# Chapter 3: Fourier Representation of Signals and LTI Systems

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# Outline

- Introduction
- Complex Sinusoids and Frequency Response
- Fourier Representations for Four Classes of Signals
- Discrete-time Periodic Signals *Fourier Series*
- Continuous-time Periodic Signals
- Discrete-time Nonperiodic Signals *Fourier Transform*
- Continuous-time Nonperiodic Signals
- Properties of Fourier representations
- Linearity and Symmetry Properties
- Convolution Property

# Outline

- Differentiation and Integration Properties
- Time- and Frequency-Shift Properties
- Finding Inverse Fourier Transforms
- Multiplication Property
- Scaling Properties
- Parseval Relationships
- Time-Bandwidth Product
- Duality

# Introduction

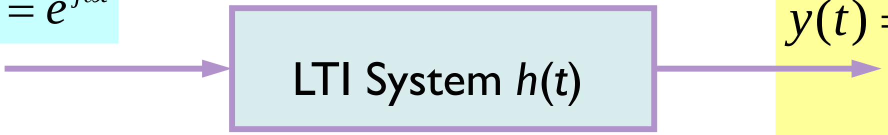
- In this chapter, we represent a signal as a weighted superposition of complex sinusoids.
  - AKA *Fourier analysis*
  - The weight associated with a sinusoid of a given frequency represents the contribution of that sinusoid to the overall signal.
  - Four distinct Fourier representations:

| Time property | Periodic | Nonperiodic |
|---|---|---|
| Continuous (t) | Fourier Series (FS) | Fourier Transform (FT) |
| Discrete [n] | Discrete-Time Fourier Series (DTFS) | Discrete-Time Fourier Transform (DTFT) |

# Frequency Response of LTI System

- The response of the LTI system to a sinusoidal input $e^{j\omega t}$ : $H\{x(t)=e^{j\omega t}\}= e^{j\omega t} H(j\omega)$

$x(t) = e^{j\omega t}$




$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

Dependent on ω, but independent on *t*

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{j\omega\tau} d\tau \\ &= e^{j\omega t} \boxed{H(j\omega)} \end{aligned}$$

constant

- For discrete-time case, the response of the LTI system to a sinusoidal input $e^{j\Omega n}$ is $H\{x[n]=e^{j\Omega n}\}= e^{j\Omega n} H(e^{j\Omega})$

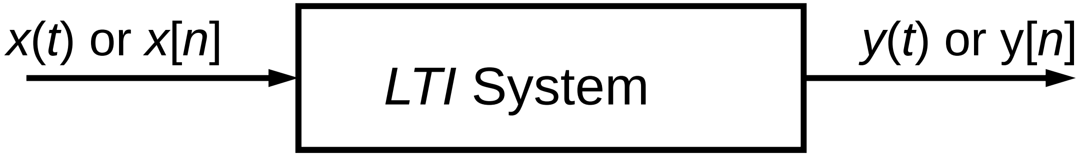
$$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}$$

Dependent on Ω, but independent on *n*

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{j\Omega(n-k)} \\ &= e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k} \\ &= e^{j\Omega n} \boxed{H(e^{j\Omega})} \end{aligned}$$

# Frequency Response of LTI System

- Frequency response of a continuous-time LTI system




$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

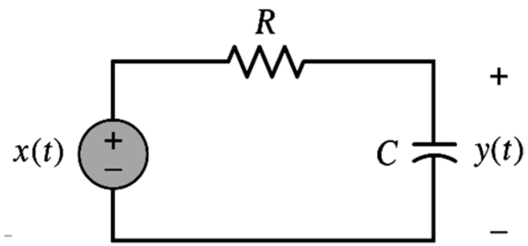
$$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}$$

- Frequency response of the LTI system can also be represented by

$$H(j\omega) = |H(j\omega)| e^{j \arg\{H(j\omega)\}}$$

  - Magnitude response $|H(j\omega)|$
  - Phase response $\arg\{H(j\omega)\}$

# Example 3.1 RC Circuit System




The impulse response of the RC circuit system is derived in Example 1.21 as

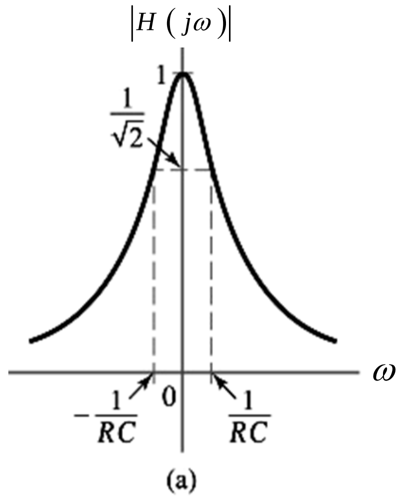
$$h(t)=\frac{1}{RC}e^{-t/RC}u(t)$$

Find an expression for the frequency response, and plot the magnitude and phase response.

<Sol.> Frequency response:

$$H(j\omega)=\frac{1}{RC}\int_{-\infty}^{\infty}e^{-\frac{\tau}{RC}}u(\tau)e^{-j\omega\tau}d\tau=\frac{1}{RC}\int_{0}^{\infty}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}d\tau$$

$$=\frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}\Bigg|_{0}^{\infty}=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$




Low-pass filter

Magnitude response:

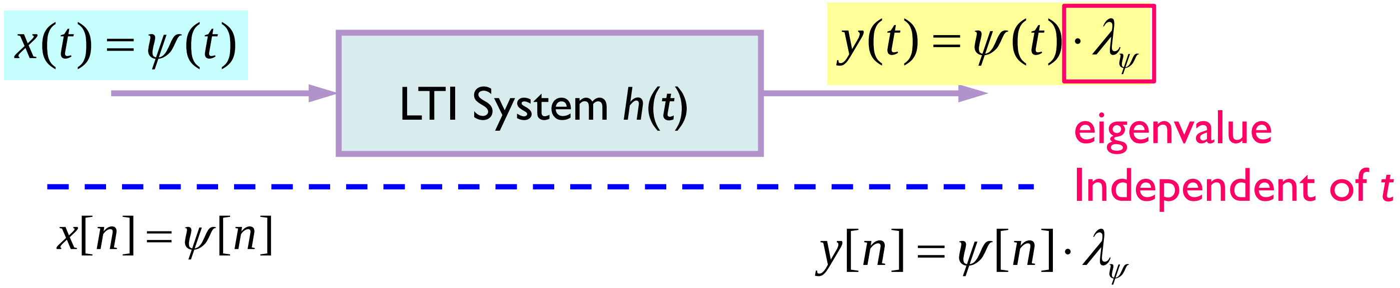
$$|H(j\omega)|=\frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

Phase response:

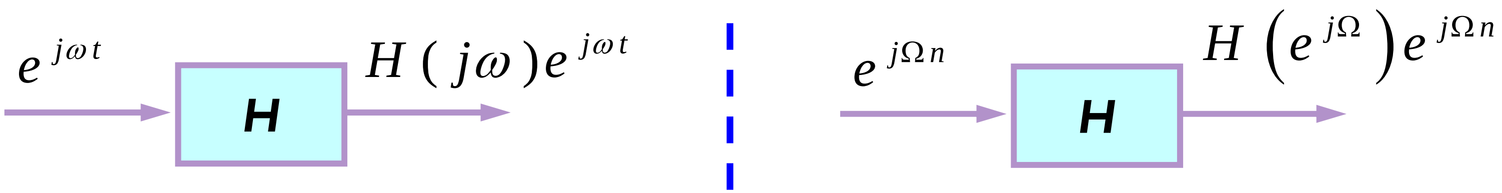
$$\arg\{H(j\omega)\}=-\arctan(\omega RC)$$

# Another Meaning for Frequency Response

- The eigenfunction of the LTI system $\psi(t)$:




- The eigen-representation of the LTI system




- By representing arbitrary signals as weighted superposition of eigenfunction $e^{j\omega t}$, then

$$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t}$$

$$y(t) = H\{x(t)\} = \sum_{k=1}^{M} a_k H(j\omega_k) e^{j\omega_k t}$$

the weights describe the signal as a function of frequency. (frequency-domain representation)

Multiplication in frequency domain, c.f. convolution in time-domain

# Fourier Analysis

- Non-periodic signals have (continuous) Fourier transform representations, while periodic signals have (discrete) Fourier series representations.
- Why Fourier series representations for **Periodic signals**
  - Periodic signal can be considered as a weighted superposition of (periodic) complex sinusoids (using periodic signals to construct a periodic signal)
  - Recall that the periodic signal has a (fundamental) period, this implies that the period (or frequency) of each component sinusoid must be an integer multiple of the signal's fundamental period (or frequency) ➔ in frequency-domain analysis, the weighted complex sinusoids look like a discrete series of weighted frequency impulse ➔ Fourier series representation
  - **Question:** *Can any a periodic signal be represented or constructed by a weighted superposition of complex sinusoids?*

# Approximated Periodic Signals

- Suppose the signal $\hat{x}[n]=\sum_k A[k]e^{jk\Omega_0 n}$ is approximated to a discrete-time periodic signal $x[n]$ with fundamental period $N$, where $\Omega_0 = 2\pi/N$.
- Since $e^{j(k+N)\Omega_0 n}=e^{jN\Omega_0 n}e^{jk\Omega_0 n}=e^{j2\pi n}e^{jk\Omega_0 n}=e^{jk\Omega_0 n}$, there are only $N$ distinct sinusoids of the form $e^{jk\Omega_0 n}$ : e.g. $k$=0, 1, …, $N$-1
- Accordingly, we may rewrite the signal as $\hat{x}[n]=\sum_{k=0}^{N-1} A[k]e^{jk\Omega_0 n}$ **DTFS**

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- For continuous-time case, we then have $\hat{x}(t)=\sum_k A[k]e^{jk\omega_0 t}$, where $\omega_0 = 2\pi/T$ is the fundamental frequency of periodic signal $x(t)$
- Although $e^{jk\omega_0 t}$ is periodic, $e^{jk\omega_0 t}$ is distinct for distinct $k\omega_0$
- Hence, an infinite number of distinct terms, i.e. $\hat{x}(t)=\sum_{k=-\infty}^{\infty} A[k]e^{jk\omega_0 t}$ **FS**

# Approximation Error

- Mean-square error (MSE) performance:

$$MSE = \frac{1}{N}\sum_{n=0}^{N-1}\left|x[n]-\hat{x}[n]\right|^2 dt$$

$$MSE = \frac{1}{T}\int_0^T \left|x(t)-\hat{x}(t)\right|^2 dt$$

  - We seek the weights or coefficients *A*[*k*] such that the MSE is minimum
  - The DTFS and FS coefficients (Fourier analysis) achieve the minimum MSE (MMSE) performance.

# Fourier Analysis

- Why Fourier transform representations for **Non-periodic signals**
  - Using periodic sinusoids (the same approach) to construct a non-periodic signal, there are no restrictions on the period (or frequency) of the component sinusoids ➔ there are generally having a continuum of frequencies in frequency-domain analysis➔ Fourier transform representation
  - **Fourier transform:**
    - Continuous-time case

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega t}d\omega$$

**FT**

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} A[k]e^{jk\omega_0 t}$$

**FS**

- Discrete-time case

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

**DTFT**

$$\hat{x}[n] = \sum_{k=0}^{N-1} A[k]e^{jk\Omega_0 n}$$

**DTFS**

Frequencies separated by an integer multiple of $2\pi$ are identical

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# Discrete-Time Fourier Series (DTFS)

- The DTFS-pair of a periodic signal $x[n]$ with fundamental period $N$ and fundamental frequency $\Omega_0=2\pi/N$ is

$$x[n] \xleftrightarrow{DTFS;\ \Omega_0} X[k]$$

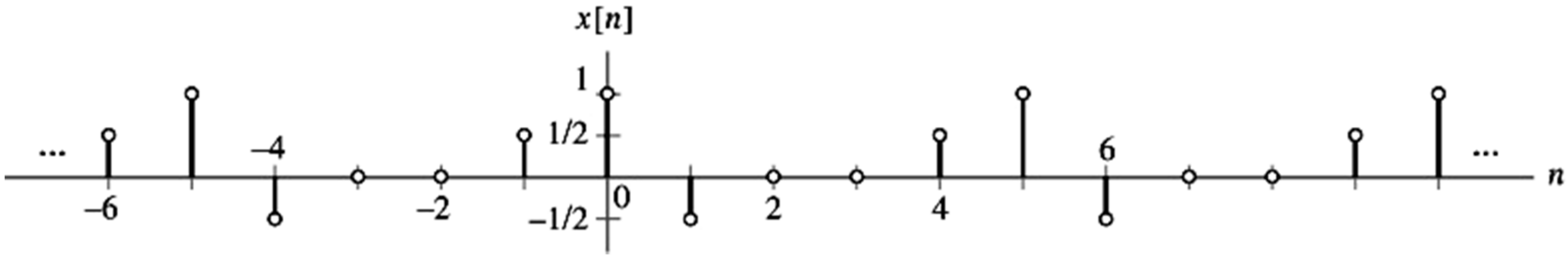
$$x[n]=\sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$$

$$X[k]=\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$$

  - The DTFS coefficients $X[k]$ are called the frequency-domain representation for $x[n]$
  - The value $k$ determines the frequency of the sinusoid associated with $X[k]$
  - The DTFS is exact. (Any periodic discrete-time signal can be described in terms of DTFS coefficients exactly)
  - The DTFS is the only one of Fourier analysis that can be evaluated and manipulated in computer for a finite set of $N$ numbers.

# Example 3.2 DTFS Coefficients

Find the frequency domain representation of the signal depicted in Fig. 3.5.




<Sol.>

1. Period: $N = 5$ ➡ $\Omega_o = 2\pi/5$
2. Odd symmetry ➡ *We choose n = – 2 to n = 2*
3. Fourier coefficient:

$$X[k] = \frac{1}{5}\sum_{n=-2}^{2} x[n]e^{-jk2\pi n/5}$$

$$= \frac{1}{5}\left\{x[-2]e^{jk4\pi/5} + x[-1]e^{jk2\pi/5} + x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5}\right\}$$

➡

$$X[k] = \frac{1}{5}\{1 + \frac{1}{2}e^{jk2\pi/5} - \frac{1}{2}e^{-jk2\pi/5}\}$$

$$= \frac{1}{5}\{1 + j\sin(k2\pi/5)\}$$

# Example 3.2 (conti.)

If we calculate $X[k]$ using $n = 0$ to $n = 4$:

$$X[k] = \frac{1}{5}\left\{x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5} + x[3]e^{-jk6\pi/5} + x[4]e^{-jk8\pi/5}\right\}$$

$$= \frac{1}{5}\left\{1 - \frac{1}{2}e^{-jk2\pi/5} + \frac{1}{2}e^{-jk8\pi/5}\right\}$$

since $e^{-jk8\pi/5} = e^{-jk2\pi}e^{jk2\pi/5} = e^{jk2\pi/5}$

$$X[k] = \frac{1}{5}\{1 + \frac{1}{2}e^{jk2\pi/5} - \frac{1}{2}e^{-jk2\pi/5}\}$$

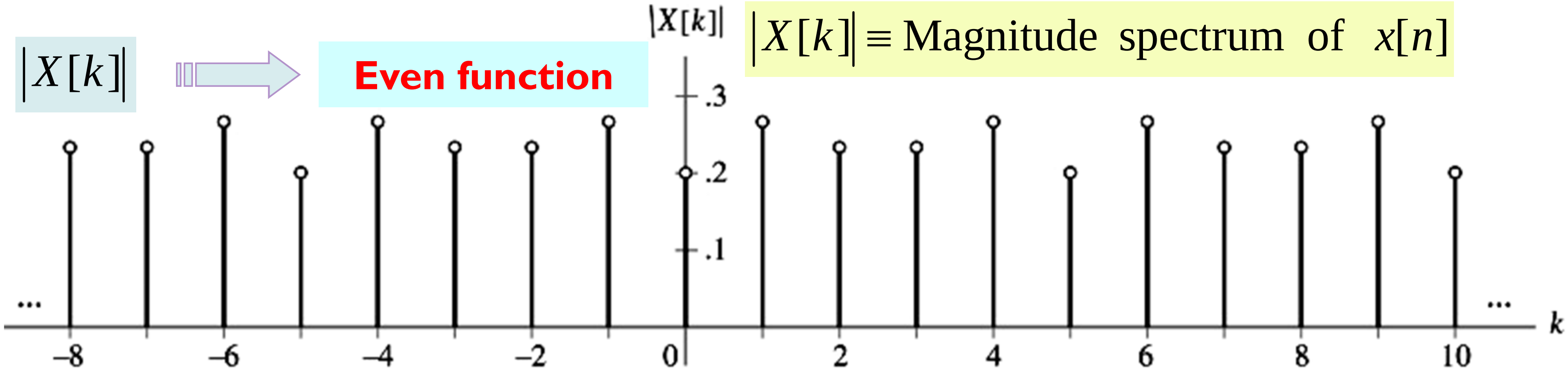
$$= \frac{1}{5}\{1 + j\sin(k2\pi/5)\}$$

The same expression for the DTFS coefficients !!!

# Example 3.2 (conti.)

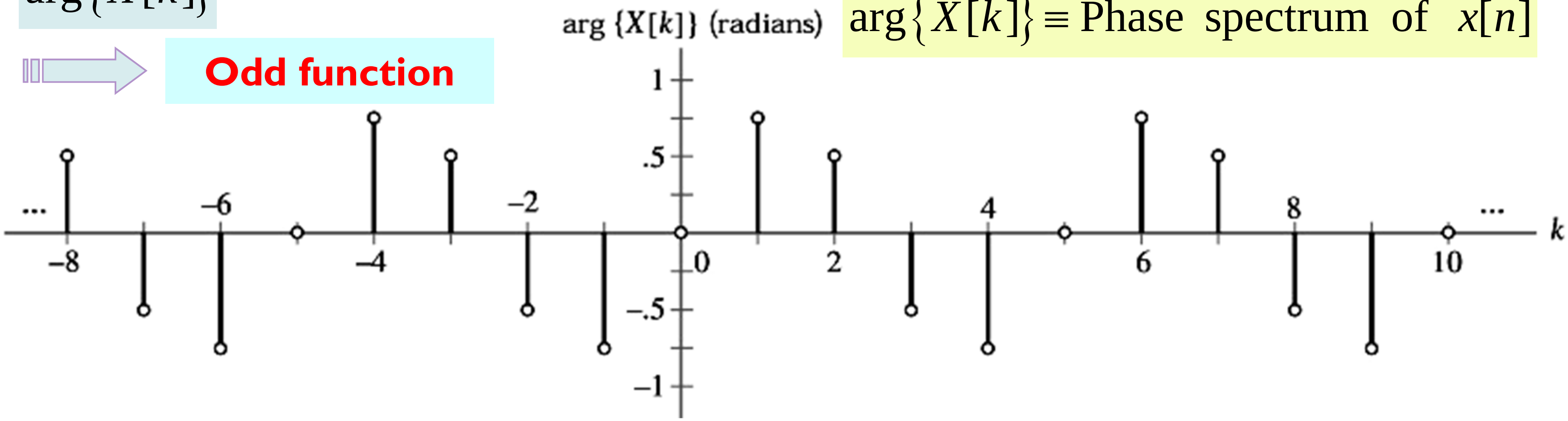
$|X[k]|$ ⇒ **Even function**

$|X[k]| \equiv \text{Magnitude spectrum of } x[n]$




$\arg\{X[k]\}$ ⇒ **Odd function**

$\arg\{X[k]\} \equiv \text{Phase spectrum of } x[n]$



# Example 3.3 Computation by Inspection

Determine the DTFS coefficients of $x[n] = \cos(n\pi/3 + \phi)$, using the method of inspection.

<Sol.>

1. Period: $N = 6$ ➡ $\Omega_o = 2\pi/6 = \pi/3$

2. Using Euler's formula, $x[n]$ can be expressed as

$$x[n] = \frac{e^{j(\frac{\pi}{3}n+\phi)} + e^{-j(\frac{\pi}{3}n+\phi)}}{2} = \frac{1}{2}e^{-j\phi}e^{-j\frac{\pi}{3}n} + \frac{1}{2}e^{j\phi}e^{j\frac{\pi}{3}n} \qquad (3.13)$$

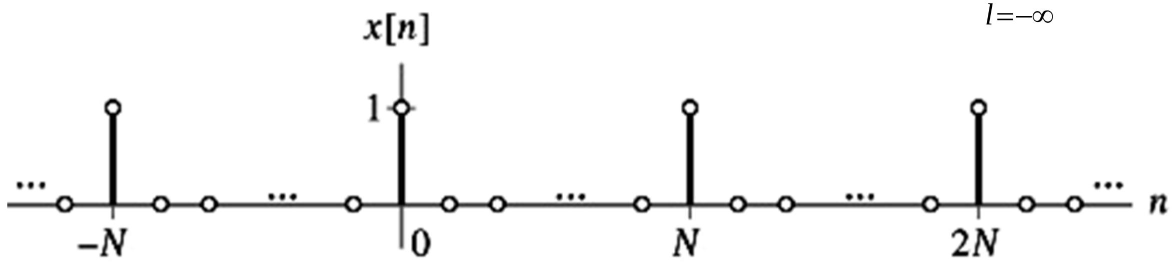
3. Compare Eq. (3.13) with the DTFS of Eq. (3.10) with $\Omega_o = \pi/3$, written by summing from $k = -2$ to $k = 3$:

$$x[n] = \sum_{k=-2}^{3} X[k]e^{jk\pi n/3}$$
$$= X[-2]e^{-j2\pi n/3} + X[-1]e^{-j\pi n/3} + X[0] + X[1]e^{j\pi n/3} + X[2]e^{j2\pi n/3} + X[3]e^{j\pi n}$$

➡ $$x[n] \xleftrightarrow{DTFS;\ \frac{\pi}{3}} X[k] = \begin{cases} e^{-j\phi}/2, & k = -1 \\ e^{j\phi}/2, & k = 1 \\ 0, & \text{otherwise on } -2 \le k \le 3 \end{cases}$$

# Example 3.4

Find the DTFS coefficients of the *N*-periodic impulse train $x[n] = \sum_{l=-\infty}^{\infty} \delta[n-lN]$.



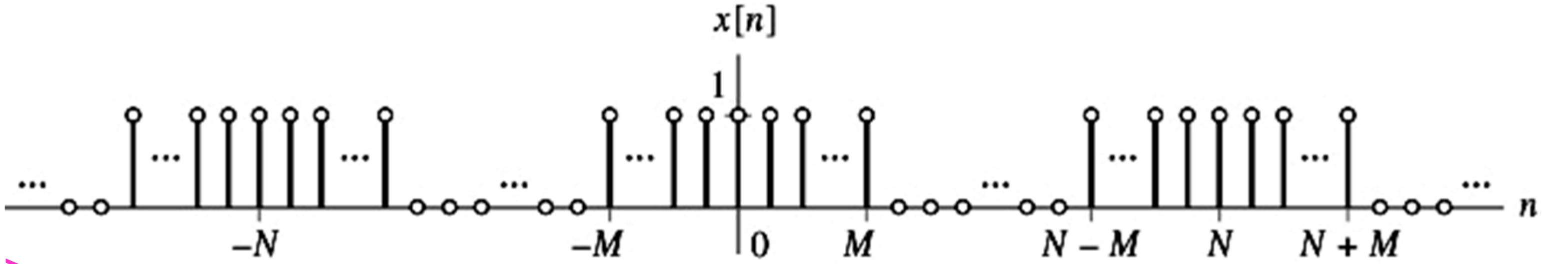

<Sol.>

1. Period: *N*.
2. By (3.11), we have

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1}\delta[n]e^{-jkn2\pi/N} = \frac{1}{N}$$

# Example 3.6

Find the DTFS coefficients for the *N*-periodic square wave given by




<Sol.>

1. Period = *N*, hence $\Omega_o = 2\pi/N$
2. It is convenient to evaluate DTFS coefficients over the interval $n = -M$ to $n = N-M-1$.

$$x[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & M < n < N-M \end{cases} \;\Rightarrow\; X[k] = \frac{1}{N}\sum_{n=-M}^{N-M-1} x[n]e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=-M}^{M} e^{-jk\Omega_0 n}$$

3. For $k = 0, \pm N, \pm 2N, \ldots$, we have $e^{jk\Omega_o} = e^{-jk\Omega_o} = 1$

$$\Rightarrow X[k] = \frac{1}{N}\sum_{n=-M}^{M} 1 = \frac{2M+1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$$

For $k \neq 0, \pm N, \pm 2N, \ldots$, we have

$$X[k] = \frac{1}{N}\sum_{n=-M}^{M} e^{-jk\Omega_0 n} = \frac{e^{jk\Omega_0 M}}{N}\left(\frac{1-e^{-jk\Omega_0(2M+1)}}{1-e^{-jk\Omega_0}}\right), \quad k \neq 0, \pm N, \pm 2N, \ldots$$

# Example 3.6 (conti.)

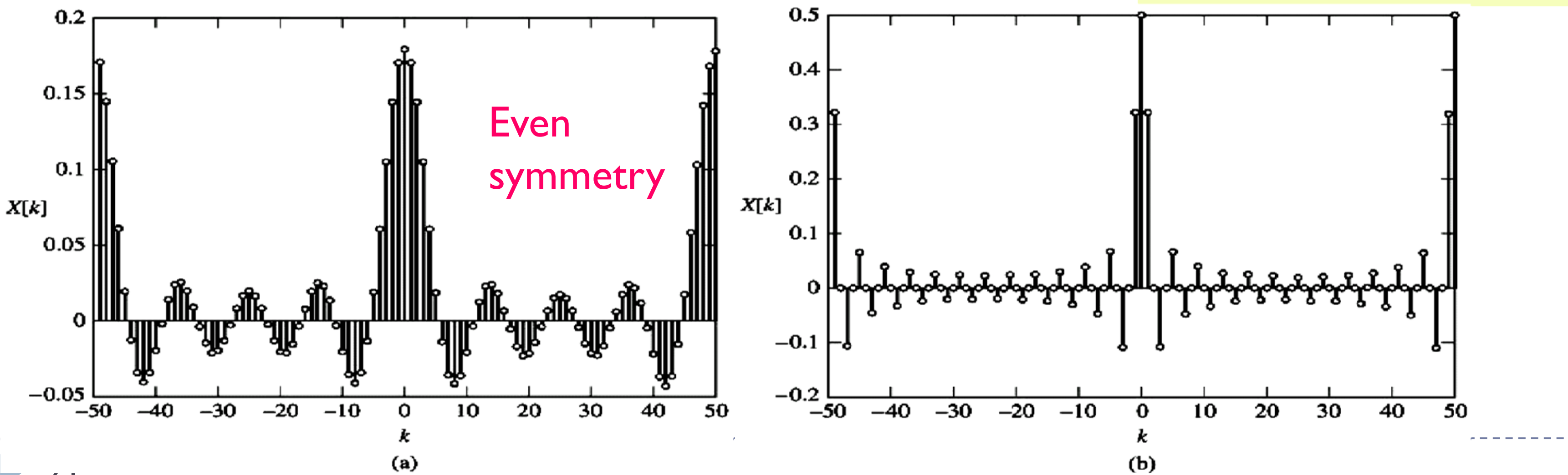
$$X[k]=\frac{e^{jk\Omega_0 M}}{N}\left(\frac{1-e^{jk\Omega_0 M(2M+1)}}{1-e^{-jk\Omega_0}}\right),\quad k\neq 0,\ \pm N,\ \pm 2N,\ \ldots\ldots$$

$$\Rightarrow\quad X[k]=\frac{1}{N}\left(\frac{e^{jk\Omega_0(2M+1)/2}}{e^{jk\Omega_0/2}}\right)\left(\frac{1-e^{jk\Omega_0(2M+1)}}{1-e^{jk\Omega_0}}\right)=\frac{1}{N}\left(\frac{e^{jk\Omega_0(2M+1)/2}-e^{-jk\Omega_0(2M+1)/2}}{e^{jk\Omega_0/2}-e^{-jk\Omega_0/2}}\right)$$

The numerator and denominator of above Eq. are divided by 2*j*

$$\Rightarrow\quad X[k]=\frac{1}{N}\frac{\sin(k\Omega_0(2M+1)/2)}{\sin(k\Omega_0/2)},$$

$$k=0,\ \pm N,\ \pm 2N,\ \ldots$$




**The DTFS coefficients for the square wave, assuming a period *N* = 50: (a) *M* = 4. (b) *M* = 12.**

# Symmetry Property of DTFS Coefficients

- If $X[k] = X[-k]$, it is instructive to consider the contribution of each term in

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$$

of period $N$

- Assume that $N$ is even, so that $N/2$ is integer. $\Omega_o = 2\pi/N$
- Rewrite the DTFS coefficients by letting $k$ range from $-N/2+1$ to $N/2$, i.e.

$$x[n] = \sum_{k=-N/2+1}^{N/2} X[k] e^{jk\Omega_0 n}$$

$$\Rightarrow x[n] = X[0] + X[N/2]e^{j\pi n} + \sum_{m=1}^{N/2-1} 2X[m]\left(\frac{e^{jm\Omega_0 n} + e^{-jm\Omega_0 n}}{2}\right)$$

$$= X[0] + X[N/2]\cos(\pi n) + \sum_{m=1}^{N/2-1} 2X[m]\cos(m\Omega_0 n)$$

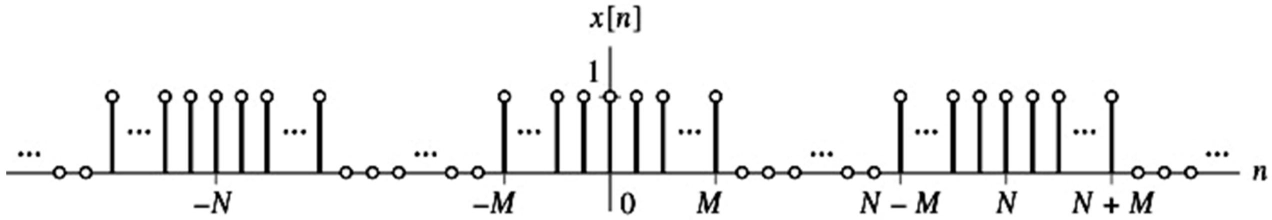
- Define new set of coefficients

$$B[k] = \begin{cases} X[k], & k = 0, \quad N/2 \\ 2X[k], & k = 1, \; 2, \; \ldots, \; N/2-1 \end{cases} \Rightarrow x[n] = \sum_{k=0}^{N/2} B[k]\cos(k\Omega_0 n)$$



A similar expression may be derived for $N$ odd.

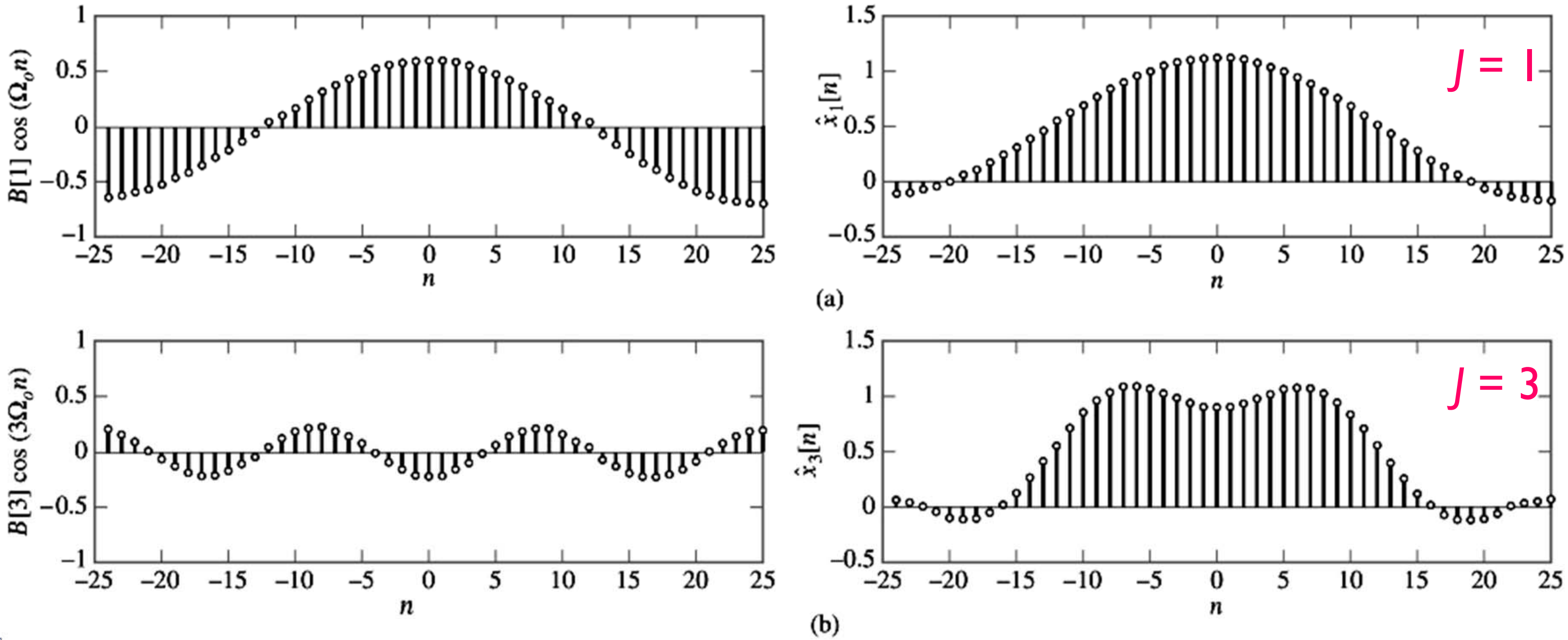
# Example 3.7

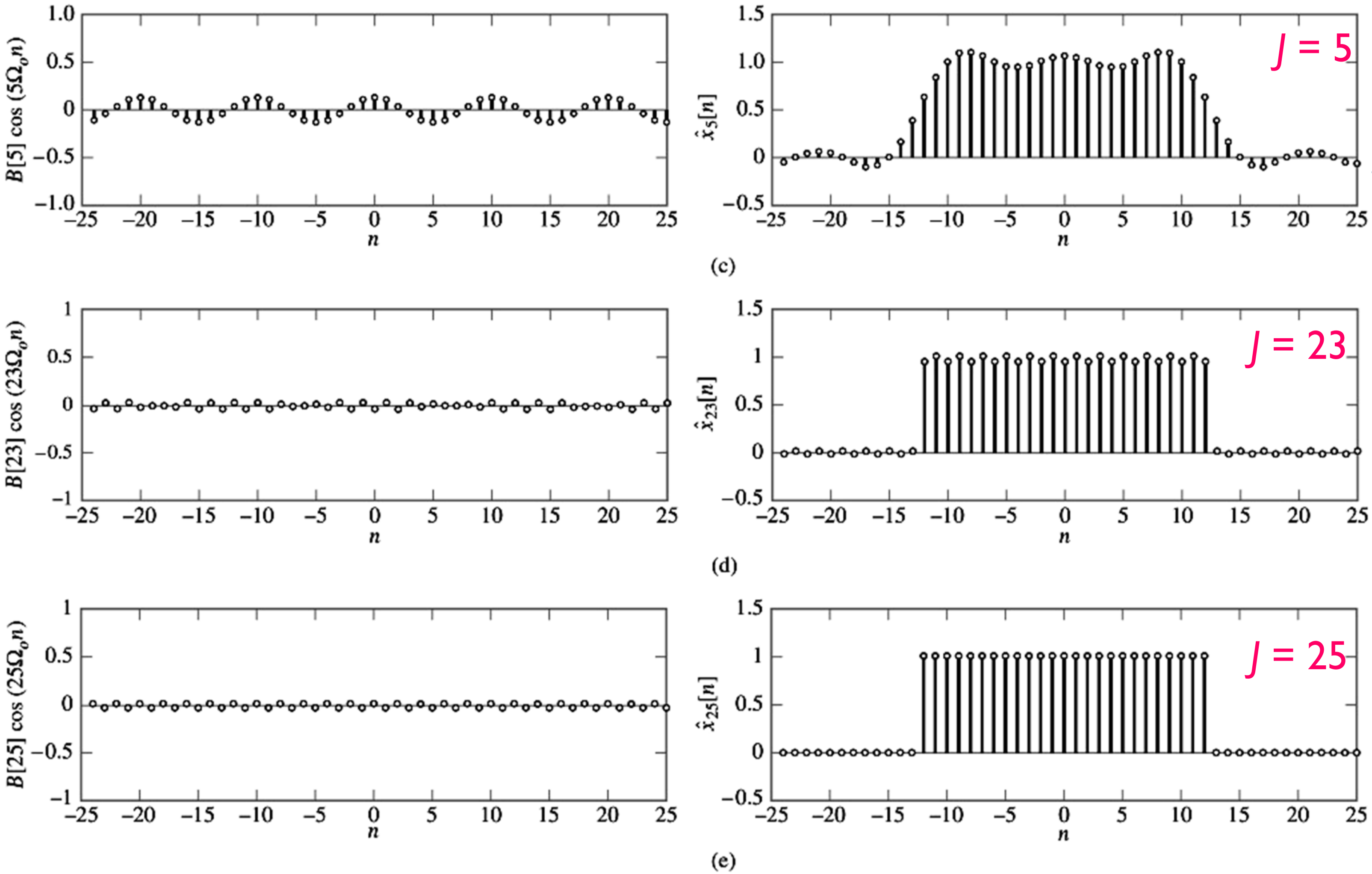



The contribution of each term in DTFS series to the square wave may be illustrated by defining the partial-sum approximation to $x[n]$ as $\hat{x}_J[n] = \sum_{k=0}^{J} B[k]\cos(k\Omega_0 n)$

where $J \leq N/2$. This approximation contains the first $2J + 1$ terms centered on $k = 0$ in the square wave above. Assume a square wave has period $N = 50$ and $M = 12$. Evaluate one period of the $J$th term and the $2J + 1$ term approximation for $J = 1, 3, 5, 23$, and 25

<Sol.>






The coefficients $B[k]$ associated with values of $k$ near zero represent the low-frequency or slowly varying features in the signal, while the coefficients associated with values of $k$ near $\pm N/2$ represent the high frequency or rapidly varying features in the signal.

# Fourier Series (FS)

- The DT-pair of a periodic signal $x(t)$ with fundamental period $T$ and fundamental frequency $\omega_0=2\pi/T$ is

$$x(t) \xleftrightarrow{FS;\omega_0} X[k]$$

$$x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

$$X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

take one period of $x(t)$

- The FS coefficients $X[k]$ are called the frequency-domain representation for $x(t)$
- The value $k$ determines the frequency of the sinusoid associated with $X[k]$
- The infinite series in $x(t)$ is not guaranteed to converge for all possible signals.
  - Suppose we define

$$\hat{x}(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \xrightarrow{\text{approach to?}} x(t)$$

If $x(t)$ is square integrable, then

$$\frac{1}{T}\int_0^T \left|x(t)-\hat{x}(t)\right|^2 dt = 0$$

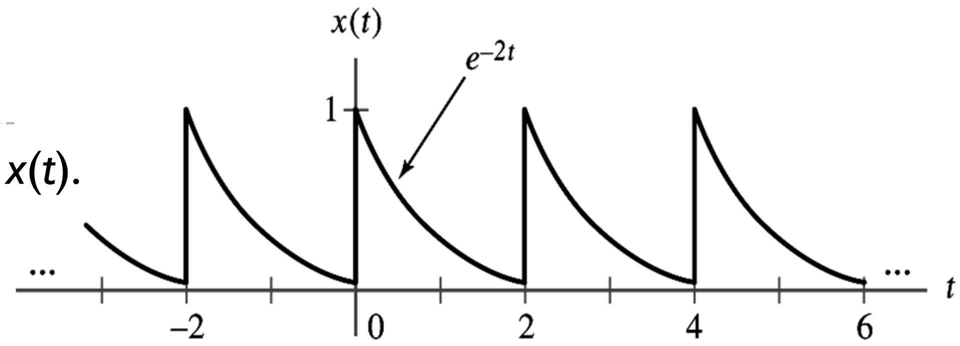
a zero power in their differences.

# Remarks

- A zero MSE does not imply that the two signals are equal pointwise.
- Dirichlet's conditions:
  1. ***x*(*t*) is bounded**
  2. ***x*(*t*) has a finite number of maximum and minima in one period**
  3. ***x*(*t*) has a finite number of discontinuities in one period**
- Pointwise convergence of $\hat{x}(t)$ and $x(t)$ is guaranteed at all *t* except those corresponding to discontinuities satisfying Dirichlet's conditions.
- If $x(t)$ satisfies Dirichlet's conditions and is not continuous, then $\hat{x}(t)$ converges to the midpoint of the left ad right limits of $x(t)$ at each discontinuity.

# Example 3.9

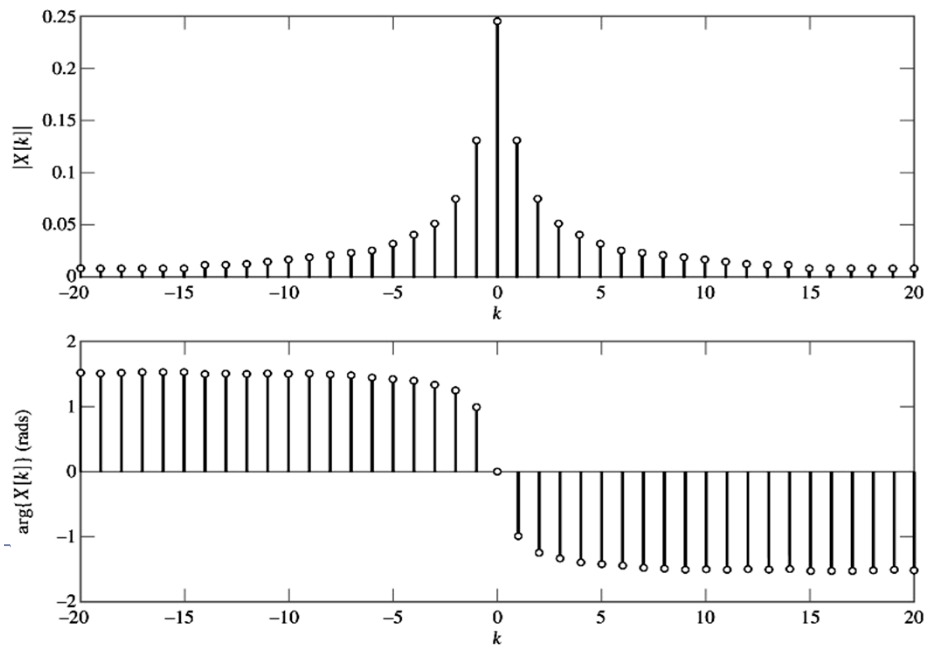
Determine the FS coefficients for the signal $x(t)$.




<Sol.>

1. The period of $x(t)$ is $T = 2$, so $\omega_o = 2\pi/2 = \pi$.
2. Take one period of $x(t)$: $x(t) = e^{-2t}$, $0 \le t \le 2$. Then

$$X[k] = \frac{1}{2}\int_0^2 e^{-2t}e^{-jk\pi t}dt = \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}dt = \frac{1}{4+jk2\pi}\left(1-e^{-4}e^{-jk2\pi}\right) = \frac{1-e^{-4}}{4+jk2\pi}$$




The Magnitude of $X[k]$ ≡ the magnitude spectrum of $x(t)$

The phase of $X[k]$ ≡ the phase spectrum of $x(t)$

# Example 3.10

Determine the FS coefficients for the signal $x(t)$ defined by $x(t)=\sum_{l=-\infty}^{\infty}\delta(t-4l)$

<Sol.>

1. Fundamental period of $x(t)$ is $T = 4$, each period contains an impulse.
2. By integrating over a period that is symmetric about the origin, $-2 < t \le 2$, to obtain $X[k]$:

$$X[k]=\frac{1}{4}\int_{-2}^{2}\delta(t)e^{-jk(\pi/2)t}dt=\frac{1}{4}$$

3. The magnitude spectrum is constant and the phase spectrum is zero.

# Example 3.11 Computation by Inspection

Determine the FS representation of the signal $x(t) = 3\cos(\pi t/2 + \pi/4)$

<Sol.>

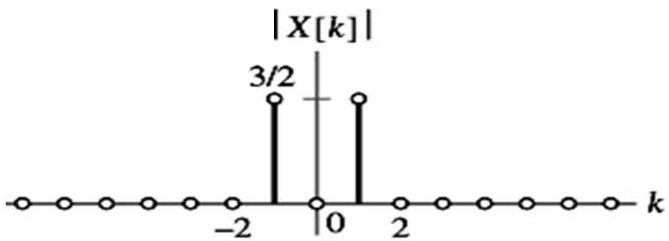
1. Fundamental frequency of $x(t)$ is $\omega_o = 2\pi/4 = \pi/2$, so $T = 4$.
2. Rewrite the $x(t)$ as

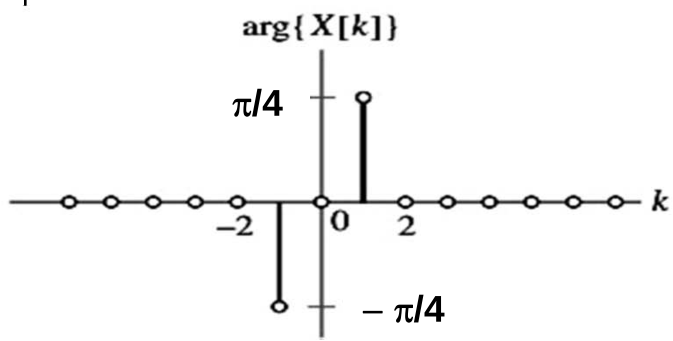
$$x(t) = 3\frac{e^{j(\pi t/2+\pi/4)} + e^{-j(\pi t/2+\pi/4)}}{2} = \frac{3}{2}e^{j\pi/4}e^{j\pi t/2} + \frac{3}{2}e^{-j\pi/4}e^{-j\pi t/2}$$

3. Compare with

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\pi t/2}$$

$$X[k] = \begin{cases} \frac{3}{2}e^{-j\pi/4}, & k=-1 \\ \frac{3}{2}e^{j\pi/4}, & k=1 \\ 0, & \text{otherwise} \end{cases}$$



# Example 3.12 Inverse FS

Find the (time-domain) signal $x(t)$ corresponding to the FS coefficients $X[k] = (1/2)^{|k|} e^{jk\pi/20}$

Assume that the fundamental period is $T$=2.

<Sol.>

1. Fundamental frequency: $\omega_o = 2\pi/T = \pi$. Then

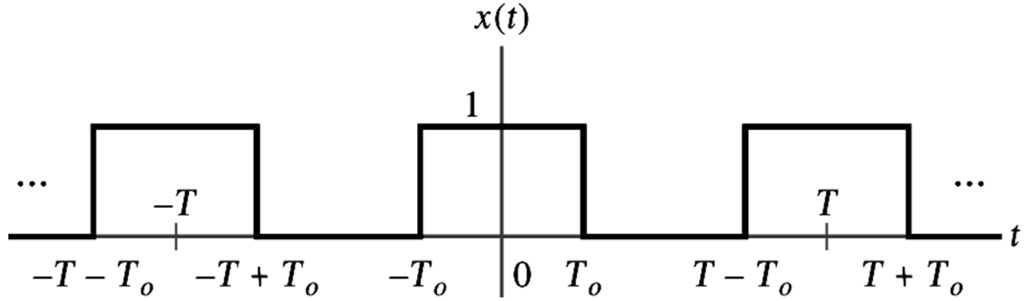
$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} = \sum_{k=0}^{\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{k=-1}^{-\infty} (1/2)^{-k} e^{jk\pi/20} e^{jk\pi t}$$

$$= \sum_{k=0}^{\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{l=1}^{\infty} (1/2)^l e^{-jl\pi/20} e^{-jl\pi t}$$

$$= \frac{1}{1-(1/2)e^{j(\pi t+\pi/20)}} + \frac{1}{1-(1/2)e^{-j(\pi t+\pi/20)}} - 1$$

# Example 3.13

Determine the FS representation of the square wave:




<Sol.>

1. The period is $T$, so the fundamental frequency $\omega_o = 2\pi/T$.
2. We consider the interval $-T/2 \le t \le T/2$ to obtain the FS coefficients. Then

(1) For $k \neq 0$, we have

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\omega_0 t} dt$$

$$= \frac{-1}{Tjk\omega_0} e^{-jk\omega_0 t}\Big|_{-T_0}^{T_0}, \quad k \neq 0$$

$$= \frac{2}{Tk\omega_0}\left(\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{2j}\right), \quad k \neq 0$$

$$= \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0}, \quad k \neq 0$$

(2) For $k = 0$, we have

$$X[0] = \frac{1}{T}\int_{-T_0}^{T_0} dt = \frac{2T_0}{T}$$

By means of L'Hôpital's rule

$$\lim_{k \to 0} \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0} = \frac{2T_0}{T}$$

$$X[k] = \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0}$$

# Example 3.13 (conti.)

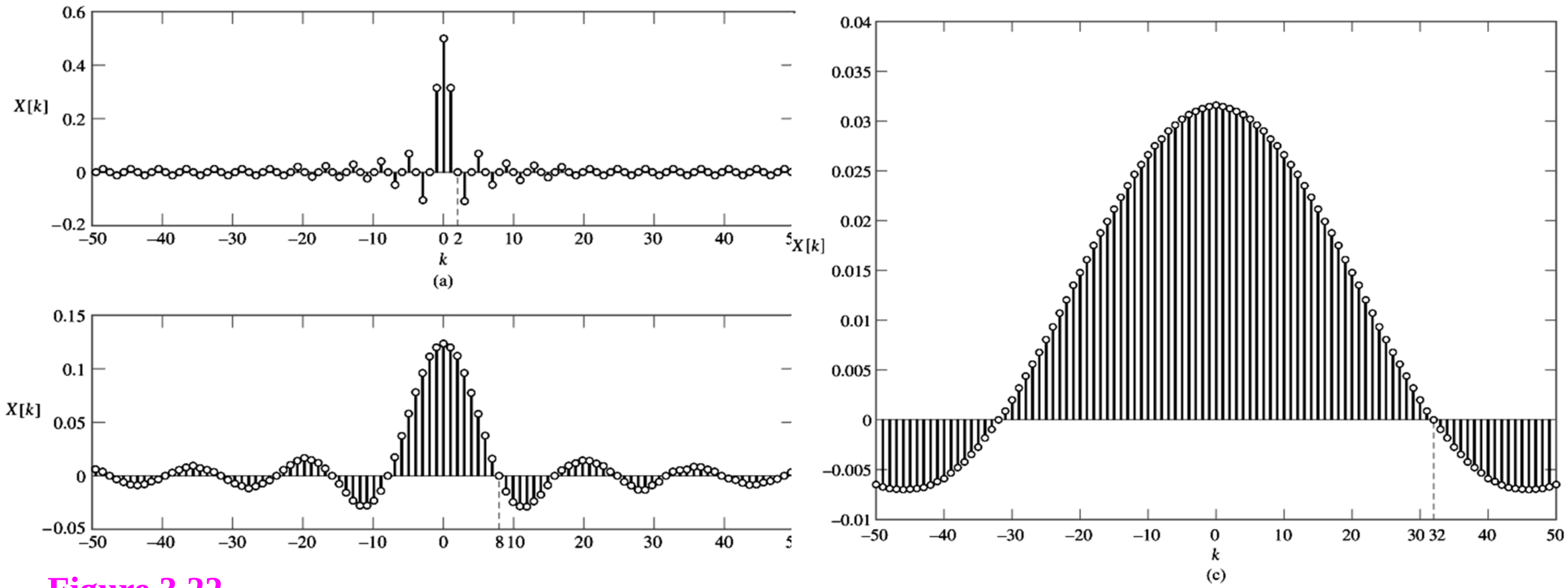
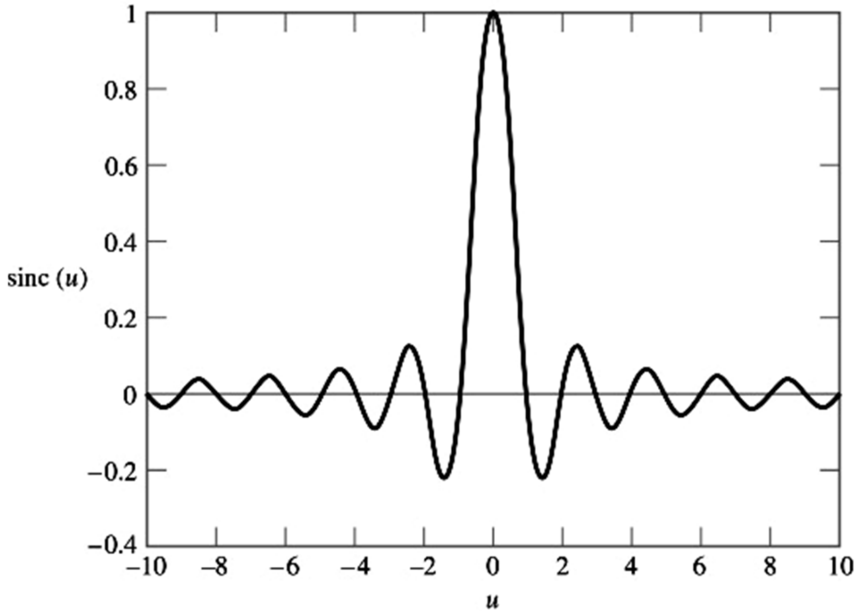



**Figure 3.22**

**The FS coefficients, $X[k]$, $-50 \le k \le 50$, for three square waves. (a) $T_o/T = 1/4$ . (b) $T_o/T = 1/16$. (c) $T_o/T = 1/64$.**

# Sinc Function $\mathrm{sinc}(u) = \dfrac{\sin(\pi u)}{\pi u}$




- Maximum of sinc($u$) is unity at $u$ = 0, the zero crossing occur at integer values of $u$, and the amplitude dies off as $1/u$.
- The portion of sinc($u$) between the zero crossings at $u = \pm 1$ is known as the *mainlobe* of the sinc function.
- The smaller ripples outside the mainlobe are termed *sidelobes*

# More on the FS Pairs

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

- The original FS pairs are described in exponential form:
- Let's consider the trigonometric form

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} = X[0] + \sum_{k=1}^{\infty} \left( X[k] e^{jk\omega_0 t} + X[-k] e^{-jk\omega_0 t} \right)$$

$$= X[0] + \sum_{k=1}^{\infty} \left( |X[k]| e^{jk\omega_0 t + j\arg\{X[k]\}} + |X[-k]| e^{-jk\omega_0 t + j\arg\lceil X[-k] \rceil} \right)$$

- For real-valued signal $x(t)$: $|X[k]| = |X[-k]|$ and $\arg\{X[-k]\} = -\arg\{X[k]\}$

$$\Rightarrow x(t) = X[0] + \sum_{k=1}^{\infty} |X[k]| \left( e^{j(k\omega_0 t + \arg\{X[k]\})} + e^{-j(k\omega_0 t + \arg\{X[k]\})} \right)$$

$$= X[0] + \sum_{k=1}^{\infty} 2|X[k]| \cos(k\omega_0 t + \arg\{X[k]\})$$

$$= X[0] + \sum_{k=1}^{\infty} 2|X[k]| \left( \cos(\arg\{X[k]\}) \cos(k\omega_0 t) - \sin(\arg\{X[k]\}) \sin(k\omega_0 t) \right)$$

Rewrite the signal as $x(t) = B[0] + \sum_{k=1}^{\infty} B[k] \cos(k\omega_0 t) + A[k] \sin(k\omega_0 t)$ we have

# Trigonometric FS Pair for Real Signals

$$x(t) = B[0] + \sum_{k=1}^{\infty} B[k]\cos(k\omega_0 t) + A[k]\sin(k\omega_0 t)$$

where

$$B[0] = X[0],$$

$$B[k] = 2|X[k]|\cos(\arg\{X[k]\}) = 2\,\mathrm{Re}\{X[k]\}$$

$$A[k] = -2|X[k]|\sin(\arg\{X[k]\}) = -2\,\mathrm{Im}\{X[k]\}$$

Or, if we use trigonometric FS representation for a real-valued periodic signal $x(t)$ with period $T$, then

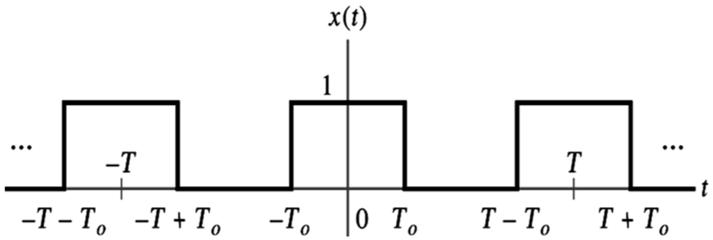
$$B[0] = \frac{1}{T}\int_0^T x(t)dt$$

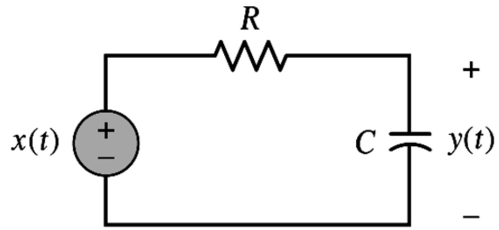
$$B[k] = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)dt$$

$$A[k] = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)dt$$

# Example 3.15

Let us find the FS representation for the output $y(t)$ of the $RC$ circuit in response to the square-wave input depicted in Fig. 3.21, assuming that $T_o/T = ¼$, $T = 1$ s, and $RC = 0.1$ s.




<Sol.>

1. If the input to an *LTI* system is expressed as a weighted sum of sinusoids (eigenfunctions), then the output is also a weighted sum of sinusoids.

2. Input:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$X[k] = \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0}$$

3. Output:

$$y(t) \xleftrightarrow{FS;\omega_0} Y[k] = H(jk\omega_0) X[k]$$

$$y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) X[k] e^{jk\omega_0 t}$$

4. Frequency response of the *RC* circuit:

$$H(j\omega) = \frac{1/RC}{j\omega + 1/RC}$$

5. Substituting for $H(jk\omega_o)$ with $RC = 0.1$ s and $\omega_o = 2\pi$, and $T_o/T = ¼$

$$Y[k] = \frac{10}{j2\pi k + 10} \frac{\sin(k\pi/2)}{k\pi}$$