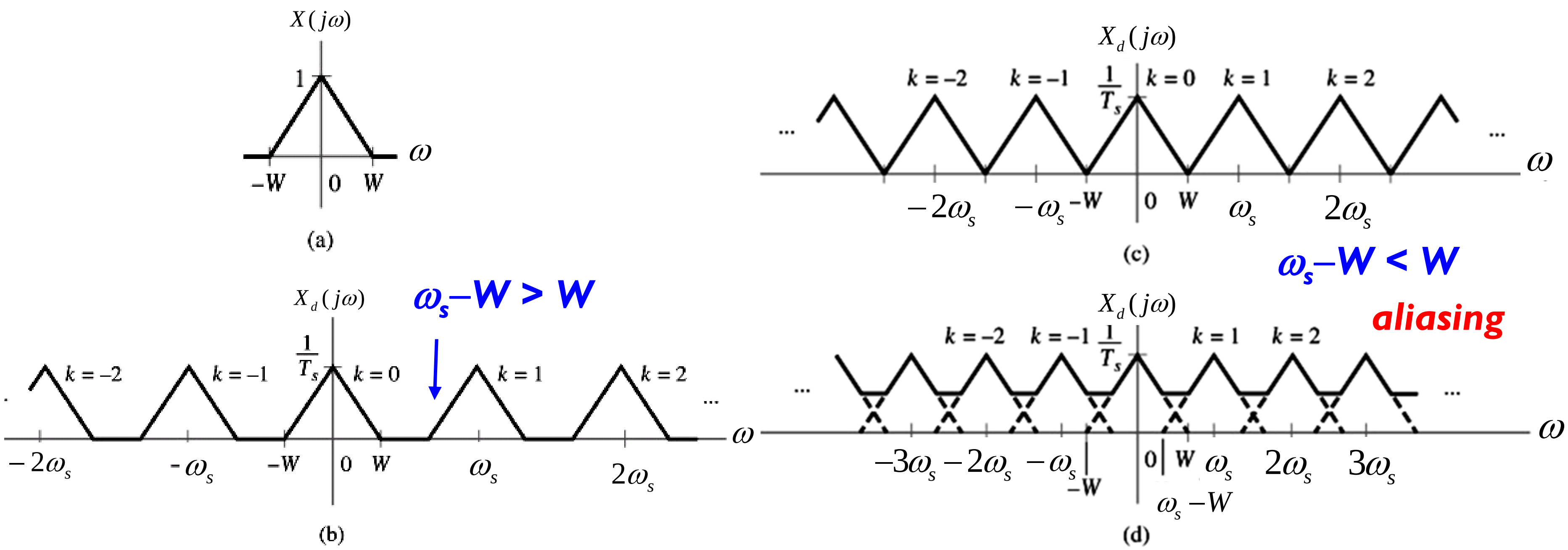
$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

An infinite sum of shifted versions of the original signal, $X(j\omega)$

Sampling Theorem

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

- The shifted version of $X(j\omega)$ may overlap with each other if ω_s is not large enough, compared with the bandwidth of $X(j\omega)$, i.e. W.



- As T_s increases or ω_s decreases, the shifted replicas of $X(j\omega)$ moves closer together, finally overlapping one another when $\omega_s < 2W$.

Example 4.9

Consider the effect of sampling the sinusoidal signal $x(t)=\cos(\pi t)$

Determine the FT of the sampled signal for T_s = 1 / 4.

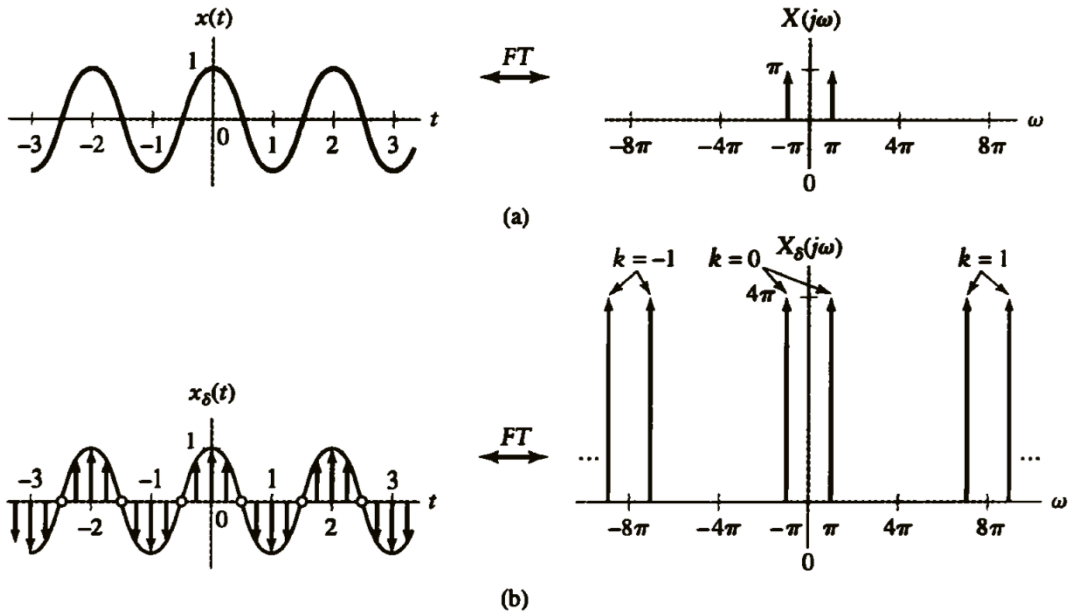
<Sol.>

Recall that $x_\delta(t)=\sum_{n=-\infty}^{\infty}x[n]\delta(t-nT_s) \xleftrightarrow{FT} X_\delta(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X(j(\omega-k\omega_s))$

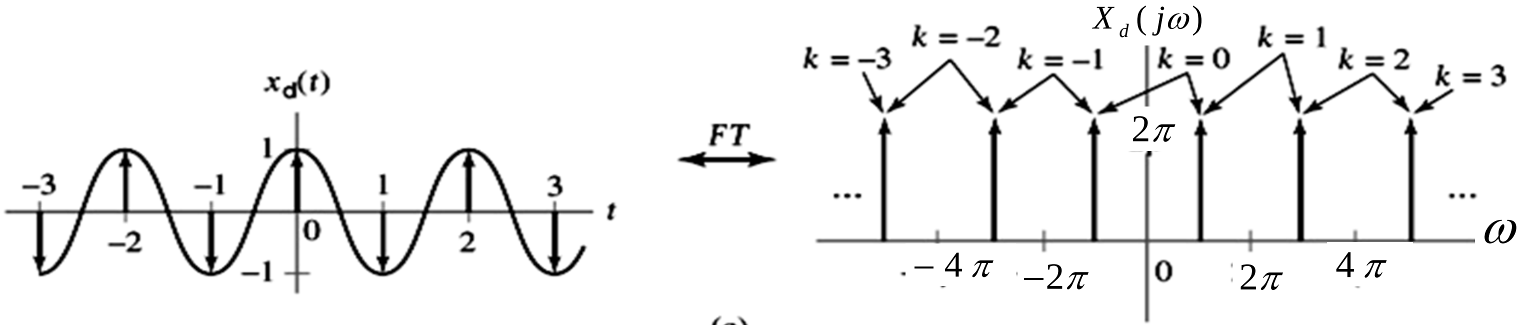
1. First find that $X(j\omega)=\pi\delta(\omega+\pi)+\pi\delta(\omega-\pi)$

2. Then, we have $X_\delta(j\omega)=\frac{\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega+\pi-k\omega_s)+\delta(\omega-\pi-k\omega_s)$

which consists of pairs of impulses separated by 2π, centered on integer multiples of $\omega_s = 2\pi/T_s$



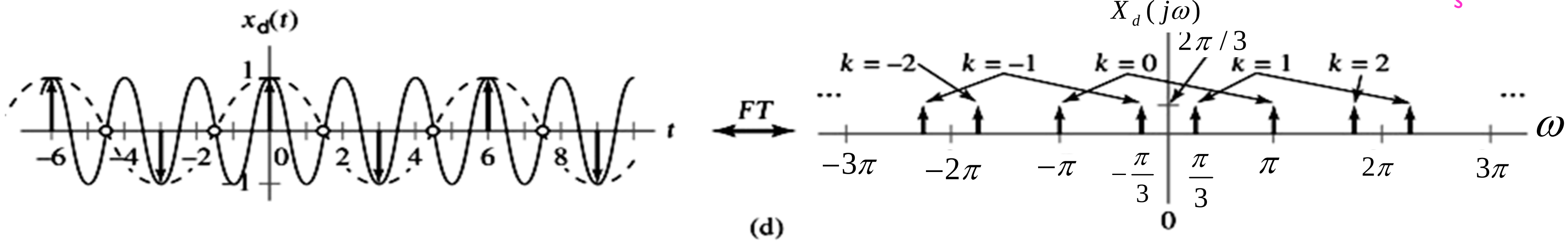
T_s = ¼.



(c)

$T_s = 1$

for $T_s = 3/2$



(d)

Down-Sampling: Sampling Discrete-Time Signals

- Recall $$x_\delta(t)=\sum_{n=-\infty}^{\infty}x[n]\delta(t-nT_s)\xleftrightarrow{FT}X_\delta(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X(j(\omega-k\omega_s))$$

- Let $y[n] = x[nq]$ be a subsampled version of $x[n]$, q is positive integer.

$$y_\delta(t)=\sum_{n=-\infty}^{\infty}y[n]\delta(t-nT_s')$$

$$\omega_s'=\omega_s/q$$

$$=\sum_{n=-\infty}^{\infty}x[nq]\delta(t-nqT_s)\xleftrightarrow{FT}Y_\delta(j\omega)=\frac{1}{qT_s}\sum_{k=-\infty}^{\infty}X(j(\omega-\frac{k}{q}\omega_s))$$

- Let $\frac{k}{q}=l+\frac{m}{q},\quad 0\le m\le q-1$, then

$$Y_\delta(j\omega)=\frac{1}{qT_s}\sum_{k=-\infty}^{\infty}X(j(\omega-\frac{k}{q}\omega_s))=\frac{1}{q}\sum_{m=0}^{q-1}\left\{\frac{1}{T_s}\sum_{l=-\infty}^{\infty}X(j(\omega-l\omega_s-\frac{m}{q}\omega_s)\right\}$$

$$=\frac{1}{q}\sum_{m=0}^{q-1}X_\delta(j(\omega-\frac{m}{q}\omega_s))$$

Down-Sampling: Sampling Discrete-Time Signals

- Since $y[n] \xleftrightarrow{DTFT} Y\left(e^{j\Omega}\right) = Y_\delta(j\omega)\Big|_{\omega=\frac{\Omega}{T_s'}} = Y_\delta(j\omega)\Big|_{\omega=\frac{\Omega}{qT_s}}$, therefore

$$Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X_\delta(j(\omega - \frac{m}{q}\omega_s))\Bigg|_{\omega=\frac{\Omega}{qT_s}} = \frac{1}{q}\sum_{m=0}^{q-1} X_\delta(j(\frac{\Omega}{qT_s} - \frac{m}{q}\omega_s))$$

$$= \frac{1}{q}\sum_{m=0}^{q-1} X_\delta(j(\frac{\Omega}{qT_s} - \frac{m}{q}\frac{2\pi}{T_s})) = \frac{1}{q}\sum_{m=0}^{q-1} X_\delta(\frac{j}{T_s}(\frac{\Omega}{q} - \frac{m}{q}2\pi))$$

- Use $X\left(e^{j\Omega}\right) = X_\delta(j\omega)\Big|_{\omega=\frac{\Omega}{T_s}}$, we have $Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X(e^{j\frac{1}{q}(\Omega - 2\pi m)})$

- Summary

$$y_\delta(t) = \sum_{n=-\infty}^{\infty} x[nq]\delta(t - nqT_s) \xleftrightarrow{FT} Y_\delta(j\omega) = \frac{1}{qT_s}\sum_{k=-\infty}^{\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

$$y[n] = x[qn] \xleftrightarrow{DTFT} Y\left(e^{j\Omega}\right) = \frac{1}{q}\sum_{m=0}^{q-1} X(e^{j\frac{1}{q}(\Omega - 2\pi m)})$$

Down-Sampling by q

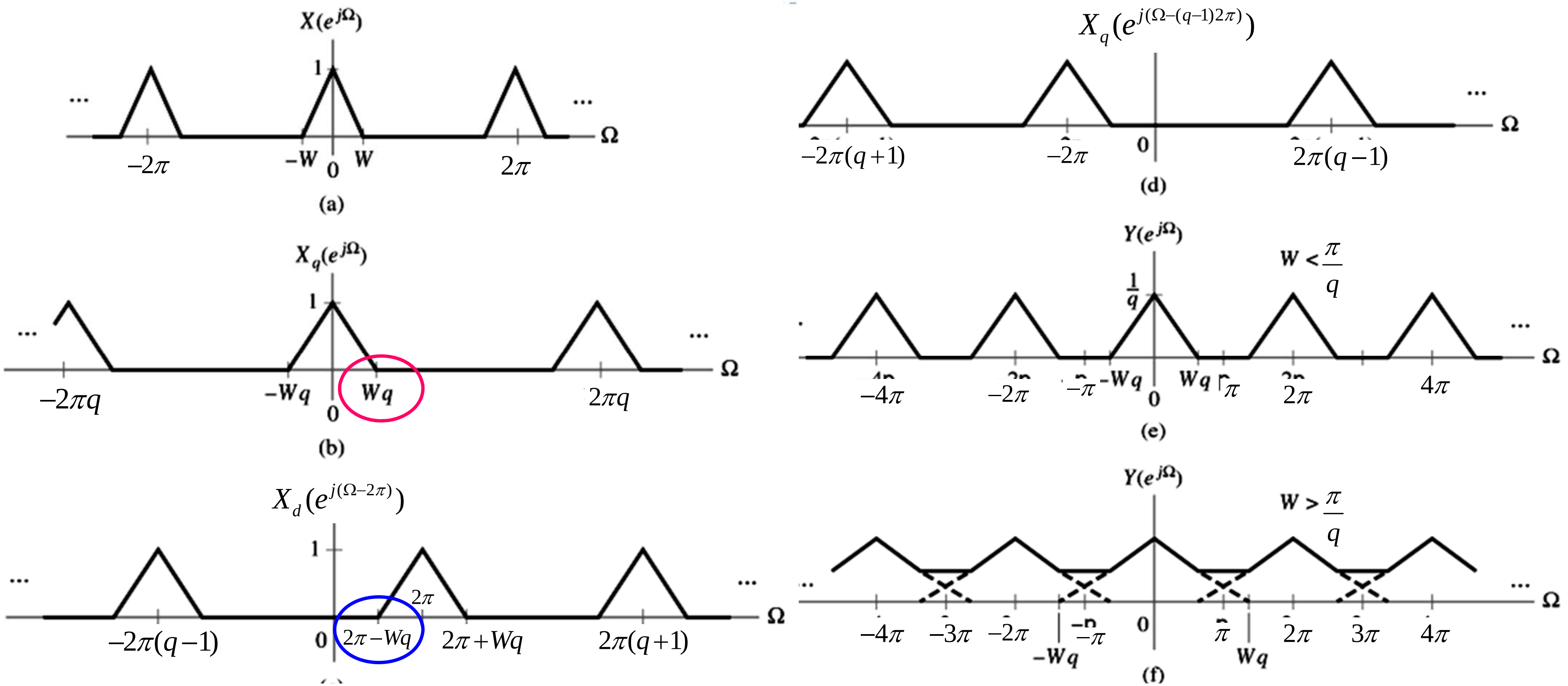
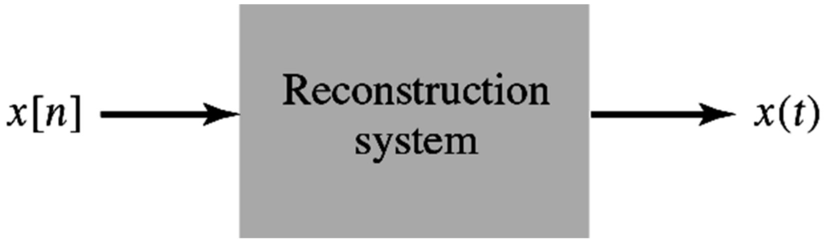


Figure 4.29
Effect of subsampling on the DTFT. (a) Original signal spectrum. (b) $m = 0$ term in Eq. (4.27) (c) $m = 1$ term in Eq. (4.27). (d) $m = q - 1$ term in Eq. (4.27). (e) $Y(e^{j\Omega})$, assuming that $W < \pi/q$. (f) $Y(e^{j\Omega})$, assuming that $W > \pi/q$.

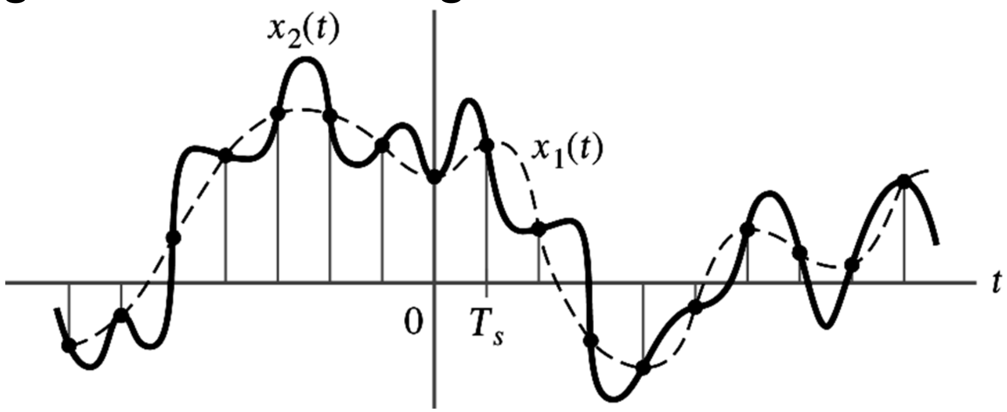
- **If the highest frequency component of $X(e^{j\Omega})$, W, is less than π/q, the aliasing can be prevented.**

Reconstruction of Sampled Signal

- Reconstruction?



- Not so easy !! The samples of a signal do not always uniquely determine the corresponding continuous-time signal



- Nyquist Sampling Theorem

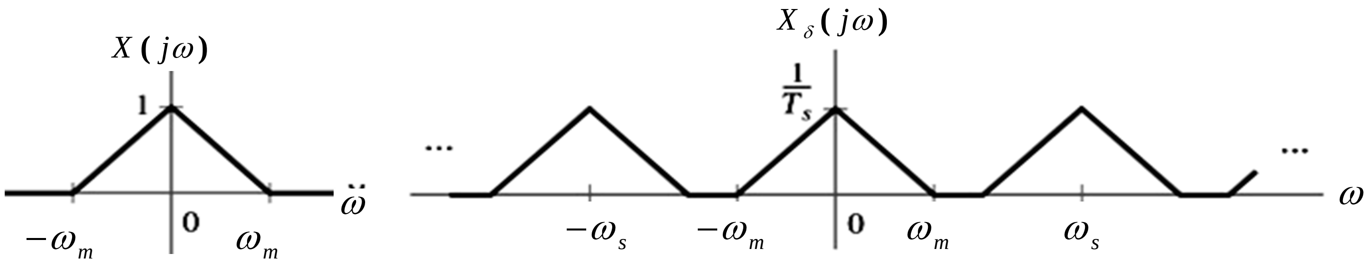
Let $x(t) \xleftrightarrow{FT} X(j\omega)$ *be a band-limited signal, i.e.* $X(j\omega)=0$ *for* $|\omega| > \omega_m$.

If $\omega_s > 2\ \omega_m$*, where* $\omega_s = 2\pi/T_s$ *is the sampling frequency, then* $x(t)$ *is uniquely determined by its samples* $x(nT_s)$, $n = 0, \pm 1, \pm 2, \ldots$.

Ideal/Perfect Reconstruction

- If the signal is not band-limited, an ***antialiasing filter*** is necessary
- Hereafter, we consider a band-limited signal
- Suppose that $x(t) \xleftrightarrow{FT} X(j\omega)$, then the FT of the sampled signal is

$$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$$



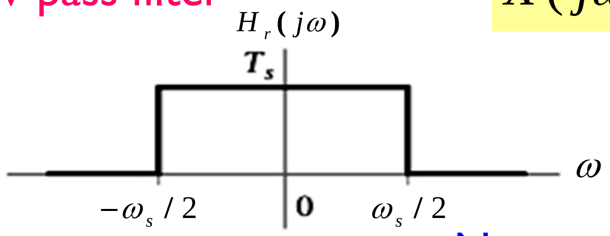
(a) (b)

Ideal low-pass filter

$$X(j\omega) = X_\delta(j\omega)H_r(j\omega)$$

- Ideal reconstruction:

$$H_r(j\omega) = \begin{cases} T_s, & |\omega| \le \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$



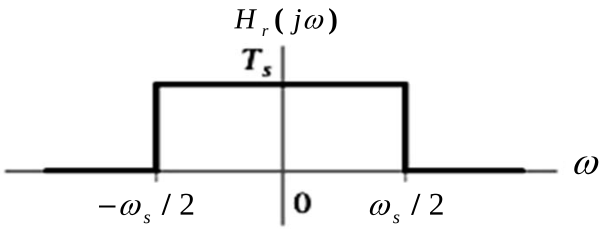
(c)

Non-causal !!

the perfect-reconstruction cannot be implemented in any practical system

ω_s: Nyquist frequency

$$X(j\omega) = X_\delta(j\omega) H_r(j\omega) \Rightarrow x(t) = x_\delta(t) * h_r(t)$$

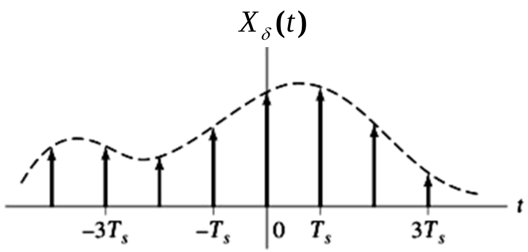


$$h_r(t) \overset{FT}{\longleftrightarrow} H_r(j\omega)$$

$$\Rightarrow h_r(t) = \frac{T_s \sin(\frac{\omega_s t}{2})}{\pi t} = \sin c(\frac{t}{T_s})$$

$$\Rightarrow x(t) = x_\delta(t) * h_r(t) = \left(\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \right) * h_r(t)$$

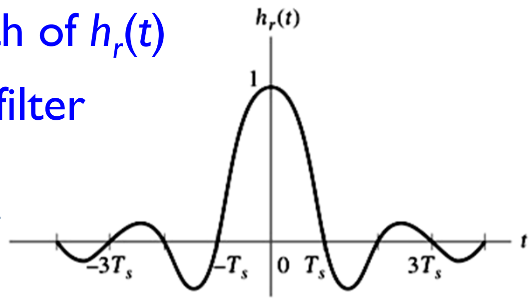
$$= \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT_s) = \sum_{n=-\infty}^{\infty} x[n] \sin c(\frac{t - nT_s}{T_s})$$



$*$

Infinite length of $h_r(t)$

Non-causal filter



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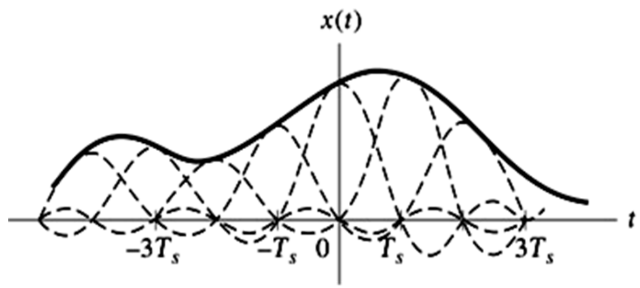
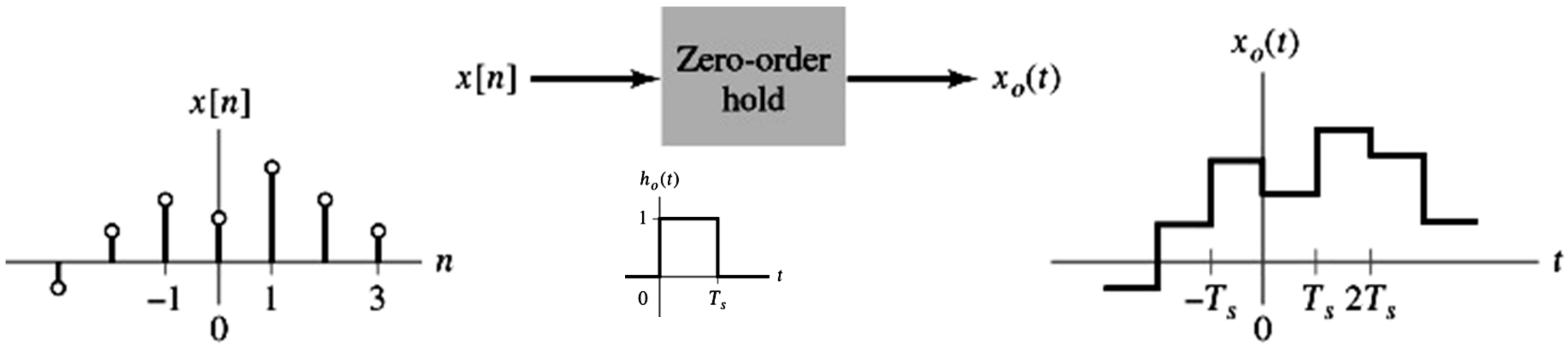


Perfect reconstruction

A Practical Reconstruction – Zero-Order Hold Filter

- The zero-order hold filter, which holds the input value for T_s seconds.

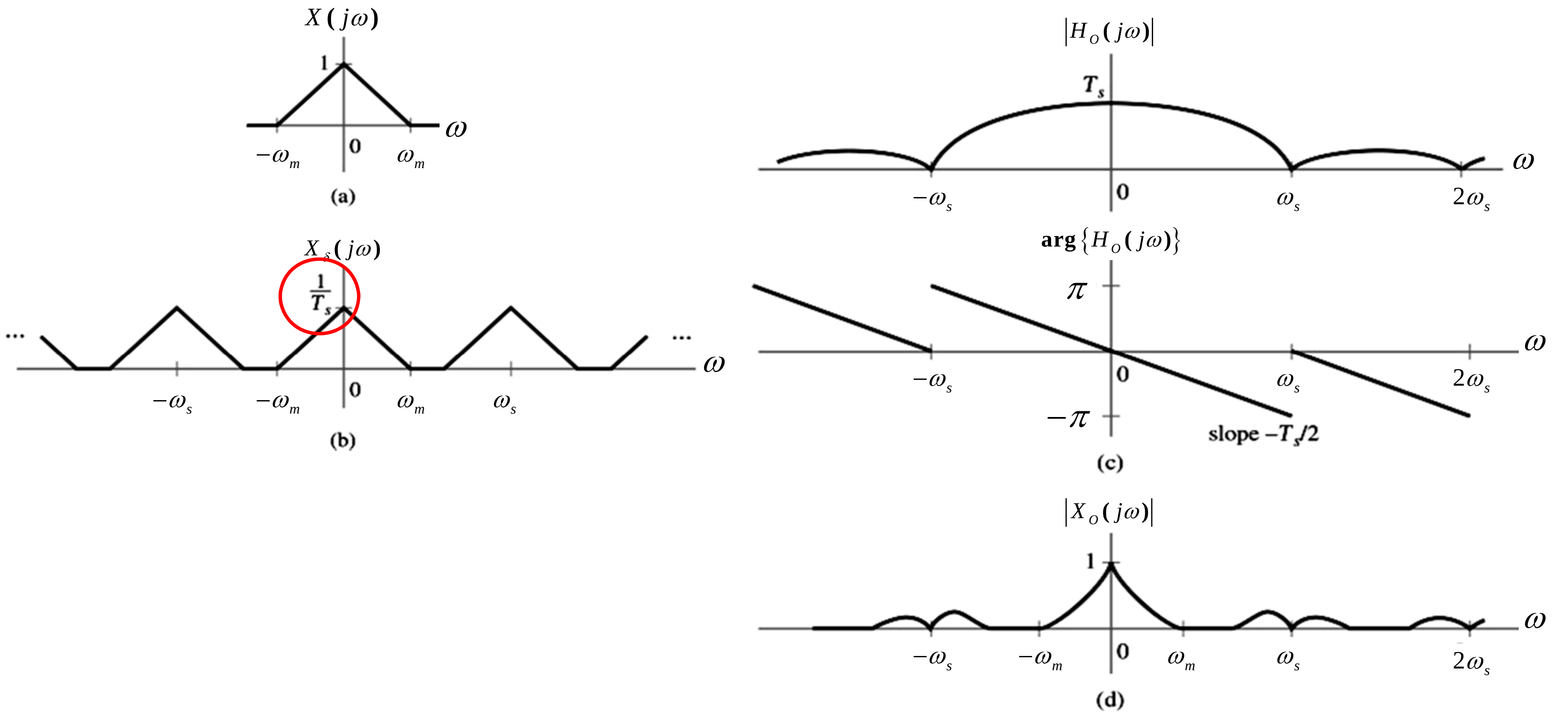


a stair-step approximation to the original signal

- Impulse response (rectangular pulse):

$$h_o(t) = \begin{cases} 1, & 0 < t < T_s \\ 0, & t < 0, t > T_s \end{cases} \xleftrightarrow{FT} H_0(j\omega) = 2e^{-j\omega T_s/2}\frac{\sin(\omega T_s/2)}{\omega}$$

1. A linear phase shift corresponding to a time delay of $T_s/2$ seconds
2. A distortion of the portion of $X_\delta(j\omega)$ between $-\omega_m$ and ω_m. [The distortion is produced by the curvature of the mainlobe of $H_o(j\omega)$.]
3. Distorted and attenuated versions of the images of $X(j\omega)$, centered at nonzero multiples of ω_s.

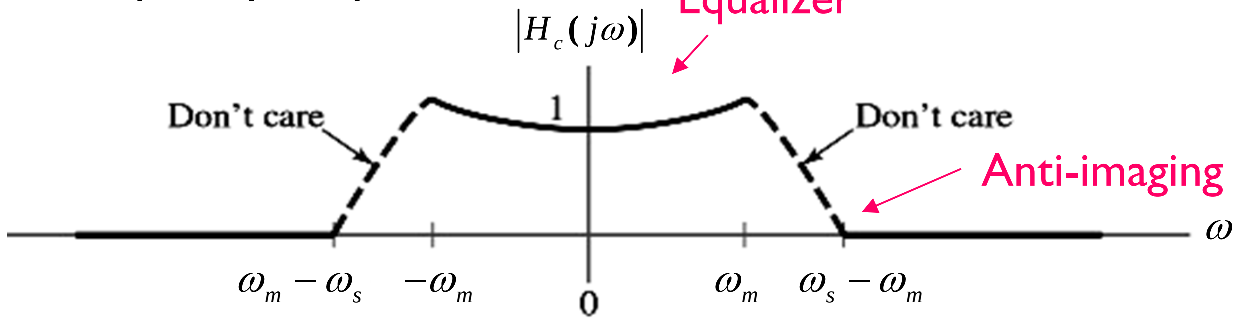


- ♣ T_s seconds holding for $x[n]$ ➔ Time shift of $T_s/2$ in $x_o(t)$ for distortion-1
- ♣ Stair-step approximation ➔ main reason for distortion-2 & 3
  ➔ 1. Using Equalizer for non-flattened mag. in passband
  2. Using anti-image filter
  ➔ Compensation filter

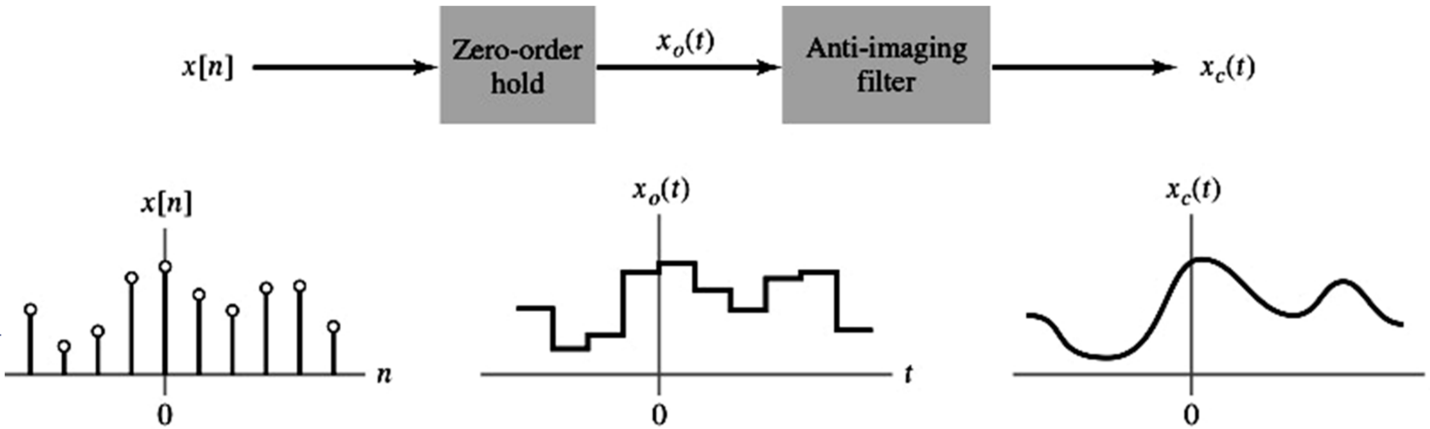
Both distortions 1 and 2 are reduced by increasing ω_s or, equivalently, decreasing T_s, e.g. oversampling !!

Compensation Filter

- The frequency response:



$$H_c(j\omega) = \begin{cases} \dfrac{\omega T_s}{2\sin(\omega T_s/2)}, & |\omega| < \omega_m \\ 0, & |\omega| > \omega_s - \omega_m \end{cases}$$



Example 4.13

In this example, we explore the benefits of oversampling in reconstructing a continuous-time audio signal using an audio compact disc player. Assume that the maximum signal frequency is f_m = 20 kHz. Consider two cases:

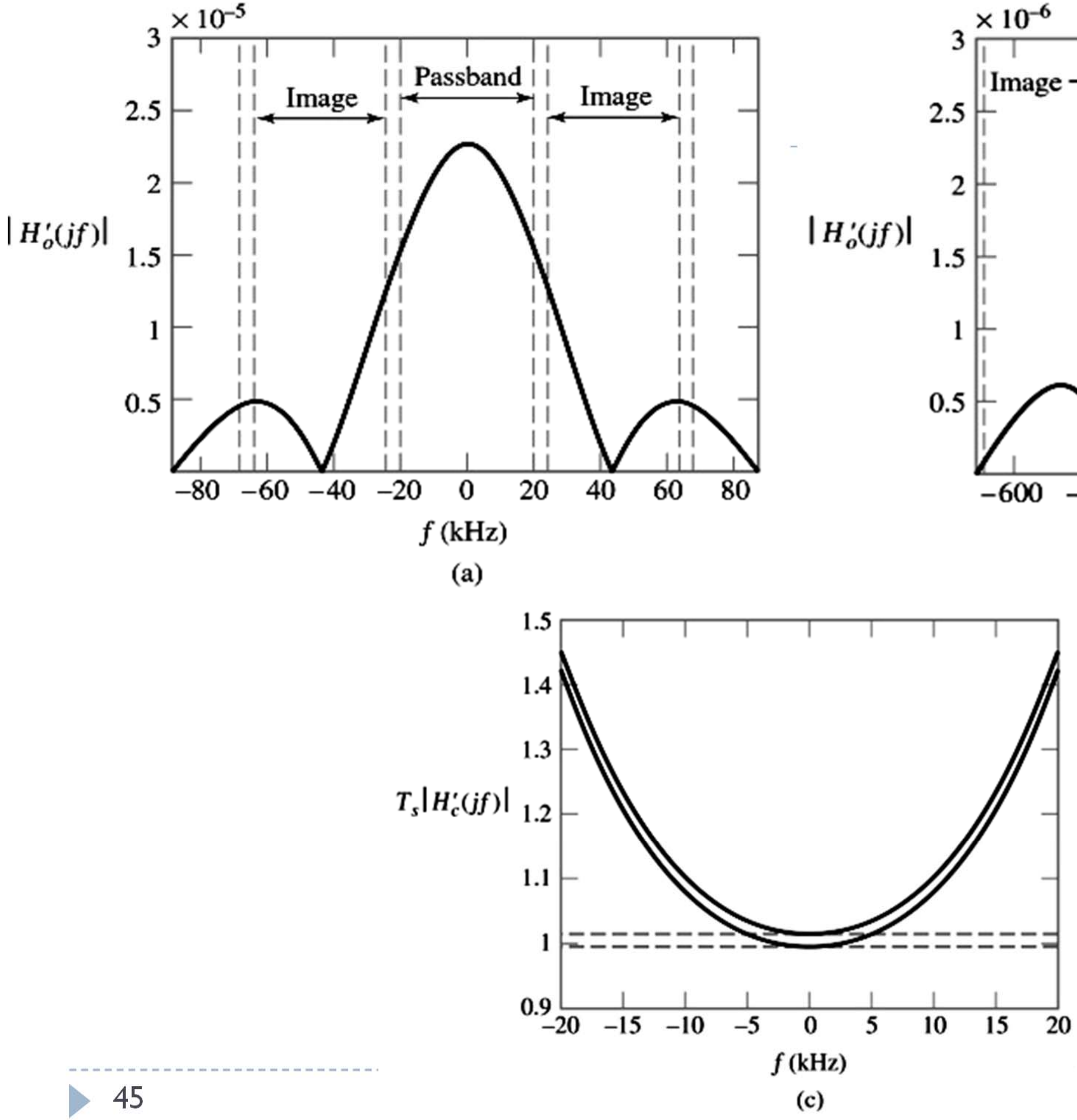
(a) reconstruction using the standard digital audio rate of $1/T_{s1}$ = 44.1 kHz, and

(b) reconstruction using eight-times oversampling, for an effective sampling rate of $1/T_{s2}$ = 352.8 kHz.

In each case, determine the constraints on the magnitude response of an anti-imaging filter so that the overall magnitude response of the zero-order hold reconstruction system is between 0.99 and 1.01 in the signal passband and the images of the original signal's spectrum centered at multiples of the sampling frequency are attenuated by a factor of 10^{-3} or more.

<Sol.>

$$\frac{0.99}{|H'_o(jf)|} < |H'_c(jf)| < \frac{1.01}{|H'_o(jf)|}, \quad -20\,\text{kHz} < f < 20\,\text{kHz}$$


× 10^−5
Image
Passband
Image
|H'_o(jf)|
f (kHz)
(a)
× 10^−6
Image
Image
Passband
|H'_o(jf)|
f (kHz)
(b)
T_s|H'_c(jf)|
f (kHz)
(c)

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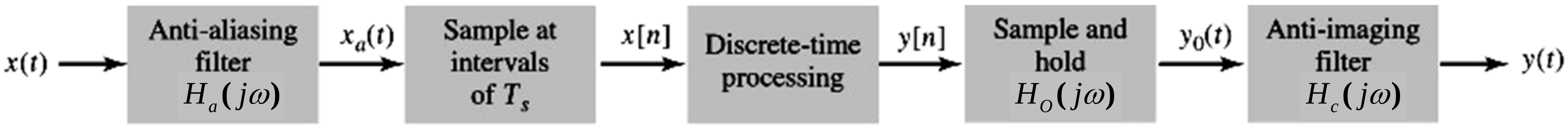
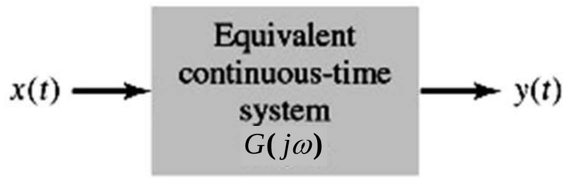
Discrete-Time Signal Processing

- Several advantages for DSP algorithm
  - Signal manipulations are more easily performed by using arithmetic operations of a computer than through the use of analog components
  - DSP systems are easily modified in real-time
  - Direct dependence of the dynamic range and signal-to-noise ratio on the number of bits used to represent the discrete-time signal
  - Easily implement the decimation and the interpolation



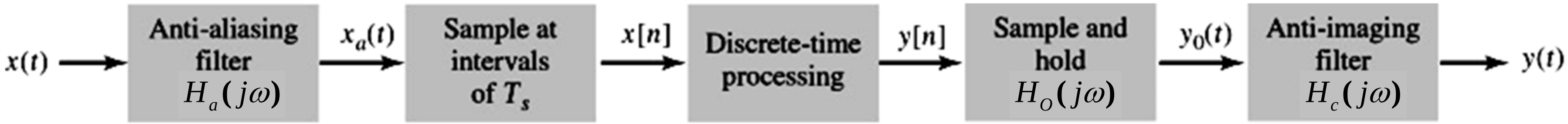
(a)



(b)

System Response Analysis

- Assume that the discrete-time processing operation is represented by a DT system with frequency response $H(e^{j\Omega})$ **$\Omega = \omega T_s$, $T_s \equiv$ sampling interval**



$$X_a(j\omega) = H_a(j\omega)X(j\omega)$$

$$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(j\left(\omega - k\omega_s\right)\right)$$
$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} H_a\left(j\left(\omega - k\omega_s\right)\right)X\left(j\left(\omega - k\omega_s\right)\right)$$

$$Y_\delta(j\omega) = \frac{1}{T_s}H\left(e^{j\omega T_s}\right)\sum_{k=-\infty}^{\infty} H_a\left(j\left(\omega - k\omega_s\right)\right)X\left(j\left(\omega - k\omega_s\right)\right)$$

$$\boxed{Y(j\omega) = \frac{1}{T_s}H_o(j\omega)H_c(j\omega)H\left(e^{j\omega T_s}\right)\sum_{k=-\infty}^{\infty} H_a\left(j\left(\omega - k\omega_s\right)\right)X\left(j\left(\omega - k\omega_s\right)\right)}$$

The anti-imaging filter $H_c(j\omega)$ eliminates frequency components above $\omega_s/2$, hence eliminating all the terms in the infinite sum except for the k = 0 term.
Therefore, we have

$$Y(j\omega) = \frac{1}{T_s} H_o(j\omega) H_c(j\omega) H\left(e^{j\omega T_s}\right) H_a(j\omega) X(j\omega)$$

The overall system is equivalent to a continuous-time LTI system having the response:

$$\boxed{G(j\omega) = \frac{1}{T_s} H_o(j\omega) H_c(j\omega) H\left(e^{j\omega T_s}\right) H_a(j\omega)}$$

♣ If we choose the anti-aliasing and anti-imaging filters such that

$$(1/T_s) H_o(j\omega) H_c(j\omega) H_a(j\omega) \approx 1 \quad \Longrightarrow \quad G(j\omega) \approx H(e^{j\omega T_s})$$

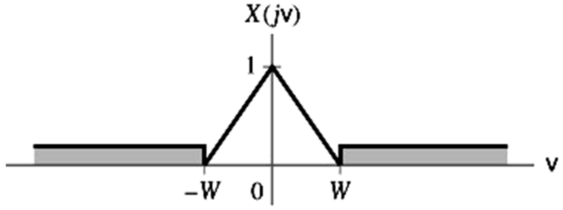
That is, we may implement a CT system in DT by choosing sampling parameters appropriately and designing a corresponding DT system.

Oversampling

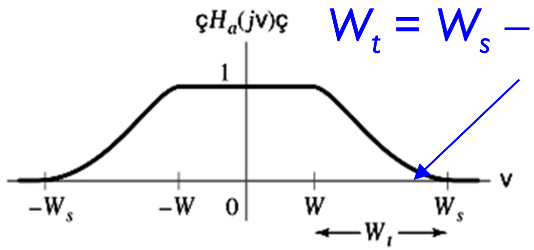
A high sampling rate leads to high computation cost

- Oversampling can relax the requirements of the anti-aliasing filter as well as the anti-image filter (a wide transition band)
- Anti-aliasing filter is used to prevent aliasing by limiting the bandwidth of the signal prior to sampling.

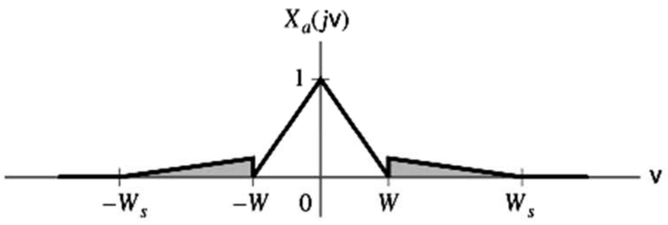
$W_s \equiv$ stopband of filter,
$W_t = W_s - W \equiv$ width of transition band.



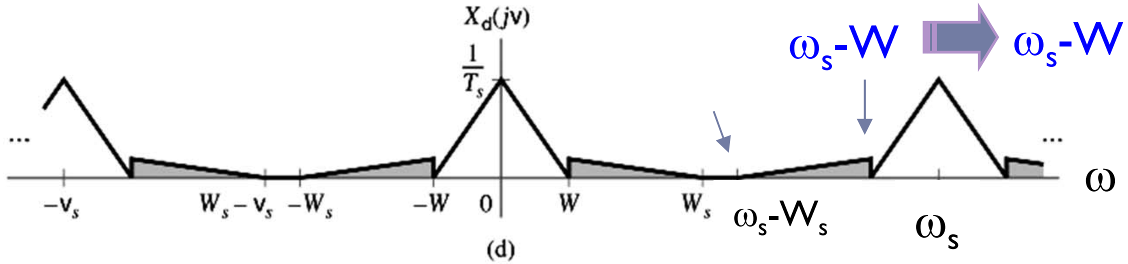
(a)



(b)



(c)

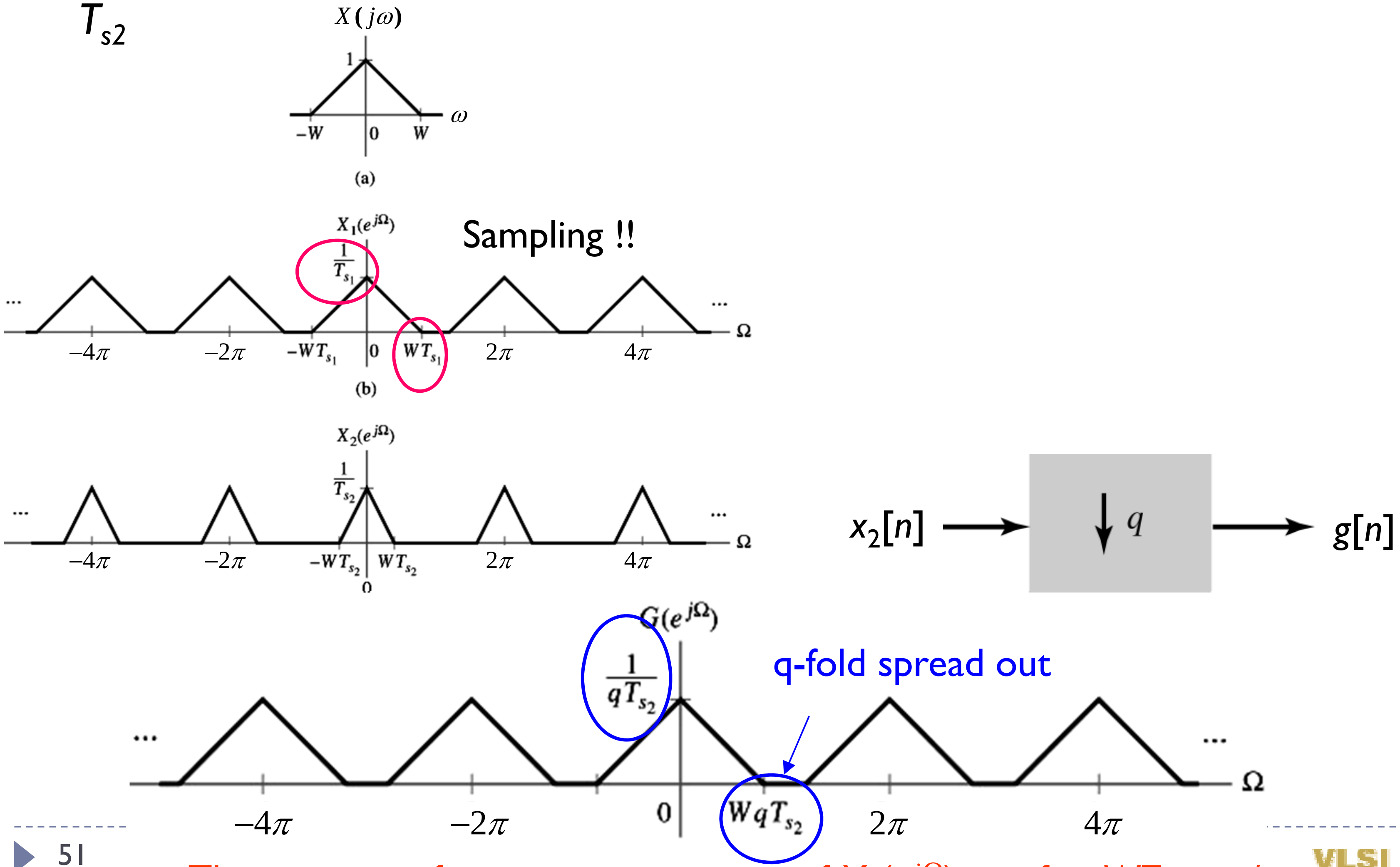


(d)

$\omega_s - W \Rightarrow \omega_s - W > W_s \Rightarrow W_t = W_s - W < \omega_s - 2W$

Decimation

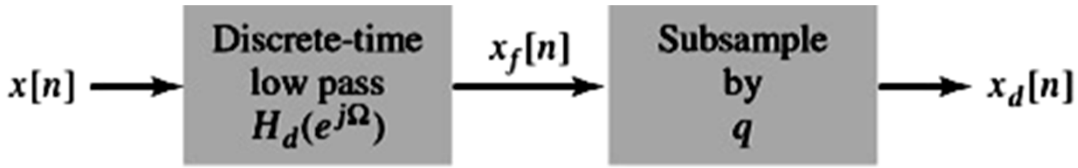
- Two sampled signals $x_1[n]$ and $x_2[n]$ with different sampling intervals T_{s1} and T_{s2}



The maximum frequency component of $X_2(e^{j\Omega})$ satisfies $WT_{s2} < \pi/q$

Decimation Filter

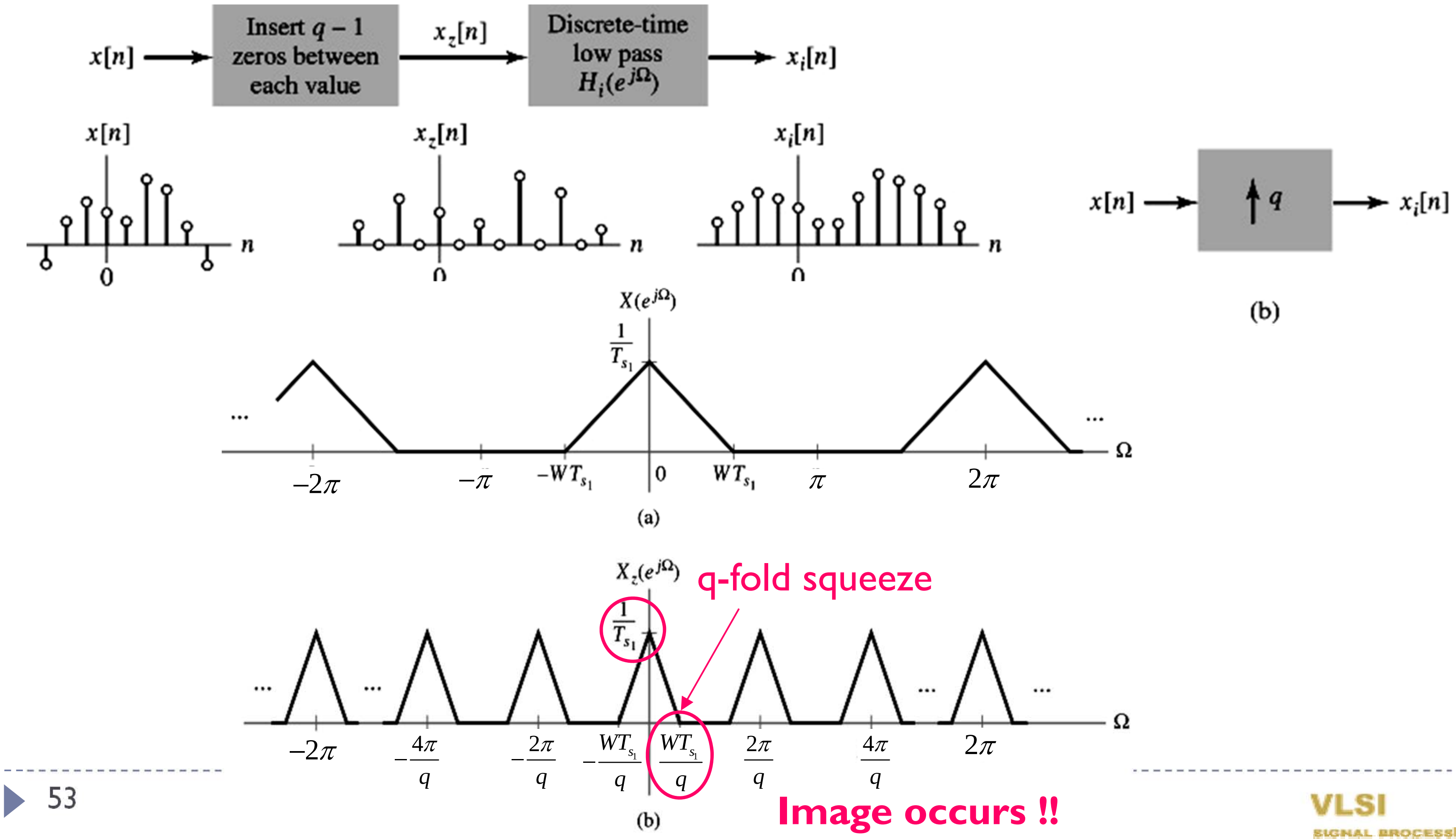
- Decimation filter $h_d[n]$: a low-pass filter prevents from aliasing problem when downsampling by q.



$$H_d(e^{j\Omega}) = \begin{cases} 1, & |\Omega T_s| \le \dfrac{\pi}{q} \\ 0, & \text{otherwise} \end{cases}$$

Interpolation

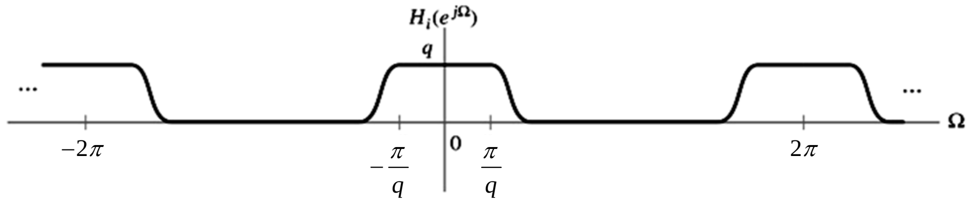
- Interpolation increases the sampling rate and requires that we somehow produce values between two samples of the signal



Interpolation Filter

- Decimation filter $h_i[n]$: a low-pass filter prevents from image problem when upsampling by q.

$$H_i(e^{j\Omega}) = \begin{cases} 1, & |\Omega T_s| \le \dfrac{\pi}{q} \\ 0, & \text{otherwise} \end{cases}$$



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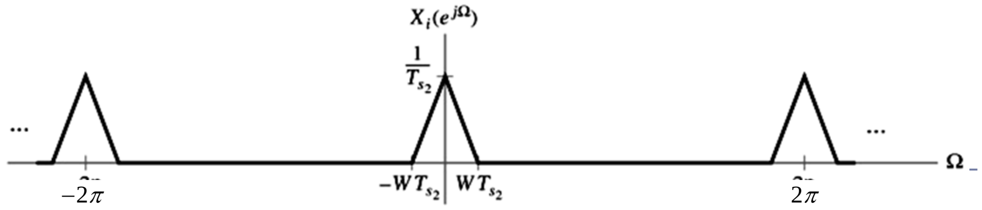

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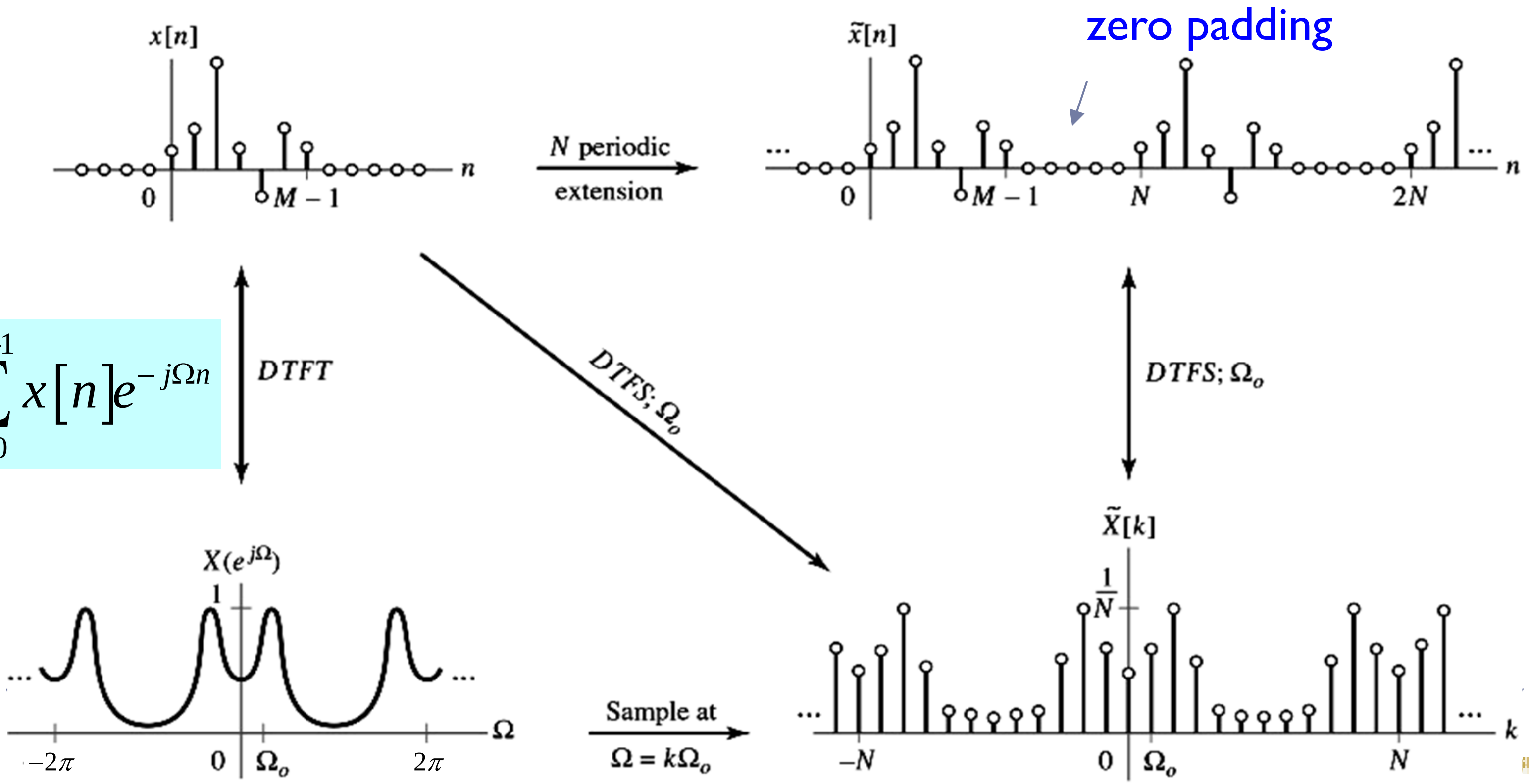
Outline

- Introduction
- Fourier Transform of Periodic Signals
- Convolution/Multiplication with (Non-)Periodic Signals
- Fourier Transform of Discrete-Time Signals
- Sampling
- Reconstruction of Continuous-Time Signals
- Discrete-Time Processing of Continuous-Time Signals
- Fourier Series of Finite-Duration Nonperiodic Signals

Why DTFS for Finite Nonperiodic Signals?

- The primary motivation is for the numerical computation of Fourier transform
- DTFS is only Fourier representation that can be evaluated numerically !!
- Let $x[n]$ be a finite-duration signal of length M
- Introduce a periodic DT signal $\tilde{x}[n]$ with period $N \geq M$

$$X(e^{j\Omega}) = \sum_{n=0}^{M-1} x[n] e^{-j\Omega n}$$



Relating the DTFS to the DTFT

- Consider a periodic DT signal $\tilde{x}[n]$ with period $N \geq M$, then

$$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk\Omega_0 n}$$

$\Omega_o = 2\pi/N$

Apply $x[n]$, we have $$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{M-1}x[n]e^{-jk\Omega_0 n}$$

- Recall that $X(e^{j\Omega}) = \sum_{n=0}^{M-1} x[n]e^{-j\Omega n}$, we conclude that

$$\tilde{X}[k] = \frac{1}{N}X(e^{j\Omega})\Big|_{\Omega=k\Omega_0}$$

The DTFS coefficients of $\tilde{x}[n]$ are samples of the DTFT of $x[n]$, divided by N and evaluated at intervals of $2\pi/N$.

- **The effect of sampling the DTFT of a finite-duration nonperiodic signal is to periodically extend the signal in the time domain**
- Dual to sampling in time domain

In order to prevent overlap in time domain, we requires $N \geq M$, or the sampling frequency $\Omega_0 \leq 2\pi/M$

Example 4.14

Consider the signal $x[n] = \begin{cases} \cos\left(\frac{3\pi}{8} n\right), & 0 \le n \le 31 \\ 0, & \text{otherwise} \end{cases}$

Derive both the DTFT, $X(e^{j\Omega})$, and the DTFS, $X[k]$, of $x[n]$, assuming a period $N > 31$. Evaluate and plot $|X(e^{j\Omega})|$ and $N|X[k]|$ for N = 32, 60, and 120.

<Sol.>

1. We rewrite the signal $x[n]$ as $g[n]w[n]$, where $g[n] = \cos(3\pi n/8)$ and a rectangular window

$$w[n] = \begin{cases} 1, & 0 \le n \le 31 \\ 0, & \text{otherwise} \end{cases} \xleftrightarrow{\text{DTFT}} W\left(e^{j\Omega}\right) = e^{-j31\Omega/2} \frac{\sin(16\Omega)}{\sin(\Omega/2)}$$

2. Then

$$X(e^{j\Omega}) = \frac{1}{2\pi} G(e^{j\Omega}) \otimes W(e^{j\Omega})$$

$$\Rightarrow \quad X\left(e^{j\Omega}\right) = \frac{e^{-j31(\Omega+3\pi/8)/2}}{2} \frac{\sin\left(16\left(\Omega+3\pi/8\right)\right)}{\sin\left(\left(\Omega+3\pi/8\right)/2\right)} + \frac{e^{-j31(\Omega-3\pi/8)/2}}{2} \frac{\sin\left(16\left(\Omega-3\pi/8\right)\right)}{\sin\left(\left(\Omega-3\pi/8\right)/2\right)}$$

3. Sample at $\Omega = k\Omega_0$ and divide by N, we have

$$X[k]=\left(\frac{e^{-j(k\Omega_o+3\pi/8)16}}{2Ne^{-j\frac{1}{2}(k\Omega_o+3\pi/8)}}\right)\frac{e^{j(k\Omega_o+3\pi/8)16}-e^{-j(k\Omega_o+3\pi/8)16}}{e^{j(k\Omega_o+3\pi/8)2}-e^{-j(k\Omega_o+3\pi/8)2}}$$

$$+\left(\frac{e^{-j(k\Omega_o-3\pi/8)16}}{2Ne^{-j(k\Omega_o-3\pi/8)/2}}\right)\frac{e^{j(k\Omega_o-3\pi/8)16}-e^{-j(k\Omega_o-3\pi/8)16}}{e^{j(k\Omega_0-3\pi/8)2}-e^{-j(k\Omega_o-3\pi/8)2}}$$

$$=\left(\frac{e^{-j31(k\Omega_o+3\pi/8)/2}}{2N}\right)\frac{\sin\left(16\left(k\Omega_o+3\pi/8\right)\right)}{\sin\left(\left(k\Omega_o+3\pi/8\right)/2\right)}$$

$$+\left(\frac{e^{-j31(k\Omega_o-3\pi/8)/2}}{2N}\right)\frac{\sin\left(16\left(k\Omega_o-3\pi/8\right)\right)}{\sin\left(\left(k\Omega_o-3\pi/8\right)/2\right)}$$

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Relating the FS to the FT

- The relationship between the FS and the FT of a finite-duration non-periodic continuous-time signal is analogous to that of discrete-time case
- Let $x(t)$ have duration T_0, so that $x(t) = 0, \quad t < 0 \quad \text{or} \quad t \geq T_0$
- Construct a periodic signal $\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t + mT)$ with $T \geq T_o$
- Consider the FS of $\tilde{x}(t)$

$$\tilde{X}[k] = \frac{1}{T}\int_0^T \tilde{x}(t)e^{jk\omega_o t}dt = \frac{1}{T}\int_0^{T_0} x(t)e^{jk\omega_0 t}dt$$

- Recall that

$$x(t) \xleftrightarrow{DT} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_0^T x(t)e^{-j\omega t}dt$$

$$\Rightarrow \boxed{\tilde{X}[k] = \frac{1}{T}X(j\omega)\Big|_{\omega = k\omega_o}}$$

The FS coefficients are samples of the FT, normalized by T