Chapter 3: Fourier Representation of Signals and LTI Systems

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Outline

- Introduction
- Complex Sinusoids and Frequency Response
- Fourier Representations for Four Classes of Signals
- Discrete-time Periodic Signals *Fourier Series*
- Continuous-time Periodic Signals
- Discrete-time Nonperiodic Signals *Fourier Transform*
- Continuous-time Nonperiodic Signals
- Properties of Fourier representations
- Linearity and Symmetry Properties
- Convolution Property

Reviews of Fourier Representations

Table 3.2 The Four Fourier Representations

Time Domain	**Periodic** (*t*, *n*)	**Non-periodic** (*t*, *n*)	
Continuous (*t*)	**Fourier Series** $x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_0 t}$ $X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$ $x(t)$ has period *T*, $\omega_o=2\pi/T$.	**Fourier Transform** $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$ $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$	*Nonperiodic* $(k,\ \omega)$
Discrete [*n*]	**Discrete-Time Fourier Series** $x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n}$ $X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{jk\Omega_0 n}$ *x*[*n*] and *X*[*k*] have period *N*, $\Omega_o=2\pi/N$.	**Discrete-Time Fourier Transform** $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\Omega})e^{j\Omega n}d\Omega$ $X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π .	*Periodic* $(k,\ \Omega)$
	Discrete [*k*]	*Continuous* $(\omega,\ \Omega)$	*Frequency Domain*

The four Fourier representations are all based on complex sinusoids

Periodicity Properties of Fourier Representations

Table 3.3 Periodicity Properties of Fourier Representations

Time-Domain Property	Frequency-Domain Property
Continuous	*Nonperiodic*
Discrete	*Periodic*
Periodic	*Discrete*
Nonperiodic	*Continuous*

- Periodic time signals have discrete frequency-domain representations, while nonperiodic time signals have continuous frequency-domain ones.
- In general, representations that are continuous/discrete in one domain are nonperiodic/periodic in the other domain.

Linearity Property of Fourier Representations

$$z(t) = ax(t) + by(t) \quad \xleftrightarrow{FT} \quad Z(j\omega) = aX(j\omega) + bY(j\omega)$$

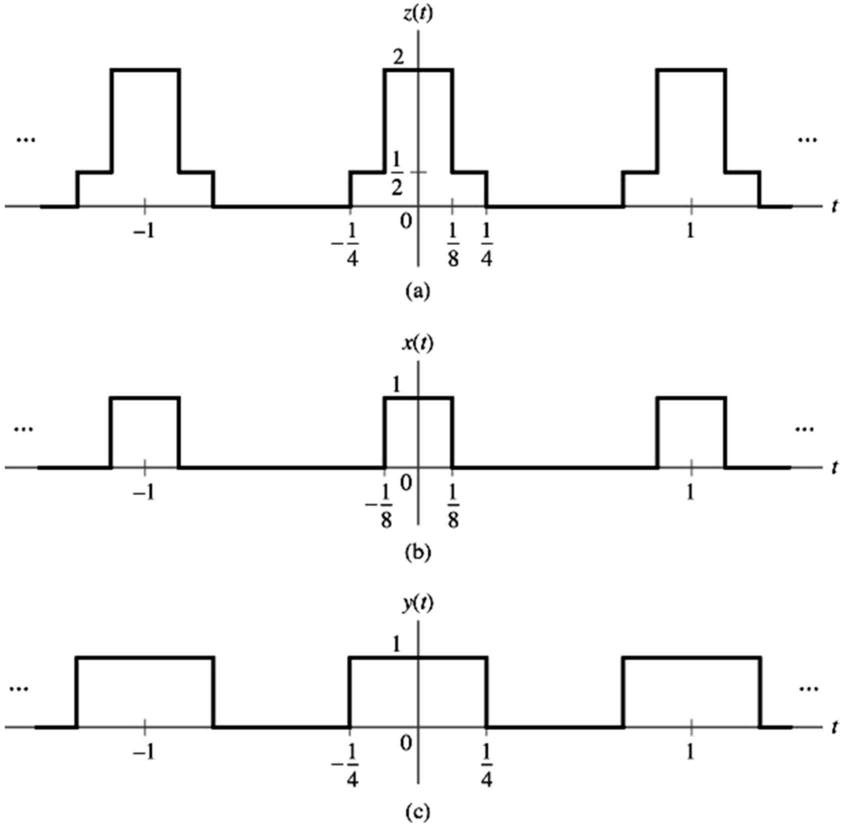
$$z(t) = ax(t) + by(t) \quad \xleftrightarrow{FS;\omega_o} \quad Z[k] = aX[k] + bY[k]$$

$$z[n] = ax[n] + by[n] \quad \xleftrightarrow{DTFT} \quad Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \quad \xleftrightarrow{DTFS;\Omega_o} \quad Z[k] = aX[k] + bY[k]$$

- Uppercase symbols denote the Fourier representation of the corresponding lowercase ones
- In case of FS and DTFS, the (two) signals being summed are assumed to have the same fundamental period

Example 3.30



$z(t) = (3/2)x(t) + (1/2)y(t)$

$$x(t) \xleftrightarrow{FS;2\pi} X[k] = \left(1/(k\pi)\right)\sin(k\pi/4)$$

$$y(t) \xleftrightarrow{FS;2\pi} Y[k] = \left(1/(k\pi)\right)\sin(k\pi/2)$$

$$z(t) \xleftrightarrow{FS;2\pi} Z[k] = \left(3/(2k\pi)\right)\sin(k\pi/2) + \left(1/(2k\pi)\right)\sin(k\pi/2)$$

Symmetry Property for Real-Valued $x(t)$

- For a real-valued signal $x(t)$, we have $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
- Consider the complex-conjugate of $X(j\omega)$:

$$X^*(j\omega)=\left(\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)^*$$

$$=\int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt=\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt$$

$$=\int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt=X(-j\omega)$$

$$\Rightarrow \quad X^*(j\omega)=X(-j\omega)$$

- For a real-valued $x(t)$, $X(j\omega)$ is complex-conjugate symmetric
- Another representation:

$$X(j\omega)=\mathrm{Re}\{X(j\omega)\}+j\,\mathrm{Im}\{X(j\omega)\}$$

$$X^*(j\omega)=\mathrm{Re}\{X(j\omega)\}-j\,\mathrm{Im}\{X(j\omega)\}$$

$$X(-j\omega)=\mathrm{Re}\{X(-j\omega)\}+j\,\mathrm{Im}\{X(-j\omega)\}$$

$\Rightarrow$

even

$$\mathrm{Re}\{X(-j\omega)\}=\mathrm{Re}\{X(j\omega)\}$$

$$\mathrm{Im}\{X(-j\omega)\}=-\mathrm{Im}\{X(j\omega)\}$$

odd

Symmetry Property for Imaginary-Valued *x*(*t*)

- For a imaginary-valued signal *x*(*t*), we have $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$
- Consider the complex-conjugate of $X(j\omega)$:

$$X^*(j\omega)=\left(\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right)^*$$

$$=\int_{-\infty}^{\infty}x^*(t)e^{j\omega t}dt=\int_{-\infty}^{\infty}-x(t)e^{j\omega t}dt$$

$$=-\int_{-\infty}^{\infty}x(t)e^{-j(-\omega)t}dt=-X(-j\omega)$$

$$\Rightarrow \quad X^*(j\omega)=-X(-j\omega)$$

- For a pure imaginary *x*(*t*), $X(j\omega)$ is conjugate anti-symmetric
- Another representation:

$$X(j\omega)=\mathrm{Re}\{X(j\omega)\}+j\,\mathrm{Im}\{X(j\omega)\}$$

$$X^*(j\omega)=\mathrm{Re}\{X(j\omega)\}-j\,\mathrm{Im}\{X(j\omega)\}$$

$$-X(-j\omega)=-\mathrm{Re}\{X(-j\omega)\}-j\,\mathrm{Im}\{X(-j\omega)\}$$

$$\Rightarrow \quad \mathrm{Re}\{X(-j\omega)\}=-\mathrm{Re}\{X(-j\omega)\} \quad \text{(odd)}$$

$$\mathrm{Im}\{X(-j\omega)\}=\mathrm{Im}\{X(j\omega)\} \quad \text{(even)}$$

Symmetry Properties of Fourier Representations

Table 3.4 Symmetry Properties for Fourier Representation of Real- and Imaginary-Valued Signals

Representation	Real-Valued Time Signals	Imaginary-Valued Time Signals
FT	$X^*(j\omega)=X(-j\omega)$	$X^*(j\omega)=-X(-j\omega)$
FS	$X^*[k]=X[-k]$	$X^*[k]=-X[-k]$
DTFT	$X^*(e^{j\Omega})=X(e^{-j\Omega})$	$X^*(e^{j\Omega})=-X(e^{-j\Omega})$
DTFS	$X^*[k]=X[-k]$	$X^*[k]=-X[-k]$

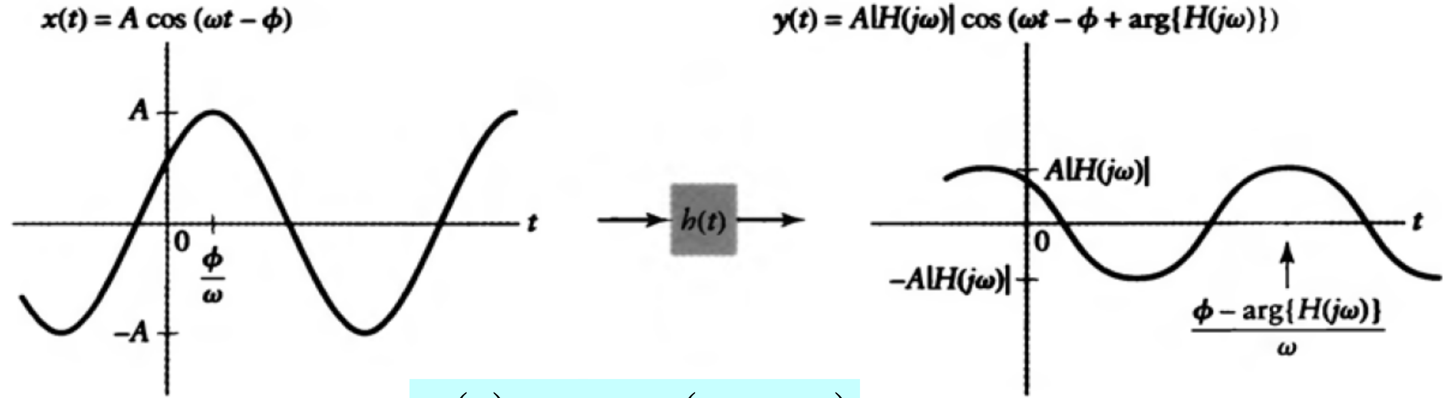
- Note that for the periodic signal with period N, $X[-k] = X[N-k]$
- If $x(t)$ is real and even, then $\boldsymbol{X^*(j\omega)= X(j\omega)}$. That is, $\boldsymbol{X(j\omega)}$ **is real**

➔ A real and even signal has a real and even frequency representation

- If $x(t)$ is real and even, then $\boldsymbol{X^*(j\omega)= -X(j\omega)}$. That is, $\boldsymbol{X(j\omega)}$ **is imaginary**

➔ A real and odd signal has a imaginary and odd frequency representation

IO-Relationship in Real-Valued LTI System



1. The real-valued input signal $x(t) = A\cos(\omega t - \phi)$

   Rewrite the input signal : $x(t) = (A/2)e^{j(\omega t - \phi)} + (A/2)e^{-j(\omega t - \phi)}$ Two eigenfunctions $e^{j\omega t}$, $e^{-j\omega t}$

2. The real-valued impulse response of LTI system: $h(t)$.
3. Applied the linear property of the LTI sytem to obtain the output signal:

$$y(t) = \tfrac{A}{2}e^{-j\phi}H(j\omega)e^{j\omega t} + \tfrac{A}{2}e^{j\phi}H(-j\omega)e^{-j\omega t}$$

$$\Rightarrow y(t) = |H(j\omega)|(A/2)e^{j(\omega t - \phi + \arg\{H(j\omega)\})} + |H(-j\omega)|(A/2)e^{-j(\omega t - \phi - \arg\{H(j\omega)\})}$$

Exploiting the symmetry conditions: $|H(j\omega)| = |H(-j\omega)|$ $\arg\{H(j\omega)\} = -\arg\{H(-j\omega)\}$

$$\Rightarrow y(t) = |H(j\omega)|A\cos(\omega t - \phi + \arg\{H(j\omega)\})$$

Convolution of Nonperiodic Signals

- The convolution property is a consequence of complex sinusoids being eigenfunctions of LTI system

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

Since $x(t-\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega(t-\tau)}d\omega$

$$\Rightarrow y(t)=\int_{-\infty}^{\infty}h(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau\right]X(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

$$\Rightarrow y(t)=h(t)*x(t)\xleftrightarrow{FT}Y(j\omega)=X(j\omega)H(j\omega)$$

$$y[n]=x[n]*h[n]\xleftrightarrow{DTFT}Y(e^{j\Omega})=X(e^{j\Omega})H(e^{j\Omega})$$

- The convolution in time-domain corresponds to the multiplication in frequency-domain

Example 3.31

Let $x(t)=(1/(\pi t))\sin(\pi t)$ be the input to a system with impulse response $h(t)=(1/(\pi t))\sin(2\pi t)$. Find the output $y(t) = x(t) * h(t)$.

<Sol.>

From Example 3.26, we have

$$x(t) \xleftrightarrow{FT} X(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases} \qquad h(t) \xleftrightarrow{FT} H(j\omega) = \begin{cases} 1, & |\omega| < 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$$

Since $y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$

$$\Rightarrow Y(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases} \qquad \Rightarrow \quad y(t) = (1/(\pi t))\sin(\pi t)$$

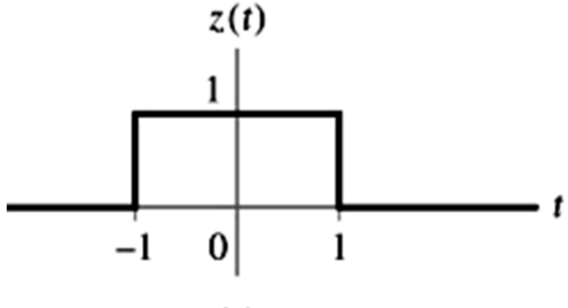
Example 3.32

Use the convolution property to find $x(t)$, where $x(t) \xleftrightarrow{FT} X(j\omega) = \frac{4}{\omega^2}\sin^2(\omega)$

<Sol.>

1. Write $X(j\omega) = Z(j\omega)\,Z(j\omega)$, where $Z(j\omega) = \frac{2}{\omega}\sin(\omega)$

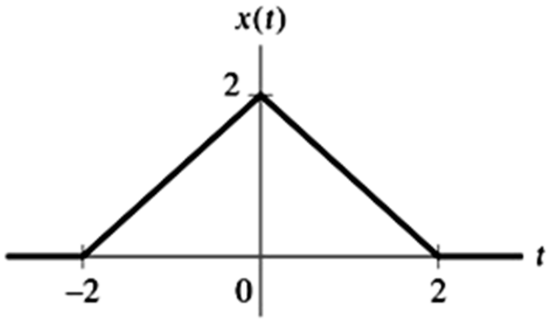
$$\Rightarrow z(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$



(a)

2. Apply the convolution property, we have $x(t) = z(t) * z(t)$

Hence



(b)

Filtering

- The multiplication in frequency domain, i.e. $X(j\omega)H(j\omega)$, gives rise to the "filtering"
- "Filtering" implies that some frequency components of the input signal are eliminated (*stopband*) while other are passed by the system (*passband*)
- Ideal filters:

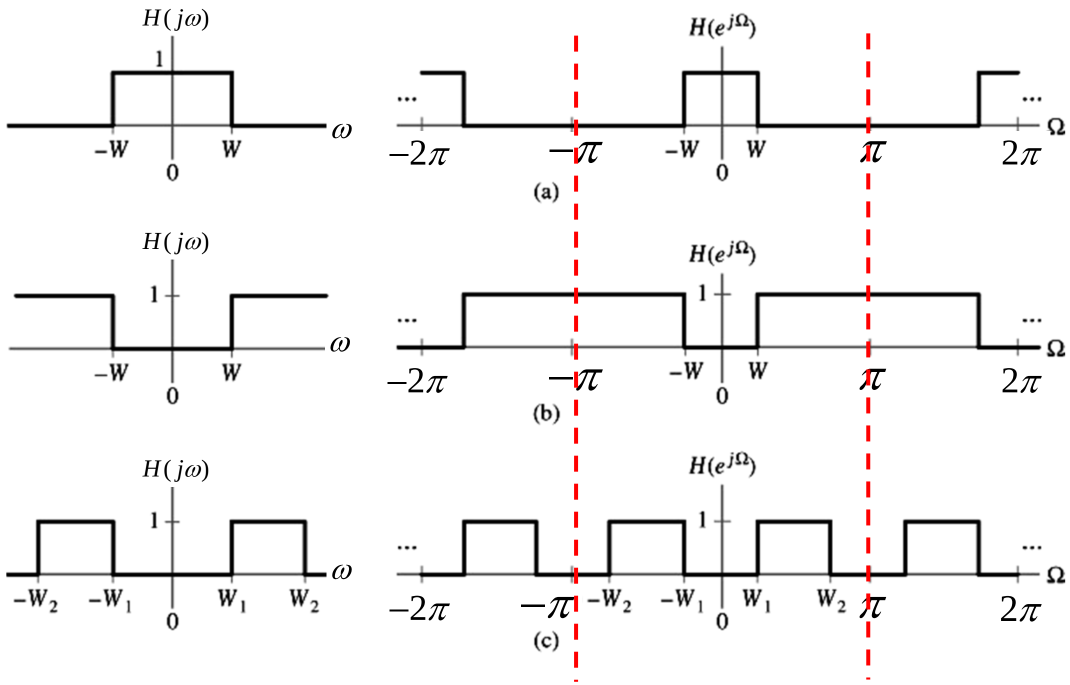


Figure 3.53
Frequency response of ideal continuous- and discrete-time filters:
(a) Low-pass filters.
(b) High-pass filters
(c) Band-pass filters.

The frequency response of discrete-time filters is based on its characteristic in the range $-\pi < \Omega \leq \pi$ because it is 2π-periodic

The Magnitude Response of Filters

- The magnitude response of the filter is defined by

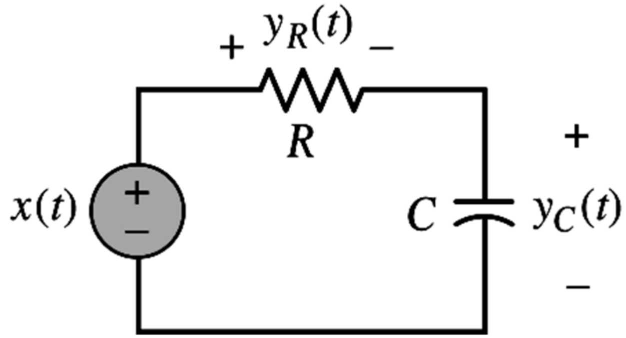
$$20\log\left|H\left(j\omega\right)\right| \quad \text{or} \quad 20\log\left|H\left(e^{j\Omega}\right)\right| \quad \text{[dB]}$$

  - Being described in units of decibels (or dB)
  - The unit gain is corresponding to 0 dB.
  - The passband of the filter is normally closed to 0 dB
  - The edge of the passband is usually defined by the frequencies for which the response is -3 dB (corresponding to a magnitude response of 1/√2)
  - Note that $\left|Y\left(j\omega\right)\right|^2 = \left|H\left(j\omega\right)\right|^2\left|X\left(j\omega\right)\right|^2$, -3 dB points correspond to frequencies at which the filter passes only half of the input power
  - -3 dB points are usually termed as the cutoff frequencies of the filter

Example 3.33 RC Circuit Filtering

For the *RC* circuit, the impulse response for the case where $y_C(t)$ is the output is given by

$$h_C(t) = \frac{1}{RC} e^{-t/RC} u(t).$$ Since $y_R(t) = x(t) - y_C(t)$,



the impulse response for the case where $y_R(t)$ is the output is given by

$$h_R(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} u(t)$$

Plot the magnitude responses of both systems on a linear scale and in dB, and characterize the filtering properties of the systems.

<Sol.>

The frequency response corresponding to $h_C(t)$: $H_C(j\omega) = \dfrac{1}{1 + j\omega RC}$

Hence, $H_R(j\omega) = 1 - H_C(j\omega) = \dfrac{j\omega RC}{1 + j\omega RC}$

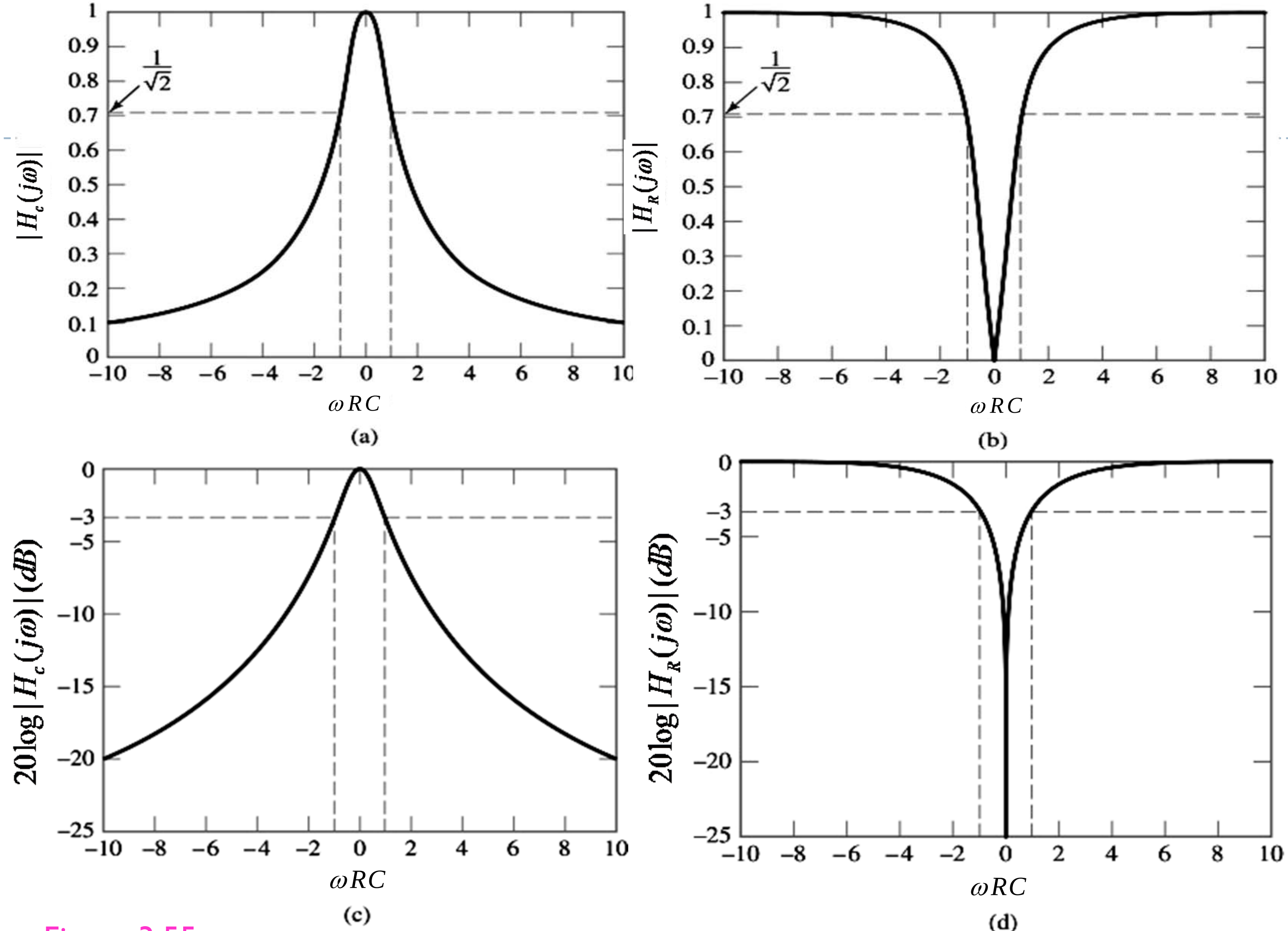


Figure 3.55

(a)-(b) Frequency response of the system corresponding to $y_C(t)$ and $y_R(t)$, linear scale.

(c)-(d) Frequency response of the system corresponding to $y_C(t)$ and $y_R(t)$, dB scale.

Frequency Response of LTI Systems

- From $y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$,
  - the frequency response of a system can be expressed as the ratio of the FT or DTFT of the output to that of the input.
  - If the input spectrum is nonzero at all frequencies, the frequency response of a system may be determined from the input and output spectra
    - Continuous-time system, $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$
    - Discrete-time system, $H(e^{j\Omega}) = \dfrac{Y(e^{j\Omega})}{X(e^{j\Omega})}$
  - .

Example 3.34

The output of an LTI system in response to an input $x(t)=e^{-2t}u(t)$ is $y(t)=e^{-t}u(t)$. Find the frequency response and the impulse response of this system.

<Sol.>

1. First find the FT of $x(t)$ and $y(t)$: $X(j\omega)=\frac{1}{j\omega+2}$ $Y(j\omega)=\frac{1}{j\omega+1}$

2. Then, the frequency response of the system is

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega+2}{j\omega+1}=1+\frac{1}{j\omega+1}$$

3. The impulse response of the system is the inverse FT of $H(j\omega)$:

$$\Rightarrow \quad h(t)=\delta(t)+e^{-t}u(t)$$

Recovery or Equalizer (Inverse Frequency Response)

- Recover the input of the system from the output
  - Continuous-Time
    - $X(j\omega) = H^{\text{inv}}(j\omega)Y(j\omega)$, where $H^{\text{inv}}(j\omega) = 1/H(j\omega)$
  - Discrete-Time
    - $X(e^{j\Omega}) = H^{\text{inv}}(e^{j\Omega})Y(e^{j\Omega})$, where $H^{\text{inv}}(e^{j\Omega}) = 1/H(e^{j\Omega})$
  - An inverse system is also known as an equalizer, and the process of recovering the input from the output is known as equalization.
    - **Causality** restrictions make it difficult to build an exact inverse system. (Time delay in a system need an equalizer to introduce a time advance)

Usually, equalizer is a noncausal system and cannot be implemented in real-time ! ⟹ Approximation!

E.g. Compensate for all but time delay

Example 3.35

Consider the multipath communication channel, where a (distorted) received signal $y[n]$ is expressed in terms of a transmitted signal $x[n]$ as

$$y[n] = x[n] + ax[n-1], \qquad |a| < 1$$

Use the convolution property to find the impulse response of an inverse system that will recover $x[n]$ from $y[n]$.

<Sol.>

1. Take DTFT on both side of $y[n] = x[n] + ax[n-1], \qquad |a| < 1$

$$\sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} ax[n-1]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} + ae^{-j\Omega} \sum_{n=-\infty}^{\infty} x[n-1]e^{-j\Omega(n-1)}$$

$$\Rightarrow \quad Y(e^{j\Omega}) = X(e^{j\Omega}) + ae^{-j\Omega}X(e^{j\Omega}) \quad \Rightarrow \quad H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = 1 + ae^{-j\Omega}$$

2. The frequency response of the inverse system is then obtained as $H^{\text{inv}}\left(e^{j\Omega}\right) = 1/H(e^{j\Omega})$

$$\Rightarrow \quad H^{\text{inv}}\left(e^{j\Omega}\right) = \frac{1}{1 + ae^{-j\Omega}} \quad \Rightarrow \quad h^{\text{inv}}[n] = (-a)^n u[n]$$

Convolution of CT Periodic Signals

- Recall that if the impulse response $h(t)$ of the LTI system is periodic, then the system is unstable. (since $h(t)$ is not absolutely integrable)

➔the convolution of periodic signals does not occur naturally

- Convolution of periodic signals often occurs in the context of signal analysis and manipulation
- Definition:

The periodic convolution of two CT signals $x(t)$ and $z(t)$, each with period T, is defined as the following integral over a single period T:

$$y(t) = x(t) \otimes z(t) = \int_0^T x(\tau) z(t-\tau) d\tau$$ also with period T

- Take FS on the both sides with $\omega_0 = 2\pi/T$, we have

$$\frac{1}{T}\int_0^T y(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_0^T \left(\int_0^T x(\tau) z(t-\tau) d\tau \right) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_0^T \left(\int_0^T z(t-\tau) e^{-jk\omega_0 (t-\tau)} dt \right) x(\tau) e^{-jk\omega_0 \tau} d\tau$$

➡ $Y[k] = TZ[k]X[k]$ ➡ $y(t) = x(t) \otimes z(t) \xleftrightarrow{FS;\frac{2\pi}{T}} Y[k] = TX[k]Z[k]$

Convolution of DT Periodic Signals

- Definition:

The periodic convolution of two DT signals *x*[*n*] and *z*[*n*], each with period *N*, is defined as the summation of length- *N*:

$$y[n] = x[n] \otimes z[n] = \sum_{k=0}^{N-1} x[k] z[n-k]$$ also with period *N*

- Take DTFS on the both sides with $\Omega_0 = 2\pi/N$, we have $Y[k] = NZ[k]X[k]$

$$y[n] = x[n] \otimes z[n] \xleftrightarrow{DTFS;\frac{2\pi}{N}} Y[k] = NX[k]Z[k]$$

◆ **Convolution in Time-Domain ↔ Multiplication in Frequency-Domain**

Convolution Property Summary

TABLE 3.5 Convolution Properties.

$$x(t) * z(t) \xleftrightarrow{FT} X(j\omega)Z(j\omega)$$

$$x(t) \circledast z(t) \xleftrightarrow{FS;\omega_o} TX[k]Z[k]$$

$$x[n] * z[n] \xleftrightarrow{DTFT} X(e^{j\Omega})Z(e^{j\Omega})$$

$$x[n] \circledast z[n] \xleftrightarrow{DTFS;\Omega_o} NX[k]Z[k]$$

Outline

- Differentiation and Integration Properties
- Time- and Frequency-Shift Properties
- Finding Inverse Fourier Transforms
- Multiplication Property
- Scaling Properties
- Parseval Relationships
- Time-Bandwidth Product
- Duality

Differentiation and Integration Properties

- Recall that differentiation and integration are operations that apply to continuous (time or frequency) functions.
- We consider CT signals in time-domain, or FT/DTFT in frequency-domain
- 1. Differentiation in Time

  Consider a **nonperiodic signal** $x(t)$ and its FT, $X(j\omega)$, representation, i.e.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \xrightarrow[\text{both sides w.r.t } t]{\text{Differentiating}} \frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,j\omega e^{j\omega t}\,d\omega$$

$$\Rightarrow \quad \frac{d}{dt}x(t) \overset{FT}{\longleftrightarrow} j\omega X(j\omega)$$

◆ Differentiation of $x(t)$ in time-domain ↔ $(j\omega) \times X(j\omega)$ in frequency-domain

- Differentiation accentuates the high-frequency components of the signal
- Differentiation destroys any dc component (i.e. ω=0) of the differentiated signal

Example 3.37

$$\frac{d}{dt}\left(e^{-at}u(t)\right) \xleftrightarrow{FT} \frac{j\omega}{a+j\omega}$$

Verify this result by differentiating and taking the FT of the result.

<Sol.>

1. Since $\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t) = -ae^{-at}u(t) + \delta(t)$

2. Taking the FT of each term and using linearity, we have

$$\frac{d}{dt}\left(e^{-at}u(t)\right) \xleftrightarrow{FT} \frac{-a}{a+j\omega} + 1$$

Frequency Response from IO-Relationship

- Given the following IO-relationship of the LTI system:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

- Take FT of both sides, we have

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) \Rightarrow \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

$$\Rightarrow H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

- The frequency response is the system's steady-state response to a sinusoid.
- The frequency response cannot represent initial conditions (it can only describes a system that is in a steady-state condition)

Differentiation and Integration Properties

- 1. Differentiation in Time

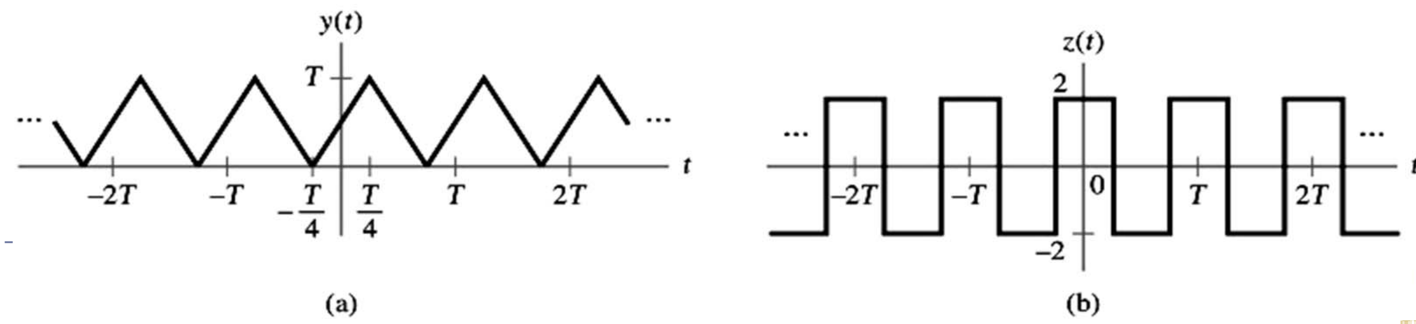
  Consider a **periodic signal** $x(t)$ and its FS, $X[k]$, representation, i.e.

$$x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \xrightarrow[\text{both sides w.r.t } t]{\text{Differentiating}} \frac{d}{dt}x(t)=\sum_{k=-\infty}^{\infty} X[k]jk\omega_0 e^{jk\omega_0 t}$$

$$\Rightarrow \quad \frac{d}{dt}x(t) \xleftrightarrow{FS;\ \omega_0} jk\omega_0 X[k]$$

◆ Differentiation of $x(t)$ in time-domain ↔ $(jk\omega_0) \times X[k]$ in frequency-domain

- Differentiation destroys the time-averaged valued (i.e. the dc component) of the differentiated signal; hence, the FS coefficient for k=0 is zero
- Example 3.39 $z(t)=\frac{d}{dt}y(t)$



Differentiation(in Frequency) Property

- 2. Differentiation in Frequency

  Beginning with the FT of the signal $x(t)$:

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \xrightarrow[\text{both sides w.r.t } \omega]{\text{Differentiating}} \frac{d}{d\omega}X(j\omega)=\int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t}dt$$

$$\Rightarrow \quad -jtx(t) \overset{FT}{\longleftrightarrow} \frac{d}{d\omega}X(j\omega)$$

◆ Differentiation of $X(j\omega)$ in frequency-domain ↔ $(-jt) \times x(t)$ in time-domain

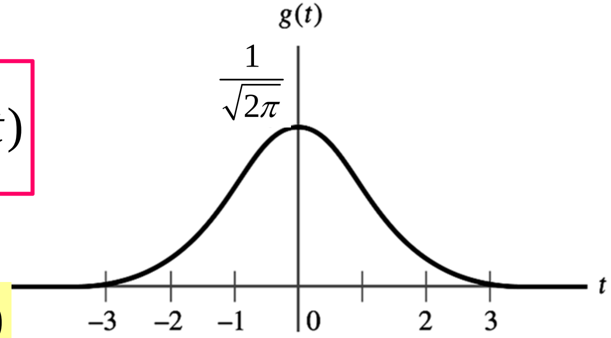
Consider the DTFT of the signal $x[n]$:

$$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \xrightarrow[\text{both sides w.r.t } \Omega]{\text{Differentiating}} \frac{d}{d\Omega}X(e^{j\Omega})=\sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\Omega n}$$

$$\Rightarrow \quad jnx[n] \overset{DTFT}{\longleftrightarrow} \frac{d}{d\Omega}X(e^{j\Omega})$$

Example 3.40 FT of a Gaussian Pulse

Use the differentiation-in-time and differentiation-in-frequency properties to determine the FT of the Gaussian pulse, $g(t) = \left(1/\sqrt{2\pi}\right)e^{-t^2/2}$



<Sol.>

1. Differentiation-in-time: $\frac{d}{dt}g(t) = (-t/\sqrt{2\pi})e^{-t^2/2} = -tg(t)$

2. Differentiation-in-time property:

$$\frac{d}{dt}g(t) \xleftrightarrow{FT} j\omega G(j\omega) \Rightarrow -tg(t) \xleftrightarrow{FT} j\omega G(j\omega)$$

3. Differentiation-in-frequency property:

$$-jtg(t) \xleftrightarrow{FT} \frac{d}{d\omega}G(j\omega) \Rightarrow -jtg(t) \xleftrightarrow{FT} \frac{d}{d\omega}G(j\omega) \Rightarrow -tg(t) \xleftrightarrow{FT} \frac{1}{j}\frac{d}{d\omega}G(j\omega)$$

$$\Rightarrow \frac{1}{j}\frac{d}{d\omega}G(j\omega) = j\omega G(j\omega) \Rightarrow \frac{d}{d\omega}G(j\omega) + \omega G(j\omega) = 0$$

$$\Rightarrow G(j\omega) = ce^{-\omega^2/2}$$

4. The integration constant *c* is determined by $G(j0) = \int_{-\infty}^{\infty}\left(1/\sqrt{2\pi}\right)e^{-t^2/2}dt = 1$

$$\left(1/\sqrt{2\pi}\right)e^{-t^2/2} \xleftrightarrow{FT} e^{-\omega^2/2}$$

The FT of a Gaussian pulse is also a Gaussian pulse!

Integration Property

- 3. Integration

  In both FT and FS, we may integrate with respect to time.

  In both FT and DTFT, we may integrate with respect to frequency.

♣ Case for nonperiodic signal

Since $y(t)=\int_{-\infty}^{t} x(\tau)d\tau$ implies $\frac{d}{dt}y(t)=x(t)$ By differentiation property, we have

$$Y(j\omega)=\frac{1}{j\omega}X(j\omega)$$

➡ This relation is indeterminate at ω=0 (also implies that $X(j0)$=0)

This is true only to signals with a zero time-averaged value, i.e. $X(j0)$=0 .

Or, it is true for all ω except ω =0.

The value at ω =0 can be modified the equation by $Y(j\omega)=\frac{1}{j\omega}X(j\omega)+c\delta(\omega)$

The constant *c* depends on the average value of $x(t)$

General Form for Nonperiodic Signal

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{FT}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

First note that

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = x(t) * u(t)$$

$$\Rightarrow Y(j\omega) = X(j\omega)FT\{u(t)\}$$

We observe that $u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$, where $\operatorname{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$

Then, $\frac{d}{dt}u(t) = \frac{d}{dt}\left(\frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)\right) = \frac{1}{2}\frac{d}{dt}\operatorname{sgn}(t)$

$$\Rightarrow 2\delta(t) = \frac{d}{dt}\operatorname{sgn}(t) \Rightarrow 2 = j\omega FT\{\operatorname{sgn}(t)\} \Rightarrow FT\{\operatorname{sgn}(t)\} = \frac{2}{j\omega}$$

Thus, $FT\{u(t)\} = FT\left\{\frac{1}{2}\right\} + FT\left\{\frac{1}{2}\operatorname{sgn}(t)\right\} = \pi\delta(\omega) + \frac{1}{j\omega}$

Hence, $Y(j\omega) = X(j\omega)FT\{u(t)\} = \frac{X(j\omega)}{j\omega} + \pi\delta(\omega)X(j\omega)$

Differentiation and Integration Properties

- Summary

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS;\ \omega_0} jk\omega_0 X[k]$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

$$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X\left(e^{j\Omega}\right)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

Time- and Frequency-Shift Properties

- 1. Time-shift property

  Let $z(t) = x(t - t_0)$ be a time-shifted version of $x(t)$.

  Take FT of $z(t)$: $Z(j\omega) = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt$ Change variable by $\tau = t - t_0$:

$$Z(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)}d\tau = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = e^{-j\omega t_0}X(j\omega)$$

$$\Rightarrow \quad x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0}X(j\omega)$$

◆ Time-shifting by t_0 in time-domain ↔ Multiply by $e^{-j\omega t_0}$ in frequency-domain

◆ Note that the mag. response and phase response are

$$|Z(j\omega)| = |X(j\omega)| \quad \text{and} \quad \arg\{Z(j\omega)\} = \arg\{X(j\omega)\} - \omega_0 t$$

◆ unchanged the mag. response but introduces a phase shift

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0}X(j\omega)$$

$$x(t-t_0) \xleftrightarrow{FT;\ \omega_0} e^{-jk\omega_0 t_0}X(k)$$

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0}X(e^{j\Omega})$$

$$x[n-n_0] \xleftrightarrow{DTFS;\Omega_0} e^{-jk\Omega_0 n_0}X[k]$$

Example:
Frequency Response of LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Taking DTFTof both sides

of this equation

$$\sum_{k=0}^{N} a_k \left(e^{-j\Omega}\right)^k Y\left(e^{j\Omega}\right) = \sum_{k=0}^{M} b_k \left(e^{-j\Omega}\right)^k X\left(e^{j\Omega}\right)$$

$$\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{\sum_{k=0}^{M} b_k \left(e^{-j\Omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k}$$

Time- and Frequency-Shift Properties

- 2. Frequency-shift property

Suppose that: $x(t) \xleftrightarrow{FT} X(j\omega)$ Consider the frequency shift: $X(j(\omega-\gamma))$

By the definition of the inverse FT, we have

$$z(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega-\gamma))e^{j\omega t}d\omega$$ Change variable by η = ω − γ, we have

$$z(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\eta)e^{j(\eta+\gamma)t}d\eta = e^{j\gamma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\eta)e^{j\eta t}d\eta = e^{j\gamma t}x(t)$$

$$\Rightarrow \quad e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$

◆ Frequency-shift by γ in frequency-domain ↔ Multiply by $e^{-\gamma t}$ in time-domain

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$

$$e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\omega_o} X[k-k_o]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$

$$e^{jk_o\Omega_o n}x[n] \xleftrightarrow{DTFS;\Omega_o} X[k-k_o]$$

Example 3.42 and 3.43

Determine the FT of the complex sinusoidal pulse: $z(t) = \begin{cases} e^{j10t}, & |t| < \pi \\ 0, & |t| > \pi \end{cases}$

<Sol.>

Recall rectangular pulse $x(t) = \begin{cases} 1, & |t| < \pi \\ 0, & |t| > \pi \end{cases}$, then $x(t) \xleftrightarrow{FT} X(j\omega) = \frac{2}{\omega}\sin(\omega\pi)$

By Frequency-shift property

$$e^{j10t}x(t) \xleftrightarrow{FT} X(j(\omega-10)) \Rightarrow z(t) \xleftrightarrow{FT} \frac{2}{\omega-10}\sin((\omega-10)\pi)$$

Find the FT of the signal $x(t) = \frac{d}{dt}\left\{\left(e^{-3t}u(t)\right) * \left(e^{-t}u(t-2)\right)\right\}$

<Sol.>

Let $w(t) = e^{-3t}u(t) \longleftrightarrow W(j\omega) = \frac{1}{3+j\omega}$

and $v(t) = e^{-t}u(t-2) \Rightarrow v(t) = e^{-2}e^{-(t-2)}u(t-2) \longleftrightarrow V(j\omega) = e^{-2}\frac{e^{-j2\omega}}{1+j\omega}$

Then, $x(t) = \frac{d}{dt}\{w(t) * v(t)\} \longleftrightarrow X(j\omega) = j\omega\{W(j\omega)V(j\omega)\}$

$$\Rightarrow X(j\omega) = e^{-2}\frac{j\omega e^{-j2\omega}}{(1+j\omega)(3+j\omega)}$$

Outline

- Differentiation and Integration Properties
- Time- and Frequency-Shift Properties
- Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions
- Multiplication Property
- Scaling Properties
- Parseval Relationships
- Time-Bandwidth Product
- Duality

Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

- Consider a ratio of polynomial in $j\omega$:

$$X(j\omega)=\frac{b_M(j\omega)^M+\cdots+b_1(j\omega)+b_0}{(j\omega)^N+a_{N-1}(j\omega)^{N-1}+\cdots a_1(j\omega)+a_0}=\frac{B(j\omega)}{A(j\omega)}$$

Assume that $M < N$

If $M \geq N$, then we may use long division to express $X(j\omega)$ in the form

$$X(j\omega)=\sum_{k=0}^{M-N} f_k(j\omega^k)+\frac{\overline{B}(j\omega)}{A(j\omega)}$$

Partial-fraction expansion is applied to this term

Applying the differentiation property and the pair $\delta(t) \xleftrightarrow{FT} 1$ to these terms

Replacing $j\omega$ with a generic variable v, then we have $v^N+a_{N-1}v^{N-1}+\cdots+a_1v+a_0=0$ for the denominator $A(j\omega)$. Suppose that we have roots d_k, $k = 1, 2, \ldots, N$.

For $M < N$, we may the write

$$X(j\omega)=\frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\prod_{k=1}^{N}(j\omega-d_k)}$$

Assuming distinct roots d_k, $k = 1, 2, \ldots, N$, we may write

$$\Rightarrow \quad X(j\omega)=\sum_{k=1}^{N}\frac{C_k}{j\omega-d_k}$$

Inverse FT for Partial-Fraction Expansions

- Recall that $e^{dt}u(t) \xleftrightarrow{FT} \dfrac{1}{j\omega - d}$ **This pair is valid only for Re{*d*} < 0.** (if Re{*d*}<0, $e^{dt}u(t)$ is not absolutely integrable)

Assuming that the real part of each d_k, k = 1, 2, …, N, is negative, then

$$x(t) = \sum_{k=1}^{N} C_k e^{d_k t} u(t) \xleftrightarrow{FT} X(j\omega) = \sum_{k=1}^{N} \frac{C_k}{j\omega - d_k}$$

For the case of repeated roots, please refer Appendix B !!

- Similarly, $X(e^{j\Omega}) = \dfrac{\beta_M e^{-j\Omega M} + \cdots + \beta_1 e^{-j\Omega} + \beta_0}{\alpha_N e^{-j\Omega N} + \alpha_{N-1} e^{-j\Omega(N-1)} + \cdots + \alpha_1 e^{-j\Omega} + 1}$ Normalized to 1

Replace $e^{j\Omega}$ with the generic variable *v* and solve the roots of the polynomial

$$v^N + \alpha_1 v^{N-1} + \alpha_2 v^{N-2} + \cdots + \alpha_{N-1} v + \alpha_N = 0$$

- Recall that $(d_k)^n u[n] \xleftrightarrow{DTFT} \dfrac{1}{1 - d_k e^{-j\Omega}}$ Assuming that all the d_k are distinct and $|d_k| < 1$, then

$$x[n] = \sum_{k=1}^{N} C_k (d_k)^n u[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \sum_{k=1}^{N} \frac{C_k}{1 - d_k e^{-j\Omega}}$$

Example 3.44

Frequency response for the MEMS accelerometer is given by $H(j\omega)=\dfrac{1}{(j\omega)^2+\dfrac{\omega_n}{Q}(j\omega)+\omega_n^2}$

Find the impulse response for the MEMS accelerometer, assuming that ω_n = 10,000 rads/s, and (a) Q = 2/5, (b) Q = 1, and (c) Q = 200.

<Sol.>

Case (a): ω_n = 10,000 rads/s and Q = 2/5, then we have $(j\omega)^2+25000(j\omega)+(10000)^2=0$

The roots of the denominator polynomial are d_1 = – 20,000 and d_2 = – 5,0000.

$$H(j\omega)=\frac{1}{(j\omega)^2+25000(j\omega)+(10000)^2}=\frac{C_1}{j\omega+20000}+\frac{C_2}{j\omega+5000}$$

$$\Rightarrow C_1(j\omega+5000)\Big|_{j\omega=-20000}=1 \qquad C_1=\frac{-1}{15000}$$

$$C_2(j\omega+20000)\Big|_{j\omega=-5000}=1 \qquad C_2=\frac{1}{15000}$$

$$\Rightarrow H(j\omega)=\frac{-1/15000}{j\omega+20000}+\frac{1/15000}{j\omega+5000}$$

$$\Rightarrow h(t)=(1/15000)\left(e^{-5000t}-e^{-20000t}\right)u(t)$$

Example 3.45

Find the inverse DTFT of $X\left(e^{j\Omega}\right)=\dfrac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}$

<Sol.>

First solve the characteristic polynomial: $v^2+\frac{1}{6}v-\frac{1}{6}=0$

The roots of the polynomial are d_1 = – 1/2 and d_2 = 1/3.

Then

$$\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}=\frac{C_1}{1+\frac{1}{2}e^{-j\Omega}}+\frac{C_2}{1-\frac{1}{3}e^{-j\Omega}}$$

➡ C_1 = 4, C_2 = 1

➡ $$x[n]=4(-1/2)^n u[n]+(1/3)^n u[n]$$

Outline

- Differentiation and Integration Properties
- Time- and Frequency-Shift Properties
- Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions
- Multiplication Property
- Scaling Properties
- Parseval Relationships
- Time-Bandwidth Product
- Duality

Multiplication Property

- Case of nonperiodic continuous-time signals
- Consider two nonperiodic signals: $x(t)$ and $z(t)$. Let's $y(t) = x(t)z(t)$.

  Suppose that the FT representation of $x(t)$ and $z(t)$ are:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jv)e^{jvt}dv \quad \text{and} \quad z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\eta)e^{j\eta t}d\eta$$

Then,

$$y(t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(jv)Z(j\eta)e^{j(\eta+v)t}d\eta dv$$

1. Change the integral order. 2. Change variable by $\eta + v = \omega$

$$\Rightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jv)Z(j(\omega - v))dv\right]e^{j\omega t}d\omega$$

$X(j\omega) * Z(j\omega)$

$$\Rightarrow y(t) = x(t)z(t) \quad \xleftrightarrow{FT} \quad Y(j\omega) = \frac{1}{2\pi}X(j\omega) * Z(j\omega)$$

◆ Multiplication in time-domain ↔ Convolution in Frequency-Domain × ($1/2\pi$)

Multiplication Property

- Case of nonperiodic discrete-time signals
- Consider two nonperiodic signals: $x[n]$ and $z[n]$. Let's $y[n] = x[n]z[n]$.

  Suppose that the DTFT representation of $x[n]$ and $z[n]$ are:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega \quad \text{and} \quad z[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z(e^{j\Omega})e^{j\Omega n}d\Omega$$

2π-periodic 2π-periodic

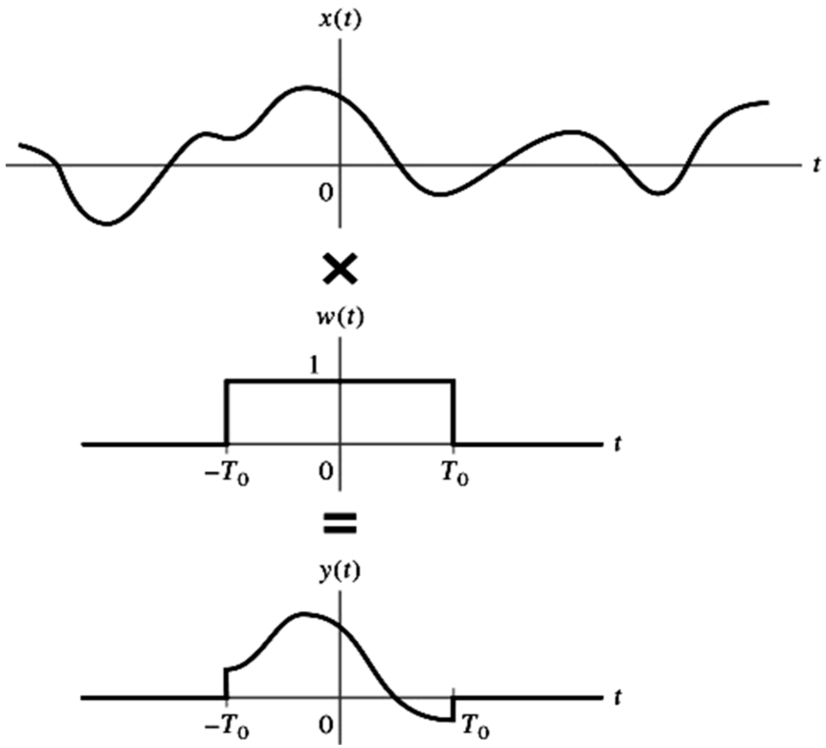
Then,

$$y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \otimes Z(e^{j\Omega})$$

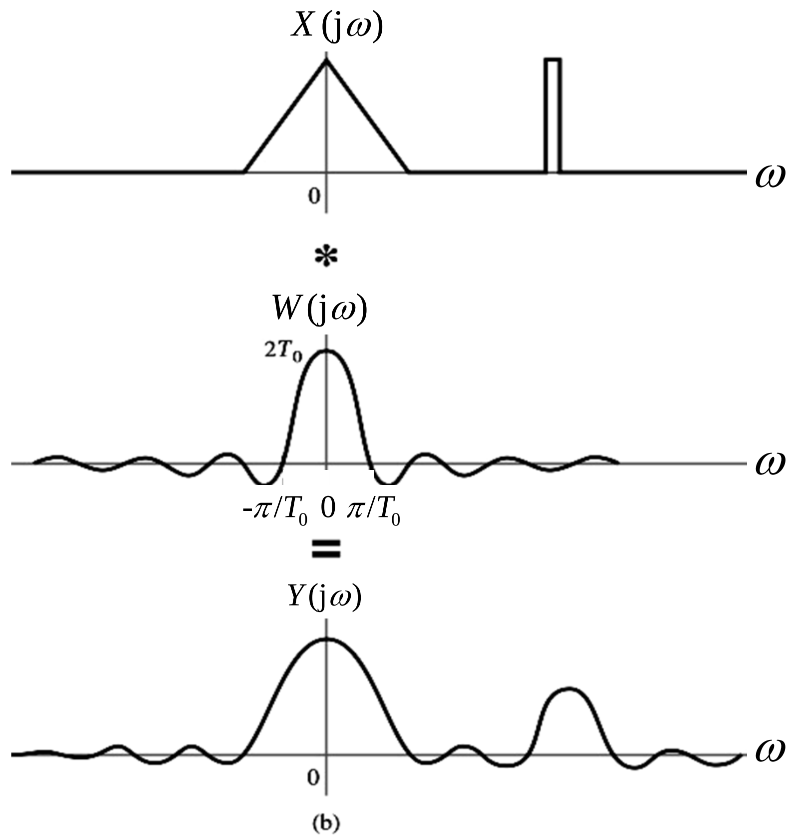
◆ Multiplication in time ↔ Periodic Convolution in Frequency × (1/2π)

Windowing Operation – Truncating a Signal

- Windowing: a signal passes through a window means only the signal within the window is visible. The other part is truncated.



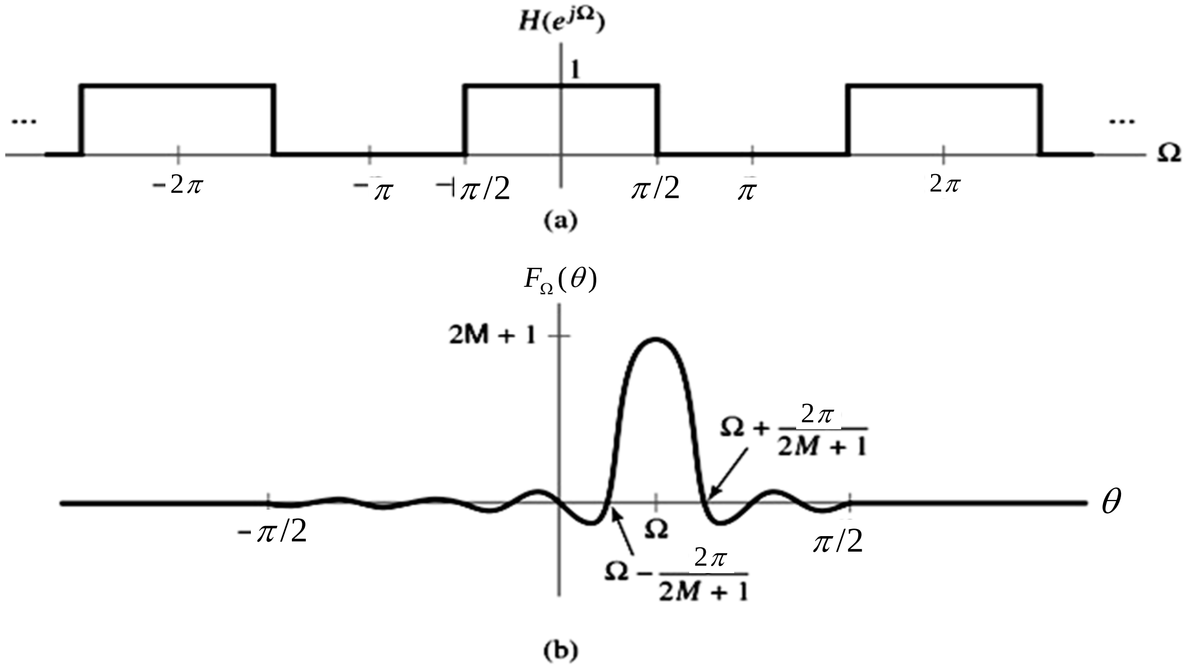
Truncated signal $y(t) = x(t)w(t)$



$$y(t) \overset{FT}{\longleftrightarrow} Y(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$$

Example 3.46 Windowing Effect (aka Gibbs Effect in Example 3.14)

The frequency response $H(e^{j\Omega})$ of an ideal discrete-time low-pass filter. Describe the frequency response of a system whose impulse response is truncated to the interval $-M \le n \le M$.



<Sol.>

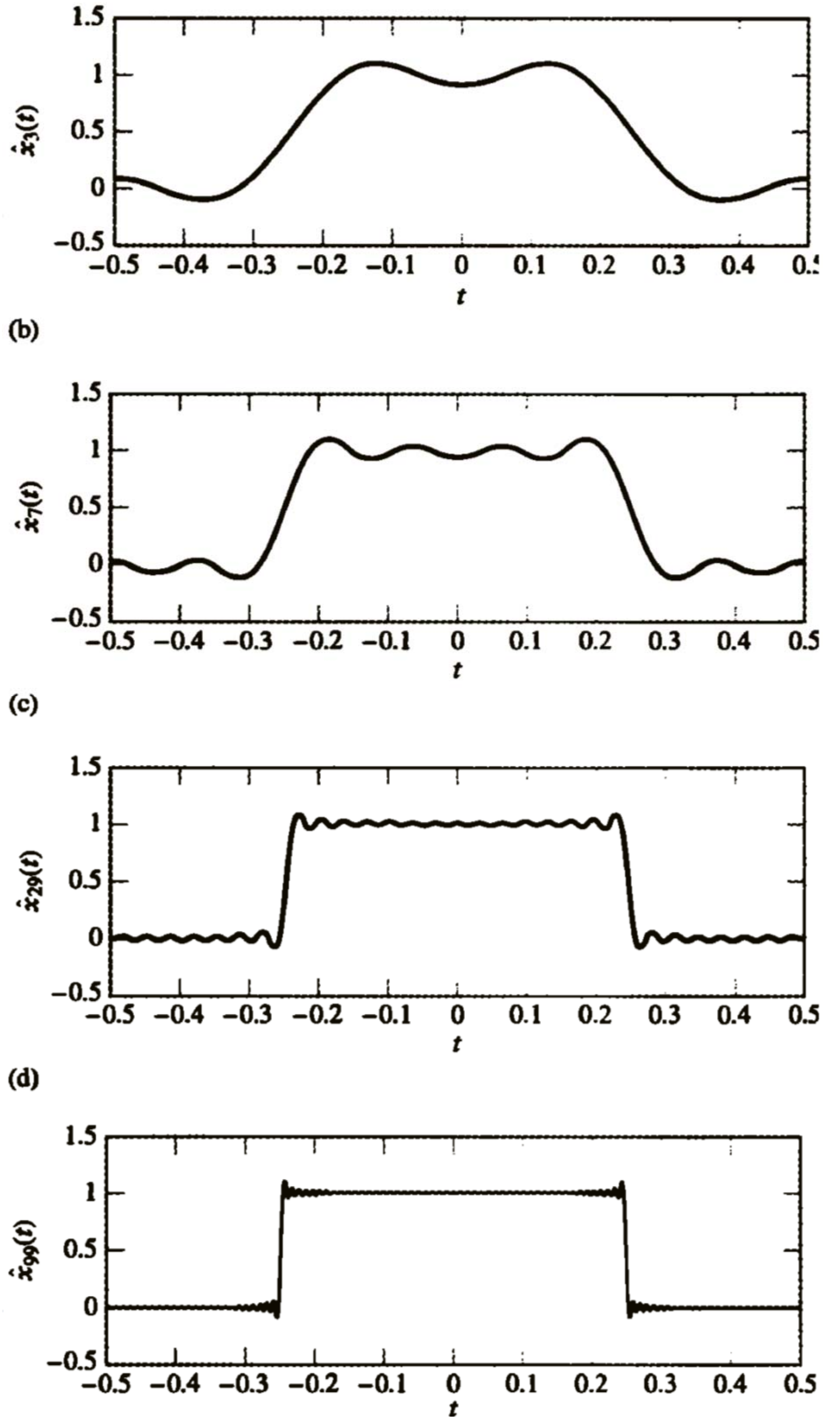
The ideal impulse response is just the inverse FT of $H(e^{j\Omega})$: $h[n] = \frac{1}{\pi n}\sin\left(\frac{\pi n}{2}\right)$

Infinite impulse response

$$h_t[n] = \begin{cases} h[n], & |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

$$h_t[n] = h[n]w[n]$$

$$w[n] = \begin{cases} 1, & |n| \le M \\ 0, & \text{otherwise} \end{cases}$$



Multiplication Property

- Case of periodic continuous-time signals

  Consider two periodic signals: $x(t)$ and $z(t)$. Let's $y(t) = x(t)z(t)$.

$$y(t) = x(t)z(t) \xleftrightarrow{FS;2\pi/T} Y[k] = X[k] * Z[k]$$

◆ Multiplication in time-domain ↔ Convolution in Frequency-Domain

This relationship is provided that $x(t)$ and $z(t)$ have a common period T

If their fundamental periods are not identical, T should be the LCM of each signal's fundamental period.

- Case of periodic continuous-time signals

  Consider two periodic signals: $x[n]$ and $z[n]$. Let's $y[n] = x[n]z[n]$.

$$y[n] = x[n]z[n] \xleftrightarrow{DTFS;2\pi/N} Y[k] = X[k] \otimes Z[k]$$

◆ Multiplication in time ↔ Periodic Convolution in Frequency

This relationship is provided that $x[n]$, $z[n]$, and $Y[k]$ have a common period N

Summary for Multiplication Property

$$x(t)z(t) \xleftrightarrow{FT} \frac{1}{2\pi} X(j\omega) * Z(j\omega)$$

$$x(t)z(t) \xleftrightarrow{FS;\omega_o} X[k] * Z[k]$$

$$x[n]z[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

$$x[n]z[n] \xleftrightarrow{DTFS;\Omega_o} X[k] \circledast Z[k]$$

Scaling Property

- Case of scaling the continuous-time signal

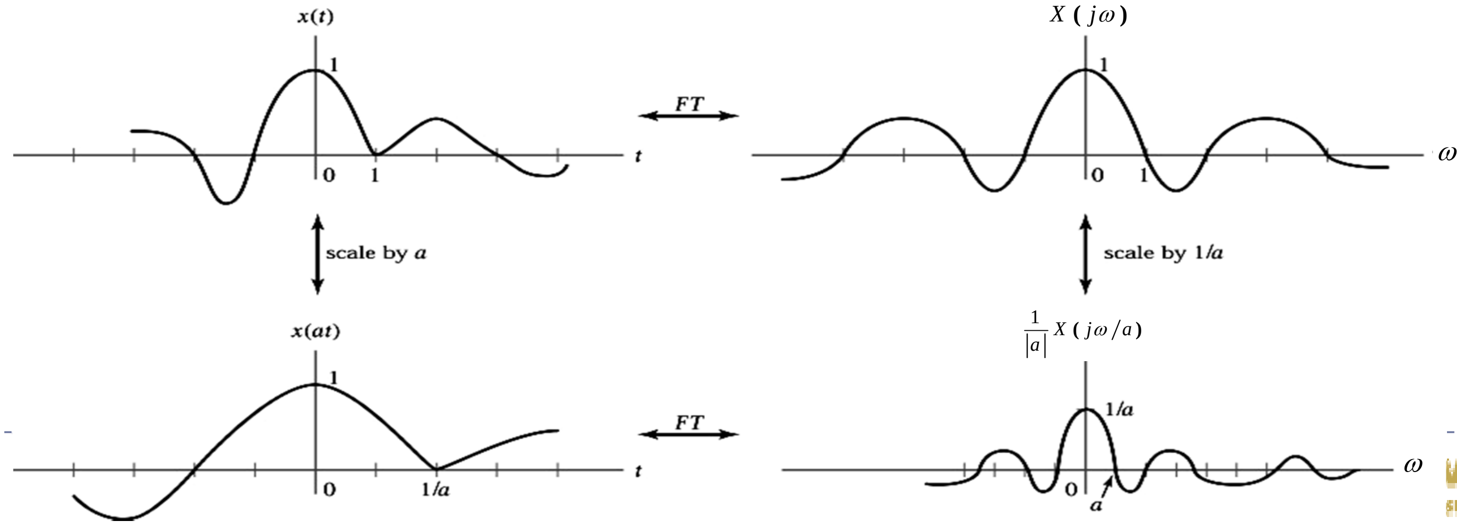
  Let z(t) = x(at). Consider the FT of z(t): $Z(j\omega)=\int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$

  Changing variable by letting $\tau = at$

$$\Rightarrow Z(j\omega)=\begin{cases}(1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a>0\\ (1/a)\int_{\infty}^{-\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a<0\end{cases} \Rightarrow Z(j\omega)=(1/|a|)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau,$$

$$\Rightarrow z(t)=x(at) \overset{FT}{\longleftrightarrow} (1/|a|)X(j\omega/a).$$

◆ Scaling in time-domain ↔ Inverse scaling in frequency-domain



Example 3.49

Find $x(t)$ if $X(j\omega) = j\dfrac{d}{d\omega}\left\{\dfrac{e^{j2\omega}}{1+j(\omega/3)}\right\}$

<Sol.>

Differentiation in frequency, time shifting, and scaling property are used to solve the problem.

First note the FT-pair: $s(t) = e^{-t}u(t) \xleftrightarrow{FT} S(j\omega) = \dfrac{1}{1+j\omega}$

Then, $X(j\omega) = j\dfrac{d}{d\omega}\left\{e^{j2\omega}S(j\omega/3)\right\}$

We apply the innermost property first: we scale, then time shift, and lastly differentiate.

Define $Y(j\omega) = S(j\omega/3)$. ➡ $y(t) = 3s(3t) = 3e^{-3t}u(3t) = 3e^{-3t}u(t)$

Define $W(j\omega) = e^{j2\omega}Y(j\omega/3)$. ➡ $w(t) = y(t+2) = 3e^{-3(t+2)}u(t+2)$

Finally $X(j\omega) = j\dfrac{d}{d\omega}W(j\omega)$ ➡ $x(t) = tw(t) = 3te^{-3(t+2)}u(t+2)$

Scaling Property

- Case of scaling the periodic continuous-time signal

  If $x(t)$ is a periodic signal, then $z(t) = x(at)$ is also periodic.

  If $x(t)$ has fundamental period T, then $z(t)$ has fundamental period T/a. Suppose that $a>0$

  If the fundamental frequency of $x(t)$ is ω_0, then the fundamental frequency of $z(t)$ is $a\omega_0$.

  FS coefficients of $z(t)$: $$Z[k] = \frac{a}{T}\int_0^{T/a} z(t)e^{-jka\omega_o t}dt$$

  $$\Rightarrow \quad z(t) = x(at) \quad \xleftrightarrow{FS;\, a\omega_0} Z[k] = X[k], \qquad a > 0$$

  ◆ Scaling in time-domain for periodic signal ↔ Same response in frequency

Scaling operation for periodic signal simply changes the harmonic spacing from ω_0 to $a\omega_0$!!

- Case of scaling the discrete-time signal

  First of all, $z[n] = x[pn]$ is defined only for **integer values of p**.

  If $|p| > 1$, then scaling operation discards information.

Parseval Relationships

- Case of continuous-time non-periodic signal

Recall the energy of $x(t)$ is defined by $W_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

Note that $|x(t)|^2 = x(t)x^*(t)$

Express $x^*(t)$ in terms of its FT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$

$$\Rightarrow x^*(t) = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega\right\}^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t} d\omega$$

Then,

$$W_x = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t} d\omega\right] dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt\right\} d\omega$$

$$\Rightarrow W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega) d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

The energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation normalized by 2π

Summary for Parseval Relationships

Table 3.10 Parseval Relationships for the Four Fourier Representations

Representations	Parseval Relationships
FT	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$
FS	$\frac{1}{T}\int_0^T \lvert x(t)\rvert^2\, dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$
DTFT	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\Omega})\rvert^2\, d\Omega$
DTFS	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert X[k]\rvert^2$

The power is defined as the integral or sum of the magnitude squared over one period, **normalized by the length of the period**

Example 3.50

Let $x[n] = \dfrac{\sin(Wn)}{\pi n}$

Use Parseval's theorem to evaluate

$$\chi = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} \frac{\sin^2(Wn)}{\pi^2 n^2}$$

Direct calculation in time-domain is very difficult!

<Sol.>

1. Using the DTFT Parseval relationship, we have $\chi = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \left|X(e^{j\Omega})\right|^2 d\Omega$

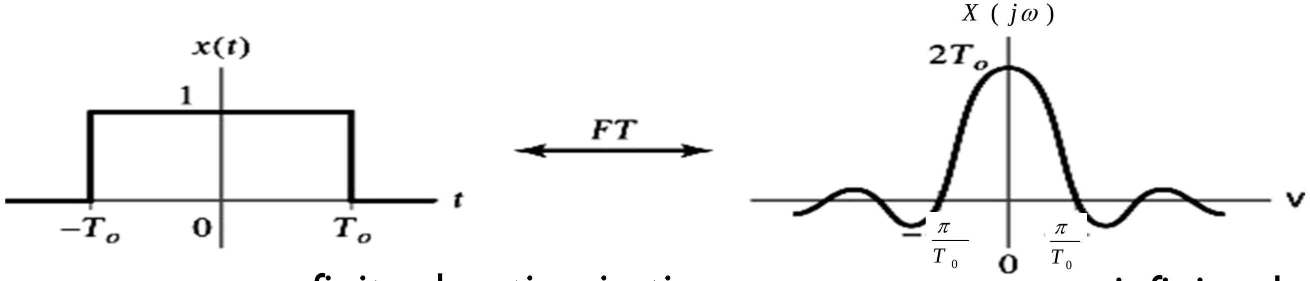
2. Since

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| \le W \\ 0, & W < |\Omega| < \pi \end{cases}$$

$$\Rightarrow \quad \chi = \frac{1}{2\pi}\int_{-W}^{W} 1 d\Omega = W/\pi$$

Time-Bandwidth Product

- Preview the fact: $x(t) = \begin{cases} 1, & |t| \le T_o \\ 0, & |t| > T_o \end{cases} \xleftrightarrow{FT} X(j\omega) = 2\sin(\omega T_o)/\omega$



finite duration in time — infinite duration in freq.

- As signal's time extent decreases (T_0 decreases), the signal's frequency extent increases.
- The product of the time extent T_0 and main-lobe width (i.e. the bandwidth) $2\pi/T_0$ is a constant.
- The bandwidth of a signal is the extent of the signal's significant frequency content.
- Compressing a signal in time leads expansion in frequency and vice versa

Time-Bandwidth Product

- Effective duration of a signal $x(t)$ is defined by

$$T_d = \left[\frac{\int_{-\infty}^{\infty} t^2 \left| x(t) \right|^2 dt}{\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt} \right]^{1/2}$$

- Effective bandwidth of a signal $x(t)$ is defined by

$$B_w = \left[\frac{\int_{-\infty}^{\infty} \omega^2 \left| X(j\omega) \right|^2 d\omega}{\int_{-\infty}^{\infty} \left| X(j\omega) \right|^2 d\omega} \right]^{1/2}$$

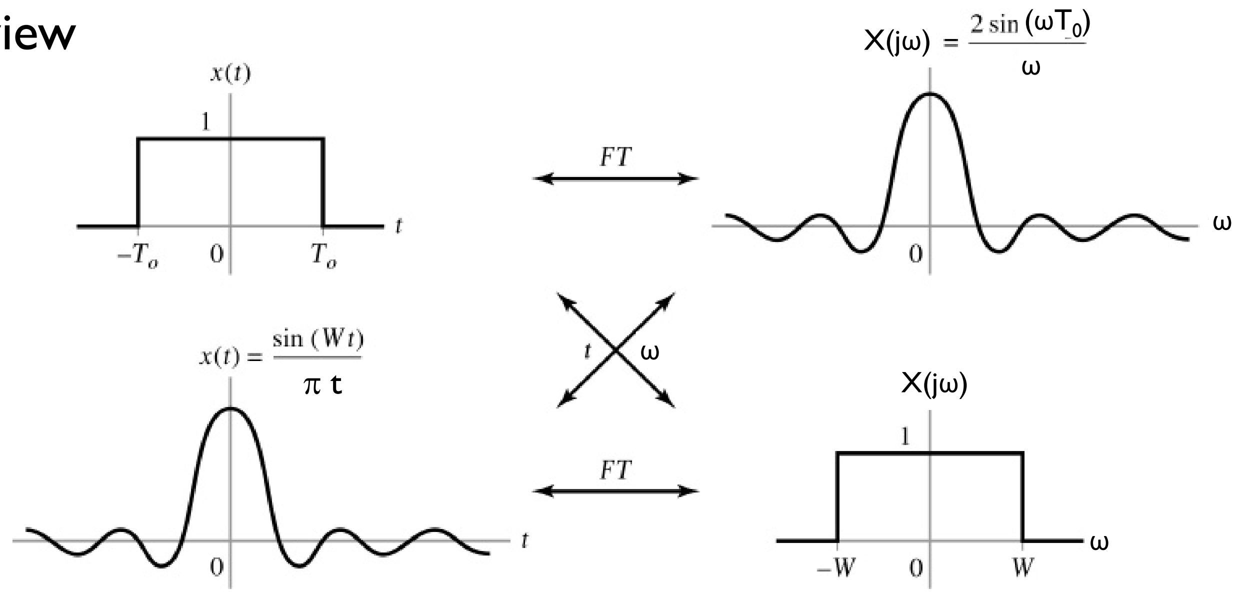
- **The uncertainty principle:**

The time-bandwidth product for any signal $x(t)$ is lower bounded by $T_d B_w \geq 1/2$

We cannot simultaneously decrease the duration and bandwidth of a signal.

Duality

- Preview



- A rectangular pulse in either time or frequency corresponds to a sinc function in either frequency or time domain.
- A impulse in either time or frequency transforms to a constant in either frequency or time domain.
- Convolution, Differentiation, …

Duality Property of the FT

- Recall that $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ and $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

  Difference in the factor 2π and the sign change in the complex sinusoid

- Duality property

$$f(t) \xleftrightarrow{FT} F(j\omega) \Leftrightarrow F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$$

- Example 3.52

  Find the FT of $x(t) = \frac{1}{1+jt}$

  <Sol.>

  Note that: $f(t) = e^{-t}u(t) \xleftrightarrow{FT} F(j\omega) = \frac{1}{1+j\omega}$

  Replacing ω by t, we obtain $F(jt) = \frac{1}{1+jt}$

  Apply duality property:

  $\Rightarrow$ $X(j\omega) = 2\pi f(-\omega) = 2\pi e^{\omega}u(-\omega)$

Duality Property of the DTFS

- FT-pair: mapping a CT nonperiodic function into a CT nonperiodic function.
- DTFS-pair: mapping a DT periodic function into a DT periodic function.
- Recall that $x[n]=\sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}$ and $X[k]=\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}$

Difference in the factor *N* and the sign change in the complex sinusoid

- Duality property

$$x[n] \xleftrightarrow{DTFS;\frac{2\pi}{N}} X[k] \quad \Leftrightarrow \quad X[n] \xleftrightarrow{DTFS;\frac{2\pi}{N}} \frac{1}{N}x[-k]$$

Duality Property of the DTFT and FS

- FS-pair: mapping a CT periodic function into a DT nonperiodic function.
- DTFT-pair: mapping a DT nonperiodic function into a CT periodic function.

- Recall that FS of a periodic continuous time signal $z(t)$: $z(t) = \sum_{k=-\infty}^{\infty} Z[k] e^{jk\omega_o t}$

- and DTFT of a nonperiodic discrete-time signal $x[n]$: $X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} x[n] e^{-j\Omega n}$

  1. Difference in the sign change in the complex sinusoid

  2. Duality relationship between $z(t)$ and $X(e^{j\Omega})$ requires $z(t)$ to have the same period as $X(e^{j\Omega})$, that is, $\boldsymbol{T = 2\pi}$

- Duality property

$$x[n] \xleftrightarrow{DTFS} X(e^{j\Omega}) \quad \Leftrightarrow \quad X(e^{jt}) \xleftrightarrow{FS;1} x[-k]$$