Chapter 2: Time-Domain Representations of Linear Time-Invariant Systems

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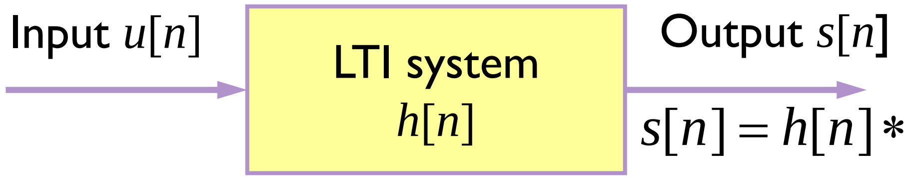
Outline

- Introduction
- The Convolution Sum
- Convolution Sum Evaluation Procedure
- The Convolution Integral
- Convolution Integral Evaluation Procedure
- Interconnections of LTI Systems
- Relations between LTI System Properties and the Impulse Response
- Step Response
- Differential and Difference Equation Representations
- Solving Differential and Difference Equations

Step Response

- Step input are often used to characterize the response of an LTI system to sudden changes in the input.
- Step response of LTI system $H\{\cdot\}$: $s[n]=H\{u[n]\}$



$$s[n]=h[n]*u[n]=\sum_{k=-\infty}^{\infty}h[k]u[n-k]$$

Since $u[n-k]=0$ for $k>n$ and $u[n-k]=1$ for $k\le n$, we have $=\sum_{k=-\infty}^{n}h[k]$

- Discrete-time case: $s[n]=\sum_{k=-\infty}^{n}h[k]$ Continuous-time case: $s(t)=\int_{-\infty}^{t}h(\tau)d\tau$

Relationships between step response and impulse response

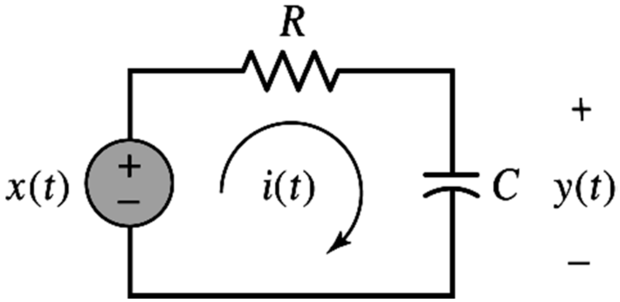
- Discrete-time case: $h[n]=H\{\delta[n]\}=H\{u[n]-u[n-1]\}=s[n]-s[n-1]$
- Continuous-time case: $h(t)=H\left\{\lim_{\Delta t\to 0}x_{\Delta t}(t)\right\}=\lim_{\Delta t\to 0}H\left\{\frac{1}{\Delta}(u(t+\tfrac{\Delta}{2})-u(t-\tfrac{\Delta}{2})\right\}$
- $$=\lim_{\Delta t\to 0}\frac{s(t+\frac{\Delta}{2})-s(t-\frac{\Delta}{2})}{\Delta}=\frac{d}{dt}s(t)$$

Example 2.14 Step Response of RC Circuit

The impulse response of the *RC* circuit depicted in Fig. 2.12 is

$$h(t)=\frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

. Find the step response of the circuit.

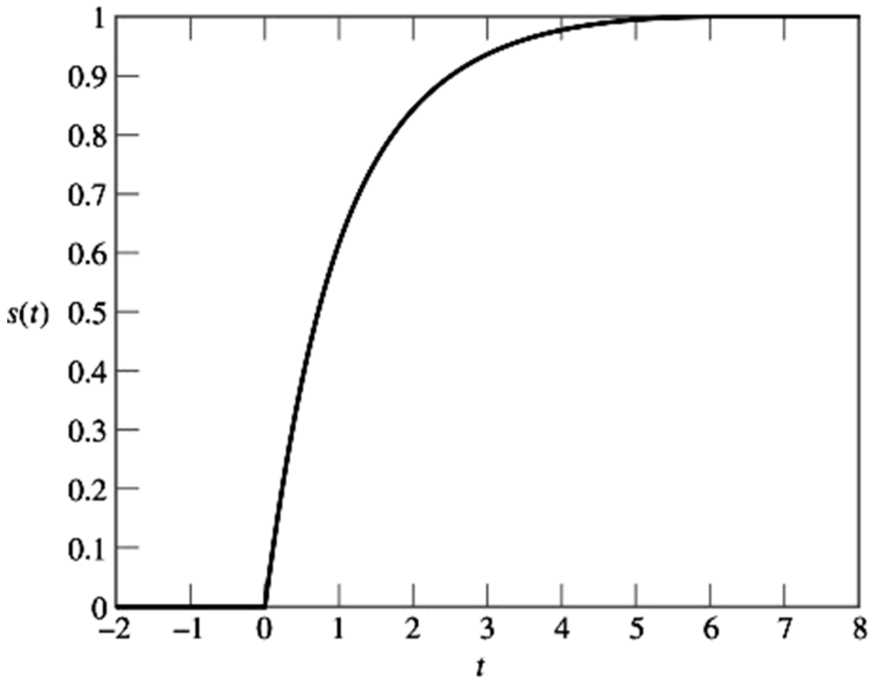


<Sol.>

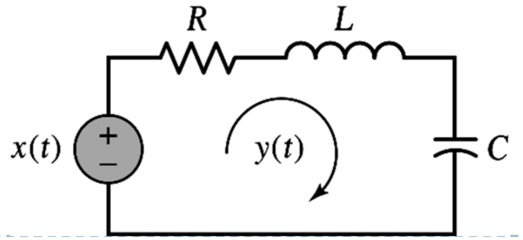
Step response: $s(t)=\int_{-\infty}^{t}\frac{1}{RC}e^{-\frac{\tau}{RC}}u(\tau)d\tau.$

$$s(t)=\begin{cases}0, & t<0\\ \frac{1}{RC}\int_{0}^{t}e^{-\frac{\tau}{RC}}d\tau & t\geq 0\end{cases}$$

$$=\begin{cases}0, & t<0\\ 1-e^{-\frac{t}{RC}}, & t\geq 0\end{cases}$$



RLC Circuit Example



Suppose that Input = voltage source $x(t)$, output = loop current $y(t)$. By KVL eq:

$$Ry(t)+L\frac{d}{dt}y(t)+\frac{1}{C}\int_{-\infty}^{t}y(\tau)\,d\tau=x(t)$$

$$\Rightarrow \quad \frac{1}{C}y(t)+R\frac{d}{dt}y(t)+L\frac{d^2}{dt^2}y(t)=\frac{d}{dt}x(t) \quad \Rightarrow \quad N=2$$

Linear constant-coefficient differential equation provide another representation for the IO-relationship of an LTI system !!!

Differential and Difference Equation Representations of LTI Systems

- The general form of a linear constant-coefficient differential equation:

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{l=0}^{M} b_l \frac{d^l}{dt^l} x(t)}$$

$x(t)$: input, $y(t)$: output
a_k, b_l : constant coefficients

- The form of a linear constant-coefficient difference equation:

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]}$$

$x[n]$: input, $y[n]$: output
a_k, b_l : constant coefficients

  - The order of the differential or difference equation is (N, M) representing the number of energy storage devices in the system
    - Often $N \geq M$, the order is described using only N.

Solving Differential Equations

- The solution for $y(t)$ may be expressed as the sum of two components: $y(t) = y^{(h)}(t) + y^{(p)}(t)$
  - Homogeneous solution: $y^{(h)}(t)$
    - Obtained by setting all terms involving the input to zero, i.e.

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0 \Rightarrow \boxed{y^{(h)}(t) = \sum_{i=1}^{N} c_i e^{r_i t}}$$

c_i is determined by I.C.

      - r_i are the N roots of system's characteristic equation $\boxed{\sum_{k=0}^{N} a_k r^k = 0}$
  - Particular solution: $y^{(p)}(t)$
    - Obtained by assuming an output of the same general form as the input

Table 2.3 ***Form of Particular Solutions Corresponding to Commonly Used Inputs***

Continuous Time		Discrete Time	
Input	Particular Solution	Input	Particular Solution
1	c	1	c
t	$c_1 t + c_2$	n	$c_1 n + c_2$
e^{-at}	ce^{-at}	α^n	$c\alpha^n$
$\cos(\omega t + \phi)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$	$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

Solving Difference Equations

- The solution for $y[n]$ may be expressed as the sum of two components: $y[n] = y^{(h)}[n] + y^{(p)}[n]$
  - Homogeneous solution: $y^{(h)}[n]$
    - Obtained by setting all terms involving the input to zero, i.e.

$$\sum_{k=0}^{N} a_k y^{(h)}[n-k] = 0 \quad \Rightarrow \quad \boxed{y^{(h)}[n] = \sum_{i=1}^{N} c_i r_i^n}$$

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Remarks

- $y^{(h)}$ is a solution for any set of constant c_i for the system's characteristic equation
- c_i is determined in order that the complete solution satisfies the initial condition (I.C.)
- The form of the homogeneous solution changes slightly when the characteristic's equation has repeated roots.
  - If a root *rj* is repeated *p* times in characteristic's equation, the corresponding solutions are

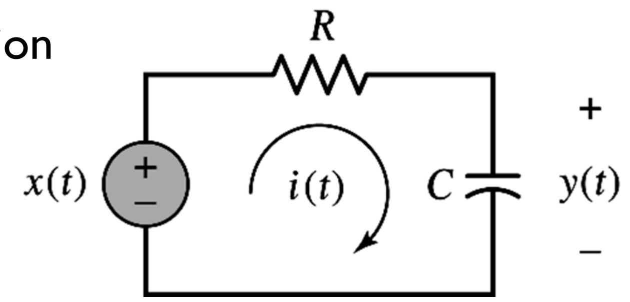
Continuous-time case: $e^{r_j t}, \quad te^{r_j t}, \quad \ldots, \quad t^{p-1}e^{r_j t}$

Discrete-time case: $r_j^n, \quad nr_j^n, \quad \ldots, \quad n^{p-1}r_j^n$

- The particular solutions given in Table 2.3 assume that the inputs exist for all time.

Example 2.17 & 2.20: RC Circuit

The RC circuit depicted is described by the differential equation



$$y(t)+RC\frac{d}{dt}y(t)=x(t)$$

Find the homogeneous solution. If an input $x(t)=\cos(\omega_0 t)$ is applied, find the particular solution

<Sol.>

1. Homogeneous solution:

Solving characteristic eq.: $1+RCr_1=0$ ⇒ $r_1=-1/RC$

$$y^{(h)}(t)=ce^{\frac{-t}{RC}}$$

2. Particular solution:

By Table 2.3 with an input $x(t)=\cos(\omega_0 t)$. ⇒ $y^{(p)}(t)=c_1\cos(\omega t)+c_2\sin(\omega t)$

Substituting $y^{(p)}(t)$ and $x(t)=\cos(\omega_0 t)$ into the given differential Eq.:

$$c_1\cos(\omega t)+c_2\sin(\omega t)-RC\omega_0 c_1\sin(\omega_0 t)+RC\omega_0 c_2\cos(\omega_0 t)=\cos(\omega_0 t)$$

$$\Rightarrow \begin{cases} c_1+RC\omega_0 c_2=1 \\ -RC\omega_0 c_1+c_2=0 \end{cases}$$

$$y^{(p)}(t)=\frac{1}{1+(RC\omega_0)^2}\cos(\omega_0 t)+\frac{RC\omega_0}{1+(RC\omega_0)^2}\sin(\omega_0 t)$$

Example 2.18 & 2.19: IIR System

Find the homogeneous solution for the first-order recursive system described by

$$y[n] - \rho y[n-1] = x[n]$$

Find a particular solution if the input is $x[n] = (1/2)^n$.

<Sol.>

1. Homogeneous solution:

Solving characteristic eq.: $r_1 - \rho = 0$ $\Rightarrow$ $r_1 = \rho$

$$y^{(h)}(t) = c\rho^n$$

2. Particular solution:

By Table 2.3 with an input $x[n] = (1/2)^n$. $\Rightarrow$ $y^{(p)}[n] = c_p \left(\frac{1}{2}\right)^n$

Substituting $y^{(p)}[n]$ and $x[n] = (1/2)^n$ into the given differential Eq.:

$$c_p \left(\tfrac{1}{2}\right)^n - \rho c_p \left(\tfrac{1}{2}\right)^{n-1} = \left(\tfrac{1}{2}\right)^n \Rightarrow c_p(1-2\rho) = 1$$

$$y^{(p)}[n] = \frac{1}{1-2\rho}\left(\frac{1}{2}\right)^n$$

Procedure for Complete Solution

1. Find the form of the homogeneous solution $y^{(h)}$ from the roots of the characteristic equation.

2. Find a particular solution $y^{(p)}$ by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution.

3. Determine the coefficients in the homogeneous solution so that the complete solution $y = y^{(h)} + y^{(p)}$ satisfies the I.C..

★ **Note that the initial translation is needed in some cases.**

Example 2.21
Complete Solution for IIR System

Find the complete solution for the first-order recursive system described by

$$y[n]-\frac{1}{4}y[n-1]=x[n]$$

if the input is $x[n]=(1/2)^n u[n]$ and the initial condition is $y[-1]=8$.

<Sol.>

1. Homogeneous sol.: $y^{(h)}[n]=c_1\left(\frac{1}{4}\right)^n$
2. Particular solution: $y^{(p)}[n]=2\left(\frac{1}{2}\right)^n$
3. Complete solution:

$$y[n]=2(\frac{1}{2})^n+c_1(\frac{1}{4})^n \quad (2.47)$$

4. Coefficient c_1 determined by I.C.:

$$y[0]=x[0]+1/4\,y[-1] \quad \Rightarrow \quad y[0]=x[0]+(1/4)\times 8=3$$

We substitute $y[0]=3$ into Eq. (2.47), yielding

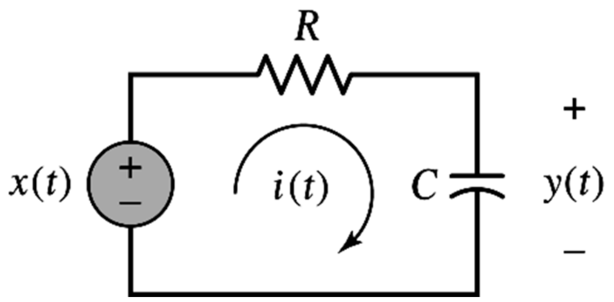
$$3=2\left(\frac{1}{2}\right)^0+c_1\left(\frac{1}{4}\right)^0 \quad \Rightarrow \quad c_1=1$$

5. Final solution:

$$y[n]=2\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n \quad \text{for } n\geq 0$$

Example 2.22 RC Circuit Example (conti)

Find the complete response of the RC circuit to an input $x(t) = \cos(t)u(t)$ V, assuming normalized values $R = 1\ \Omega$ and $C = 1$ F and assuming that the initial voltage across the capacitor is $y(0^-) = 2$ V.



<Sol.>

1. Homogeneous solution: $y^{(h)}(t) = ce^{\frac{-t}{RC}}$ V

2. Particular solution with an input $x(t) = \cos(\omega_0 t)$:

$$y^{(p)}(t) = \frac{1}{1+(RC\omega_0)^2}\cos(\omega_0 t) + \frac{RC\omega_0}{1+(RC\omega_0)^2}\sin(\omega_0 t) \quad \text{V}$$

3. For $\omega_0 = 1$, $R=1$, and $C=1$, we have the complete solution:

$$y(t) = ce^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{V} \qquad \text{for } t > 0$$

4. Coefficient c_1 determined by I.C.: $y(0^-) = y(0^+)$

$$2 = ce^{-0^+} + \frac{1}{2}\cos 0^+ + \frac{1}{2}\sin 0^+ = c + \frac{1}{2} \Longrightarrow y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{V} \qquad \text{for } t > 0$$

★ **To avoid discontinuity, the initial translation is needed in some cases.**

Example 2.23 Financial Computation

The following difference equation describes the balance of a loan $y[n] = \rho y[n-1] + x[n]$

if $x[n] < 0$ represents the principal and interest payment made at the beginning of each period and $y[n]$ is the balance after the principal and interest payment is credited. As before, if $r\%$ is the interest rate per period, then $\rho = 1 + r/100$. Assume 10% interest rate. Assume equal payments and use the complete response of the first-order difference equation to find the payment required to pay off a \$20,000 loan in 10 periods.

<Sol.>

1. We have $\rho = 1.1$ and $y[-1] = 20{,}000$. Assume that $x[n] = b$ is the payment each period. The first payment is made when $n = 0$. The loan balance is to be zero after 10 payments, i.e. we seek the payment b for which $y[9] = 0$.
2. Solving $y[n] - \rho y[n-1] = x[n]$, the homogeneous sol.: $y^{(h)}[n] = c_h(1.1)^n$
   With an input $x[n] = b$, the particular sol. (from Table 2.3): $y^{(p)}[n] = c_p$
   Substituting $y^{(p)}[n] = c_p$ and $x[n] = b$ into $y[n] - \rho y[n-1] = x[n]$, we obtain $c_p = -10b$
3. Complete solution: $y[n] = c_h(1.1)^n - 10b$
4. Coefficient c_h determined by I.C.: $y[0] = 1.1y[-1] + x[0] = 22{,}000 + b = c_h(1.1)^0 - 10b$
   $c_h = 22{,}000 + 11b \Rightarrow y[n] = (22{,}000 + 11b)(1.1)^n - 10b$
5. Payment b: By setting $y[9] = 0$, we have $b=3254.91$

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Outline

- Characteristics of Systems Described by Differential and Difference Equations
- Block Diagram Representations
- State-Variable Descriptions of LTI Systems
- Exploring Concepts with MATLAB
- Summary

Natural Response of the LTI System

- Natural Response of the LTI system: $y^{(n)}$
  - The LTI system output for zero input is called the natural response
  - It can be obtained from the homogeneous solution by choosing the coefficients satisfying I. C..
  - If the system possesses memory, $y^{(n)}$ is associated with the (non-zero) initial conditions of the LTI system
  - The $y^{(n)}$ is determined without translating initial condition forward in time
- Example 2.24

The RC circuit system is described by $y(t)+RC\frac{d}{dt}y(t)=x(t)$

Find the natural response of the system, assuming that $y(0)$= 2 V, R = 1 Ω and C = 1 F.

<Sol.>

1. Homogeneous sol.: $y^{(h)}(t)=c_1e^{-t}\quad \mathrm{V}$
2. I.C.: $y(0)=2\ \mathrm{V} \Rightarrow y^{(h)}(0)=2\ \mathrm{V} \Rightarrow c_1=2$
3. Natural Response: $y^{(n)}(t)=2e^{-t}\quad \mathrm{V}$

Forced Response of the LTI System

- Forced Response of the LTI system: $y^{(f)}$
  - The LTI system output due to the input signal assuming zero initial conditions is called the forced response
  - If the system possesses memory, $y^{(f)}$ described the system behavior that is forced by input when the system is at rest, i.e. with zero initial conditions
  - $y^{(f)}$ can be obtained from the complete solution by translating the at rest conditions: i.e. $y[-N] = 0, \ldots, y[-1] = 0$ forward to times $n = 0, 1, \ldots$
- Example 2.27

  The RC circuit system is described by $y(t) + RC\frac{d}{dt}y(t) = x(t)$

  Find the forced response of the system by assuming that $x(t) = \cos(t)u(t)$ V, $R = 1\ \Omega$ and $C = 1$ F.

  <Sol.>

  1. From Example 2.22, the complete sol.: $y(t) = ce^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{V}$
  2. "at rest" I.C.: $y(0^-) = y(0^+) = 0$ $\Rightarrow$ $c = -1/2$
  3. Forced response: $y^{(f)}(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{V}$

Characteristics of Systems Described by Differential/Difference Equations

- The output of a system can be expressed as the sum of natural response and the forced response.

  Complete solution: $y = y^{(n)} + y^{(f)}$

  $y^{(n)}$ = natural response, $y^{(f)}$ = forced response

  - One associated only with the initial conditions, i.e. $y^{(n)}$
  - The other due only to the input signal, i.e. $y^{(f)}$.

  - The sum of the natural response in Example 2.24 and the forced response in Example 2.27 is equal to the complete system response determined in Example 2.22

Complete Solution vs. Impulse Response

- Ideally, the complete solution cannot be used to find the impulse response of the LTI system directly.
- There is no provision for initial conditions when one is using the impulse response
  - Recall that $\boxed{h[n]=s[n]-s[n-1]}$ and $\boxed{h(t)=\frac{d}{dt}s(t)}$
  - It applies only to systems that are initially at rest.
- Complete solution is more flexible
  - It applies to systems either at rest or with nonzero initial conditions

Characteristics of Systems Described by Differential/Difference Equations

- The forced response is linear with respect to the input
- The natural response is linear with respect to the I. C.s
- The forced is time invariant
  - Since the system is initially at rest, a time shift in the input results in a time shift in the output.
- The complete solution, related to the initial conditions, is generally not time invariant
  - The initial conditions will not shift with a time shift of the input
- The forced response is causal
  - Again, since the system is initially at rest, the output does not begin prior to the time at which the input is applied to the system
- The roots of the characteristic equation afford considerable information about the (LTI) system behavior
  - (BIBO) stability and Response time

1. Discrete-time case: $\left|r_i^n\right|$ bounded $\Rightarrow$ $\boxed{\left|r_i\right|<1, \quad \text{for all } i}$

2. Continuous-time case: $\left|e^{r_i t}\right|$ bounded $\Rightarrow$ $\boxed{\Re e\{r_i\}<0}$