Chapter 3: Fourier Representation of Signals and LTI Systems

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Outline

- Introduction
- Complex Sinusoids and Frequency Response
- Fourier Representations for Four Classes of Signals
- Discrete-time Periodic Signals *Fourier Series*
- Continuous-time Periodic Signals
- Discrete-time Nonperiodic Signals *Fourier Transform*
- Continuous-time Nonperiodic Signals
- Properties of Fourier representations
- Linearity and Symmetry Properties
- Convolution Property

Discrete-Time Fourier Transform (DTFT)

- The DTFT-pair of a discrete-time nonperiodic signal $x[n]$ and $X(e^{j\Omega})$

$$x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

  - DTFT represents $x[n]$ as a superposition of complex sinusoids
  - Since $x[n]$ is not periodic, there are no restrictions on the periods (or frequencies) of the sinusoids to represent $x[n]$. The DTFT would involve a continuous of frequencies on $-\pi \le \Omega \le \pi$ (discrete-time sinusoids are unique only over a 2π interval of frequency)
  - $X(e^{j\Omega})$ is termed as the frequency-domain representation of $x[n]$
  - If $x[n]$ has finite duration and is finite valued, the infinite sum converges definitely
  - The infinite sum converges uniformly only if $x[n]$ is absolutely summable
  - If $x[n]$ is not absolutely summable, but squarely summable (i.e. has finite energy), the infinite sum converges in a MMSE sense, but does not converge pointwise.

Example 3.17

Find the DTFT of the sequence $x[n] = \alpha^n u[n]$.

<Sol.>

1. DTFT of $x[n]$:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}$$

2. For $\alpha < 1$, we have

$$X(e^{j\Omega}) = \sum_{n=0}^{\infty} \left(\alpha e^{-j\Omega}\right)^n = \frac{1}{1-\alpha e^{-j\Omega}}, \quad |\alpha| < 1$$

3. If α is real valued, then

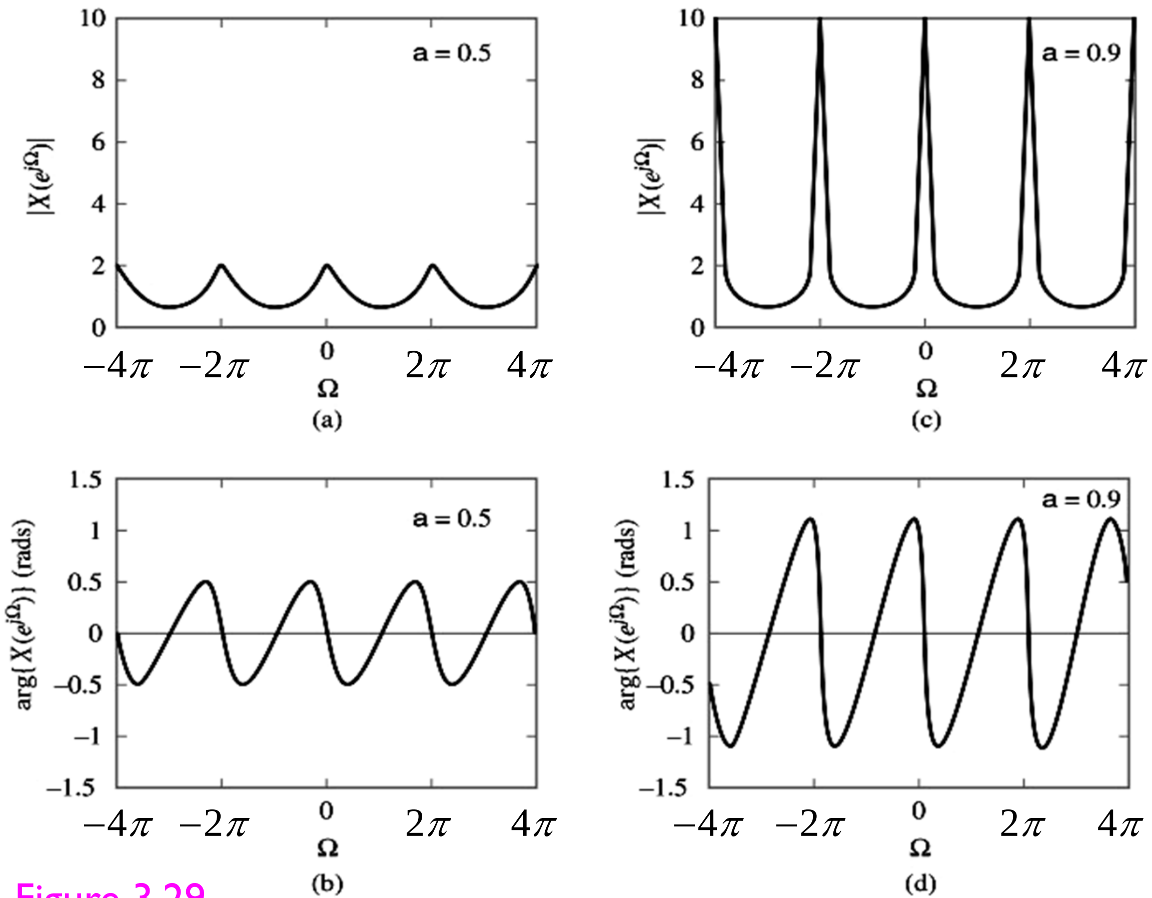
$$X(e^{j\Omega}) = \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}$$

4. Magnitude and phase spectra if α is real valued :

$$\left|X(e^{j\Omega})\right| = \frac{1}{\left((1-\alpha\cos\Omega)^2 + \alpha^2\sin^2\Omega\right)^{1/2}} = \frac{1}{\left(\alpha^2 + 1 - 2\alpha\cos\Omega\right)^{1/2}}$$

$$\arg\{X(e^{j\Omega})\} = -\arctan\left(\frac{\alpha\sin\Omega}{1-\alpha\cos\Omega}\right)$$

Example 3.17 (conti.)



Magnitude spectrum (even)

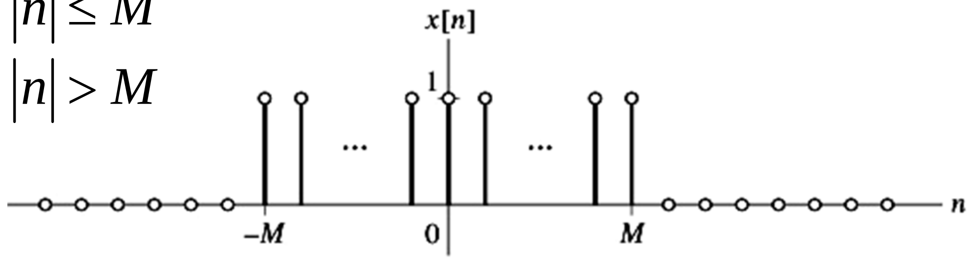
Phase spectrum (odd)

Figure 3.29
The DTFT of an exponential signal $x[n] = (\alpha)^n u[n]$.
(a) Magnitude spectrum for $\alpha = 0.5$. (b) Phase spectrum for $\alpha = 0.5$.
(c) Magnitude spectrum for $\alpha = 0.9$. (d) Phase spectrum for $\alpha = 0.9$.

Example 3.18 DTFT of a Rectangular Pulse

Find the DTFT of $x[n]$.

$$x[n]=\begin{cases}1, & |n|\le M\\ 0, & |n|>M\end{cases}$$



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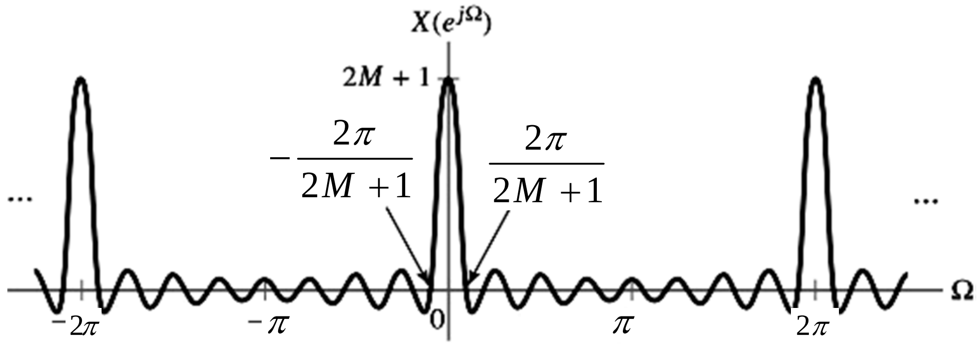


(b)

<Sol.>

1. DTFT of $x[n]$:

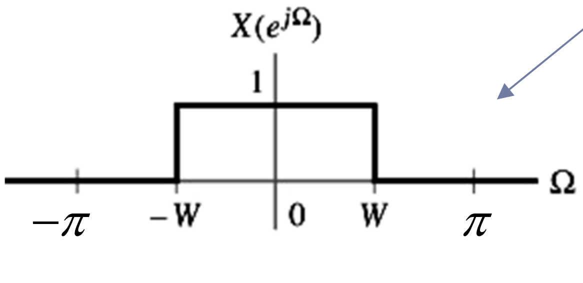
$$X(e^{j\Omega})=\sum_{n=-M}^{M}1\cdot e^{-j\Omega n}=\begin{cases}\dfrac{e^{-j\Omega M}(1-e^{-j\Omega(2M+1)})}{1-e^{-j\Omega}}, & \Omega\neq 0,\pm 2\pi,\pm 4\pi\\ 2M+1, & \Omega=0,\pm 2\pi,\pm 4\pi\end{cases}$$

Example 3.19
Inverse DTFT of a Rectangular Spectrum

Find the inverse DTFT of

$$X\left(e^{j\Omega}\right)=\begin{cases}1, & |\Omega|\le W \\ 0, & W<|n|<\pi\end{cases},$$

Note that $X(e^{j\Omega})$ is specified only for $-\pi < \Omega \le \pi$.



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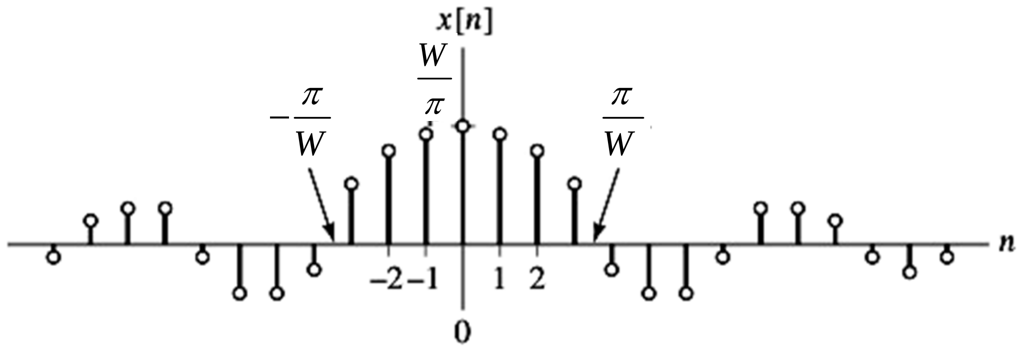


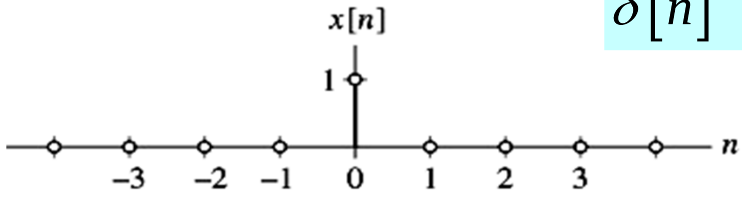
(b)

<Sol.>

$$x[n]=\frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}d\Omega \;=\frac{1}{2\pi nj}e^{j\Omega n}\Big|_{-W}^{W},\quad n\ne 0 \;=\frac{1}{\pi n}\sin(Wn),\quad n\ne 0.$$

Example 3.20, 3.21

$$\delta[n] \xleftrightarrow{DTFT} 1.$$



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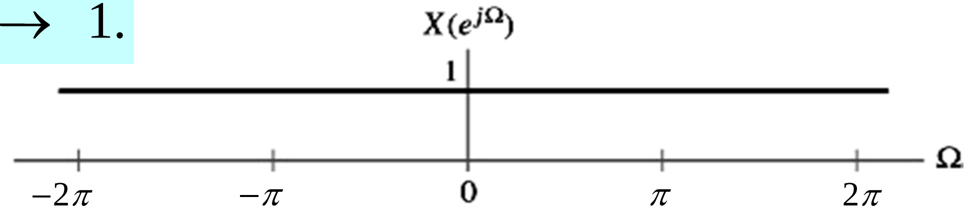

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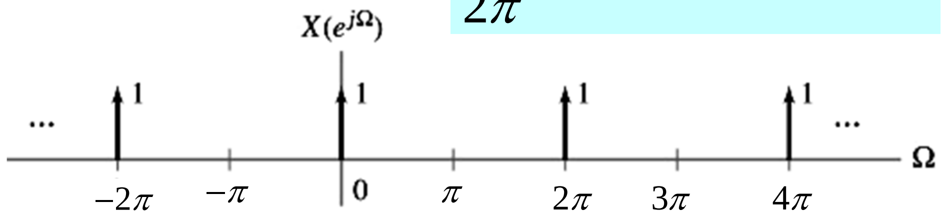
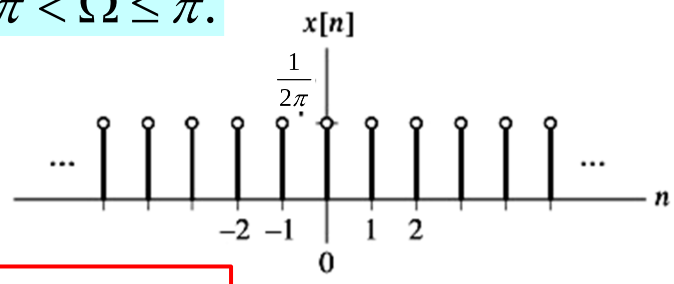
$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$$

$$\frac{1}{2\pi} \xleftrightarrow{DTFT} \delta(\Omega), \quad -\pi < \Omega \le \pi.$$



(a)



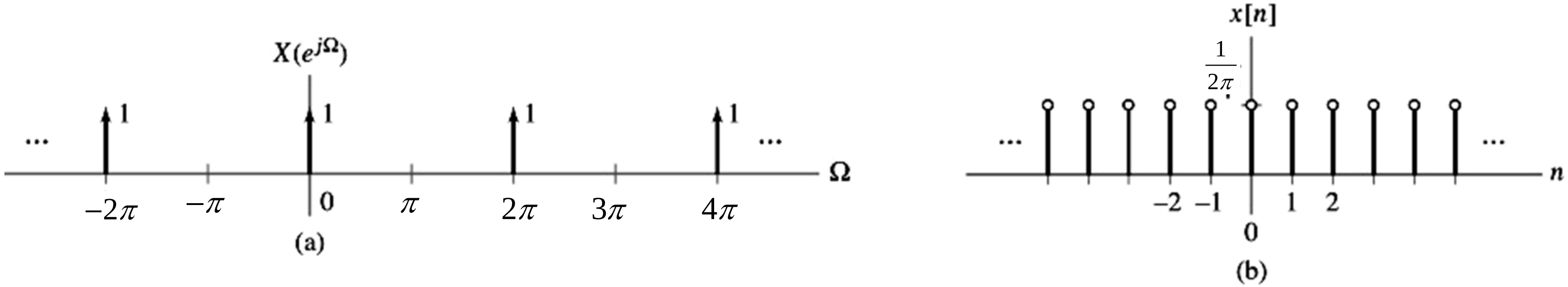
(b)

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\Omega) d\Omega = \frac{1}{2\pi}$$

$$X\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi).$$

♣ We can define $X(e^{j\Omega})$ over all Ω by writing it as an infinite sum of delta functions shifted by integer multiples of 2π.

Remarks



$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi) \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad x[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \delta(n-k)$$

- $x[n]$ is not absolutely summable (or not square summable), but it is still a valid DTFT-pair.
- Strictly speaking, the DTFT of the impulse train $x[n]$ does not exist; however, we can still identify their DTFT-pair. That is, we still can utilize the DTFT as a problem-solving tool.

Example 3.22
Simple Low-Pass and High-Pass Filters

Consider two different moving-average systems described by the input-output equations

$$y_1[n] = \frac{1}{2}\left(x[n] + x[n-1]\right) \quad \text{and} \quad y_2[n] = \frac{1}{2}\left(x[n] - x[n-1]\right)$$

Find the frequency response of each system and plot the magnitude responses.

<Sol.>

1. The frequency response is the DTFT of the impulse response.
2. For system # 1 — For system # 2 —

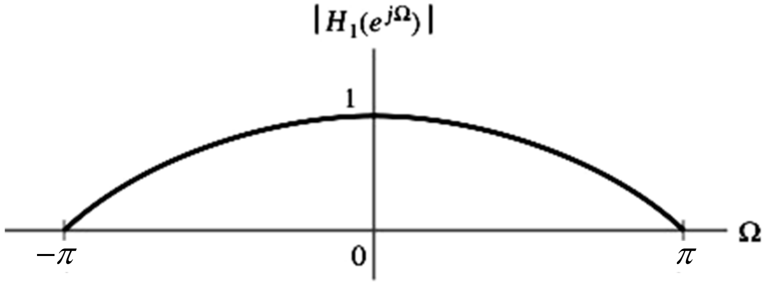
$$h_1[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1] \qquad h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]$$

The corresponding frequency response:

$$H_1(e^{j\Omega}) = e^{-j\frac{\Omega}{2}} \frac{e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}}{2} = e^{-j\frac{\Omega}{2}} \cos\left(\frac{\Omega}{2}\right).$$

$$H_2(e^{j\Omega}) = \frac{1}{2} - \frac{1}{2}e^{-j\Omega} = je^{-j\frac{\Omega}{2}} \frac{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}}{2j} = je^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right).$$

Example 3.22 (conti.)



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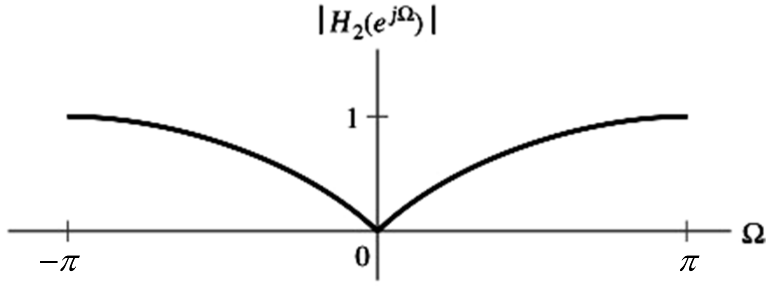


(b)

Low-pass filter

High-pass filter

Fourier Transform (FT)

- The FT-pair of a continuous-time nonperiodic signal $x(t)$ and $X(j\omega)$

$$x(t) \xleftrightarrow{FT} X(j\omega).$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - FT represents $x(t)$ as a superposition of complex sinusoids
  - Since $x(t)$ is not periodic, there are no restrictions on the periods (or frequencies) of the sinusoids to represent $x(t)$. The FT would involve a continuous of frequencies ranging from -∞ to ∞.
  - $X(j\omega)$ is termed as the frequency-domain representation of $x(t)$
  - The integral in FT-pair may not converge for all functions of $x(t)$ and $X(j\omega)$
  - if $x(t)$ is square integral, the integral converges in a MMSE sense, but not converge poitwise.
  - If $x(t)$ satisfies Dirichlet's conditions, the integral converges pointwise.

Example 3.24

Find the FT of $x(t)$ = e^{-at} $u(t)$.

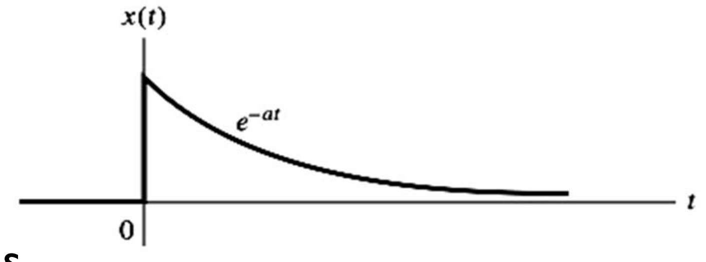
<Sol.>

For $a \le 0$, $x(t)$ is not absolutely integrable.

Therefore, we consider $a > 0$. The FT of $x(t)$ is

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$



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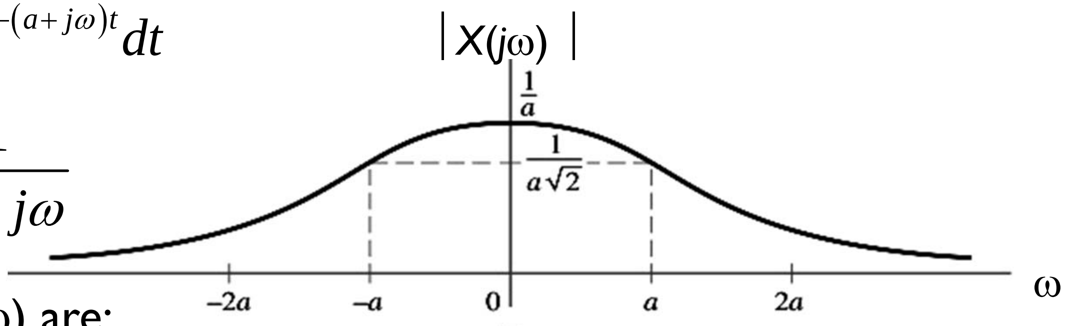
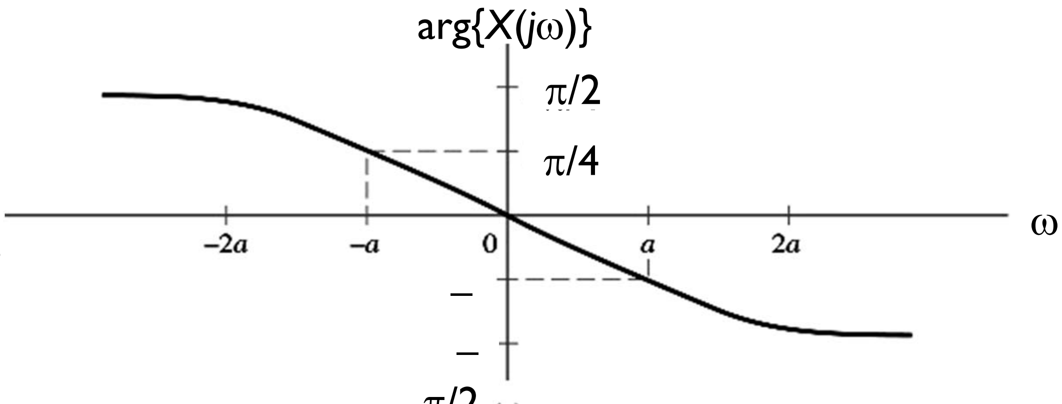


(b)

Magnitude and phase spectra of $X(j\omega)$ are:

$$|X(j\omega)| = \frac{1}{(a^2+\omega^2)^{\frac{1}{2}}}$$

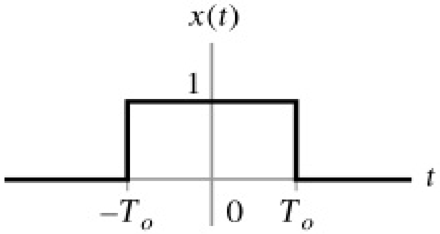
$$\arg\{X(j\omega)\} = -\arctan(\omega/a),$$



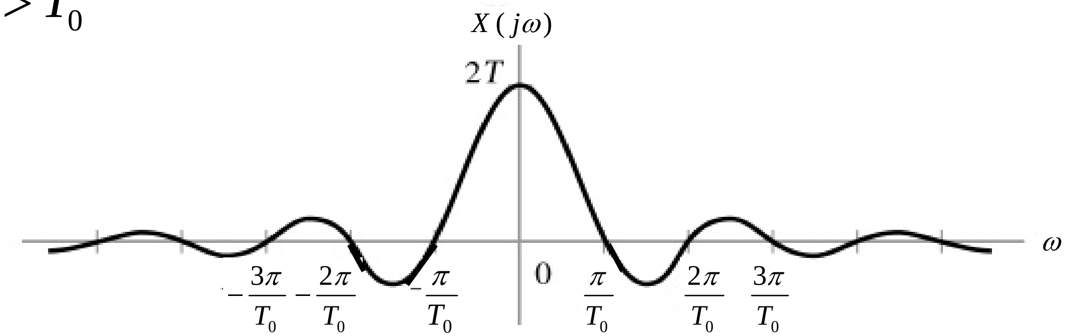
(c)

Example 3.25

Find the FT of $x(t)=\begin{cases}1, & -T_0<t<T_0\\ 0, & |t|>T_0\end{cases}$



(a)



(b)

<Sol.>

1. The rectangular pulse $x(t)$ is absolutely integrable, provided that $T_o < \infty$.

2. FT of $x(t)$:

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-T_0}^{T_0}e^{-j\omega t}dt$$

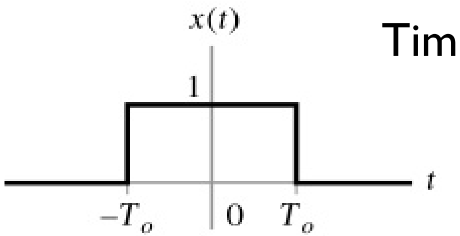
$$=-\frac{1}{j\omega}e^{-j\omega t}\Big|_{-T_0}^{T_0}=\frac{2}{\omega}\sin(\omega T_0),\quad \omega\neq 0$$

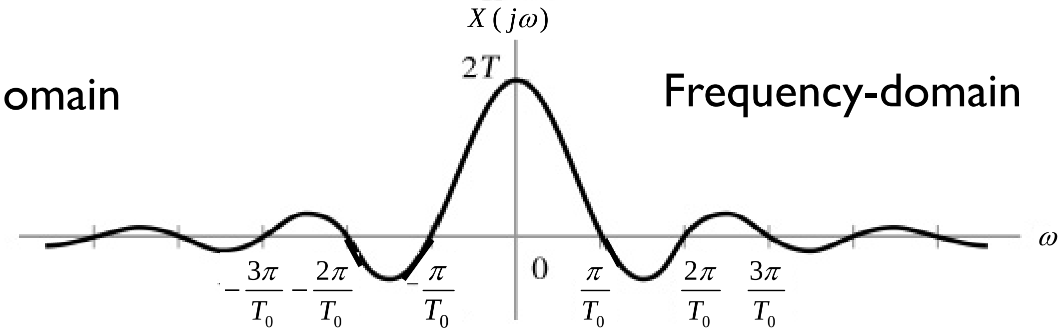
$\Rightarrow$ $X(j\omega)=\frac{2}{\omega}\sin(\omega T_0),$

$$\lim_{\omega\to 0}\frac{2}{\omega}\sin(\omega T_0)=2T_0.$$

$X(j\omega)=2T_0\,\mathrm{sinc}(\omega T_0/\pi).$

Remarks





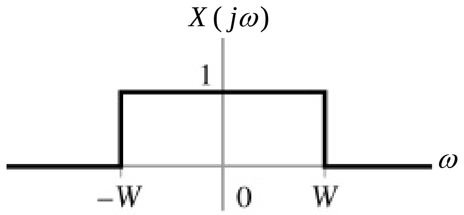
$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & |t| > T_0 \end{cases}$$

$$X(j\omega) = 2T_0 \,\mathrm{sinc}(\omega T_0/\pi).$$

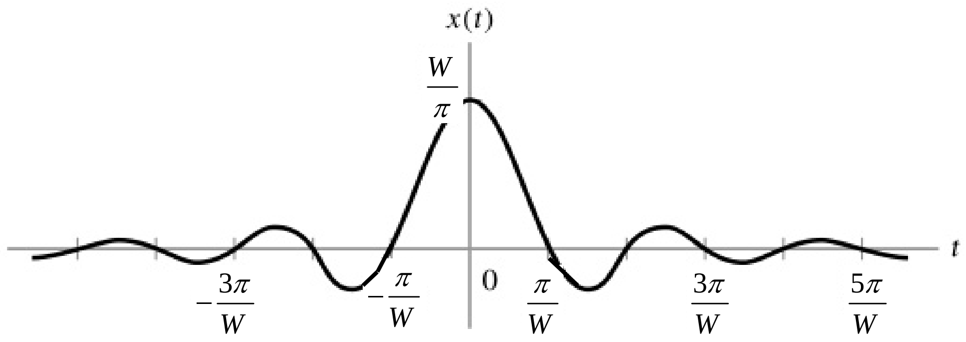
- As T_0 increases, the nonzero time extent of $x(t)$ increases, while the $X(j\omega)$ becomes more concentrated about the frequency origin
- Conversely, as T_0 decreases, the nonzero duration of $x(t)$ decreases, while the $X(j\omega)$ becomes less concentrated about the frequency origin
- The duration of $x(t)$ is inversely related to the bandwidth of $X(j\omega)$
- The signal which is concentrated in one domain is spread out in the other domain.

Example 3.26

Find the inverse FT of the rectangular spectrum $X(j\omega)=\begin{cases}1, & -W<\omega<W \\ 0, & |\omega|>W\end{cases}$



(a)



(b)

<Sol.>

The inverse FT of *x*(*t*): $x(t)=\frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega=-\frac{1}{j\pi t}e^{j\omega t}\Big|_{-W}^{W}=\frac{1}{\pi t}\sin(Wt),\quad t\neq 0$

$$\lim_{t\to 0}\frac{1}{\pi t}\sin(Wt)=W/\pi,$$

$$x(t)=\frac{1}{\pi t}\sin(Wt),$$

$$x(t)=\frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right),$$

Example 3.27, 3.28

Find the FT of $x(t) = \delta(t)$.

<Sol.>

$x(t)$ does not satisfy the Dirichlet's condition, since the discontinuity at the origin is infinite. However we attempt to process the FT of $x(t)$ directly:

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\Rightarrow \quad \delta(t) \xleftrightarrow{FT} 1,$$

Find the inverse FT of $X(j\omega) = 2\pi\delta(\omega)$.

<Sol.>

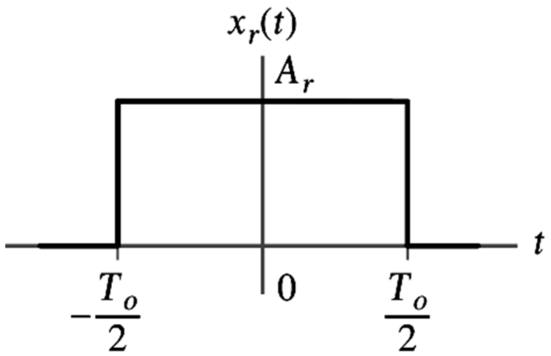
Inverse FT of $X(j\omega)$ is $x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega = 1$

$$\Rightarrow \quad 1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

Duality between Example 3.27 and 3.28

Example 3.29 BPSK Modulation

In a simple digital communication system, BPSK (binary phase-shift keying) is a common scheme to assume that the signal representing '0' is the negative of the signal and the signal representing "1" is the positive of the signal. Fig. 3.44 depicts two candidate signals for this approach: **a rectangular pulse** $x_r(t)$ and **a raised-cosine pulse** $x_c(t)$ (Note that each pulse is T_o seconds long)



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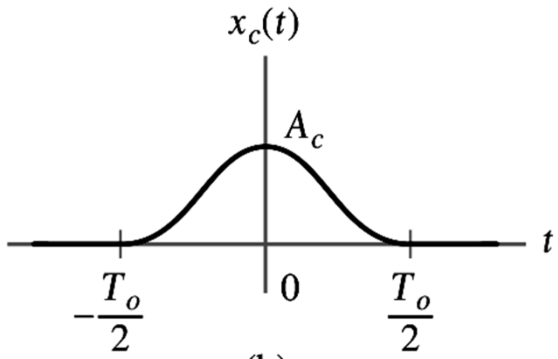
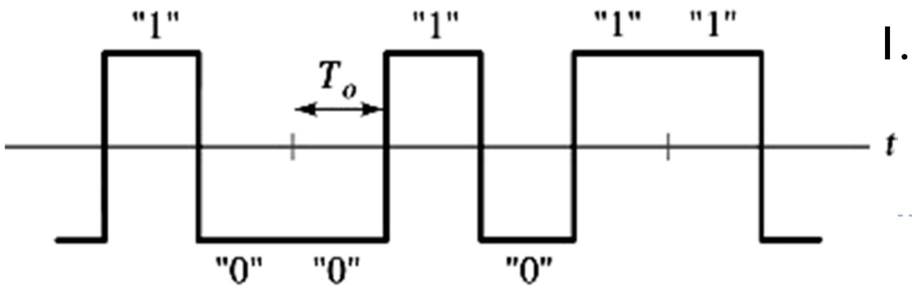


(b)

$$x_r(t) = \begin{cases} A, & |t| < T_0/2 \\ 0, & |t| > T_0/2 \end{cases}$$

$$x_c(t) = \begin{cases} (A_c/2)\left(+2\cos\left(2\pi t/T_0\right)\right), & |t| < T_0/2 \\ 0, & |t| > T_0/2 \end{cases}$$

For example for the sequence "1001011", the transmitted BPSK signals for communicating are



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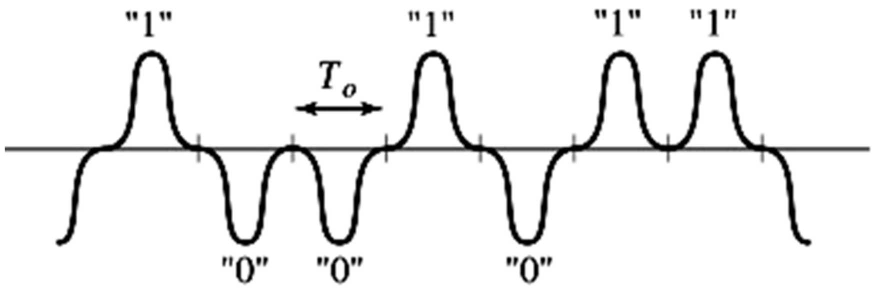


(b)

1. With BPSK, the system has a transmission rate of $1/T_o$ bits per second. (Each user's signal is transmitted within an assigned frequency band in order to prevent interference with others)
2. Suppose the frequency band assigned to each user is 20 kHz wide. Then, to prevent interference with adjacent channels, we assume that the peak value of the magnitude spectrum of the transmitted signal outside the 20-kHz band is required to be – 30 dB below the peak in-band magnitude spectrum.

Problems:

1. Choose the constant A_r and A_c so that both BPSK signals have unit power.
2. Use FT to determine the maximum number of bits per second that can be transmitted when the rectangular and raised-cosine pulse shapes are utilized.

<Sol.>

1. BPSK signal is a power signal. Powers in rectangular pulse and raised-cosine pulse are:

$$P_r = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A_r^2 dt = A_r^2$$ and

Example 3.29 (conti.)

$$P_c = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left(A_c^2/4\right)\left(1+2\cos\left(2\pi t/2\right)\right)^2 dt$$

$$= \frac{A_c^2}{4T_0}\int_{-T_0/2}^{T_0/2}\left[1+2\cos\left(2\pi t/T_0\right)+1/2+1/2\cos\left(4\pi t/T_0\right)\right]dt$$

$$= \frac{3A_c^2}{8}$$

Hence, unity transmission power is obtained by choosing $A_r = 1$ and $A_c = \sqrt{8/3}$.

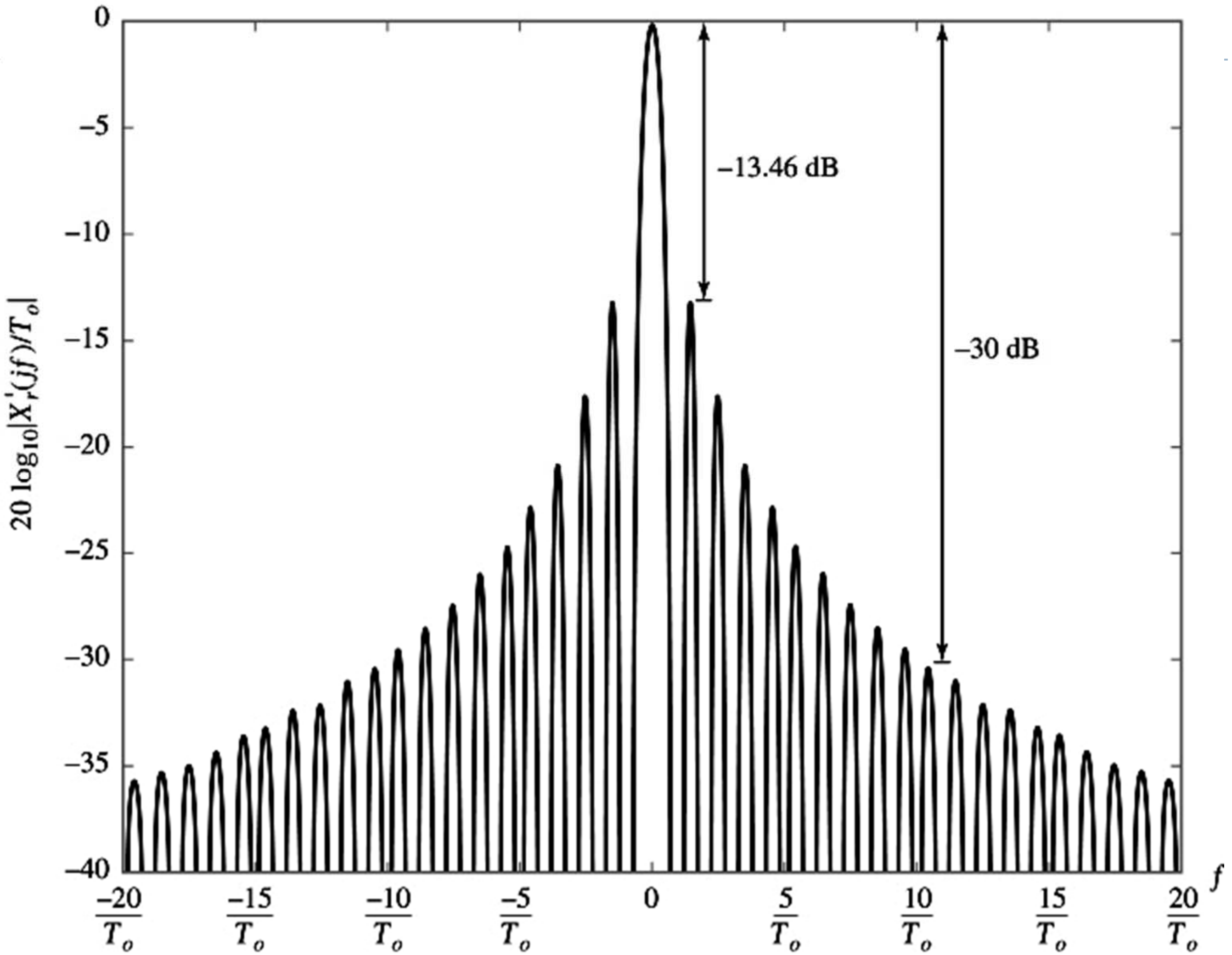
2. FT of the rectangular pulse $x_r(t)$: $X_r(j\omega) = 2\frac{\sin\left(\omega T_0/2\right)}{\omega}$ rad/sec ($\omega = 2\pi f$)

Rewrite it in terms of Hz: $X_r'(jf) = 2\frac{\sin\left(\pi f T_0\right)}{\pi f}$

The normalized magnitude spectrum of the signal is given by

$$10\log_{10}\left\{\left|X(jf)\right|^2/T_0\right\} = 20\log_{10}\left\{\left|X(jf)\right|/T_0\right\}$$

Spectrum of rectangular pulse in dB, normalized by T_0.



The normalized by T_o removed the dependence of the magnitude on T_o.

- From the figure, we find that the 10^{th} sidelobe is the first one whose peak does not exceed -30 dB.
- This implies that we must choose T_0 so that the 10^{th} zero crossing is at 10 kHz in order to satisfy the constraint.

⇨ $10K = 10/T_o$ ⇨ Data transmission rate = 1K bits/sec.

3. FT of raised-cosine pulse $x_c(t)$: $X_c(j\omega) = \frac{1}{2}\sqrt{\frac{8}{3}}\int_{-T_0/2}^{T_0/2}\left(1+2\cos\left(2\pi t/T_0\right)\right)e^{-j\omega t}dt$

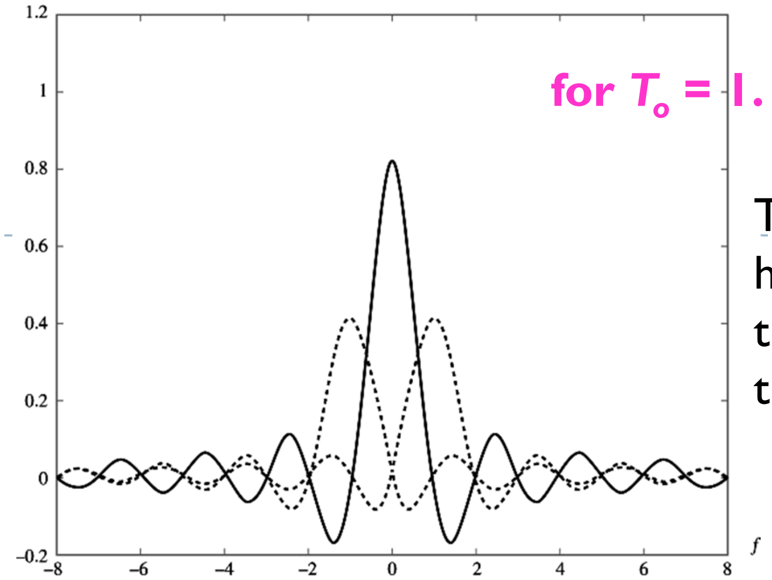
$$X_c(j\omega) = \sqrt{\frac{2}{3}}\int_{-T_0/2}^{T_0/2} e^{-j\omega t}dt + \frac{1}{2}\sqrt{\frac{2}{3}}\int_{-T_0/2}^{T_0/2} e^{-j(\omega-2\pi/T_0)t}dt + \frac{1}{2}\sqrt{\frac{2}{3}}\int_{-T_0/2}^{T_0/2} e^{-j(\omega+2\pi/T_0)t}dt$$

$$X_c(j\omega) = 2\sqrt{\frac{2}{3}}\frac{\sin\left(\omega T_0/2\right)}{\omega} + \sqrt{\frac{2}{3}}\frac{\sin\left(\left(\omega-2\pi/T_0\right)T_0/2\right)}{\omega-2\pi/T_0} + \sqrt{\frac{2}{3}}\frac{\sin\left(\left(\omega+2\pi/T_0\right)T_0/2\right)}{\omega+2\pi/T_0}$$

In terms of f (Hz), the above equation becomes

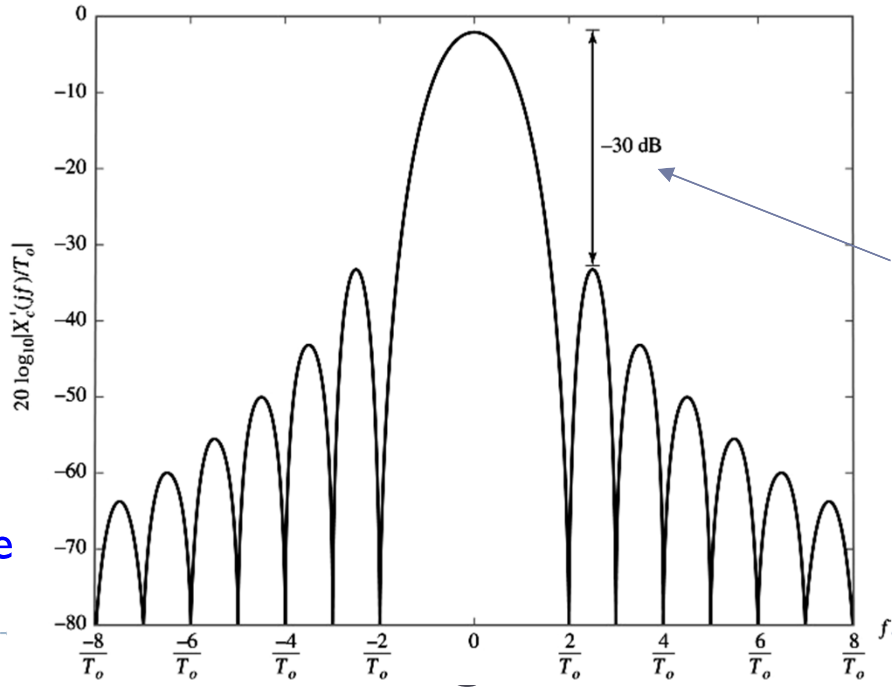
$$X_c'(jf) = \sqrt{\frac{2}{3}}\frac{\sin\left(\pi f T_0\right)}{\pi f} + 0.5\sqrt{\frac{2}{3}}\frac{\sin\left(\pi\left(f-1/T_0\right)T_0\right)}{\pi\left(f-1/T_0\right)} + 0.5\sqrt{\frac{2}{3}}\frac{\sin\left(\pi\left(f+1/T_0\right)T_0\right)}{\pi\left(f+1/T_0\right)}$$

A superposition of three sinc spectra



The sum of three spectra has lower sidelobs than that of the spectrum of the rectangular pulse

The normalized magnitude spectrum of the signal, i.e. $20\log_{10}\{|X(jf)|/T_0\}$, is



the peak value of the first sidelobe is below – 30 dB, so we may satisfy the adjacent channel interference specifications by choosing the mainlobe to be 20 kHz wide:

$\Rightarrow$ 10,000 = $2/T_o$

$T_o = 2 \times 10^{-4}$ s

Data transmission rate = 5000 bits/sec.