Chapter 2: Time-Domain Representations of Linear Time-Invariant Systems

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Outline

- Characteristics of Systems Described by Differential and Difference Equations
- Block Diagram Representations
- State-Variable Descriptions of LTI Systems
- Exploring Concepts with MATLAB
- Summary

Block Diagram Representations

- A block diagram is a graphical representation of the elementary operations acting on the input signal

1. Scalar multiplication: $y(t) = cx(t)$, and $y[n] = cx[n]$
2. Addition: $y(t) = x(t) + w(t)$, and $y[n] = x[n] + w[n]$
3. Integration and time shift: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ and $y[n] = x[n-1]$.

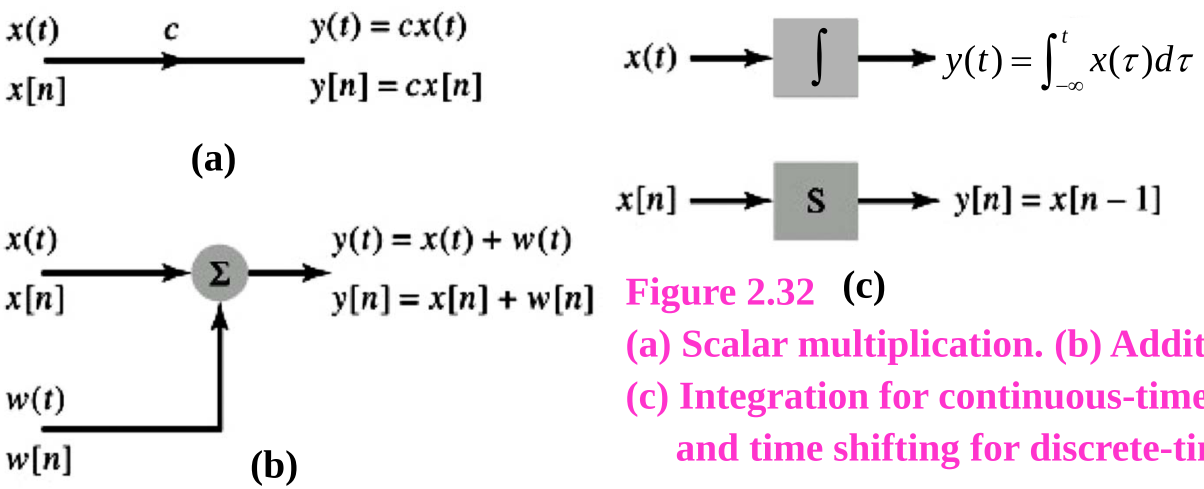
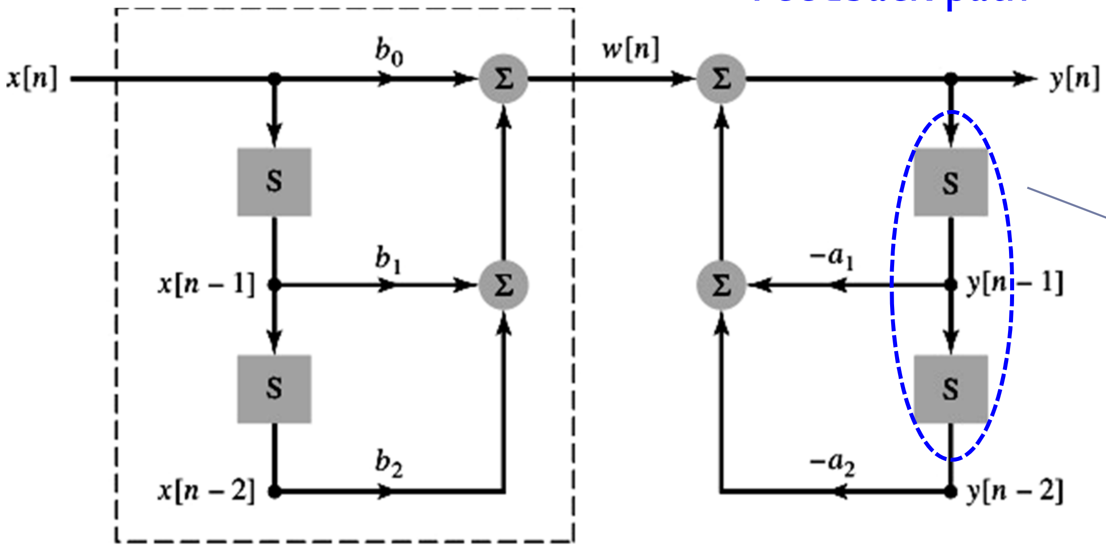


Figure 2.32
(a) Scalar multiplication. (b) Addition.
(c) Integration for continuous-time systems and time shifting for discrete-time systems.

Discrete-Time Block Diagram Representation Example

Cascade Form (Direct Form I)

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$



Second-order

4 storage elements

$$w[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$y[n] = w[n] - a_1 y[n-1] - a_2 y[n-2]$$

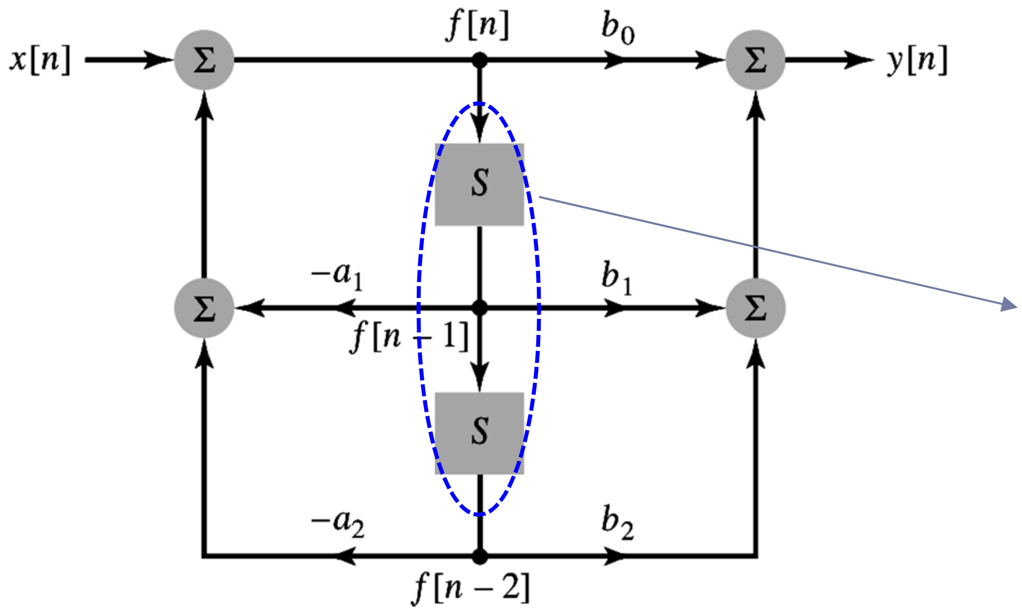
Discrete-Time Block Diagram Representation Example

Canonical Form (Direct Form II)

Interchange the order of Direct Form I.

$$y[n]+a_1y[n-1]+a_2y[n-2]=b_0x[n]+b_1x[n-1]+b_2x[n-2]$$

Feedback path Feed-forward path



Second-order

2 storage elements

$$f[n]=-a_1f[n-1]-a_2f[n-2]+x[n]$$

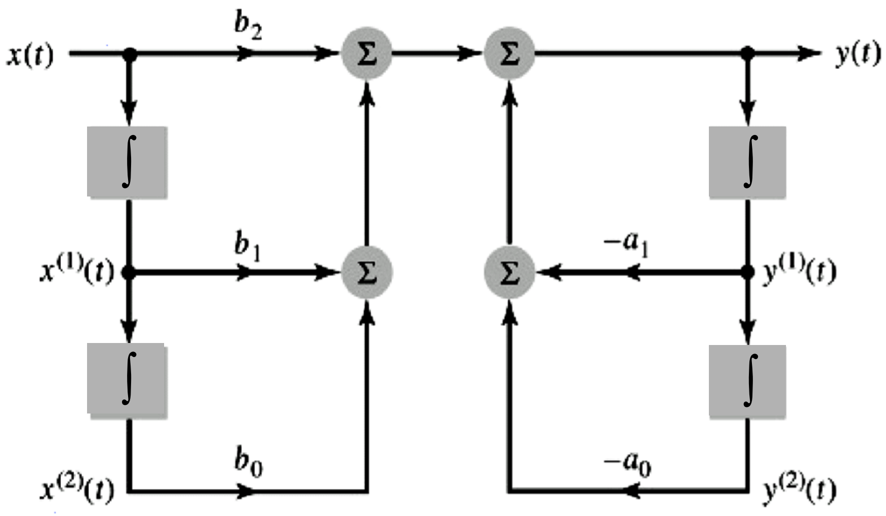
$$y[n]=b_0f[n]+b_1f[n-1]+b_2f[n-2]$$

Continuous-Time Block Diagram Representation

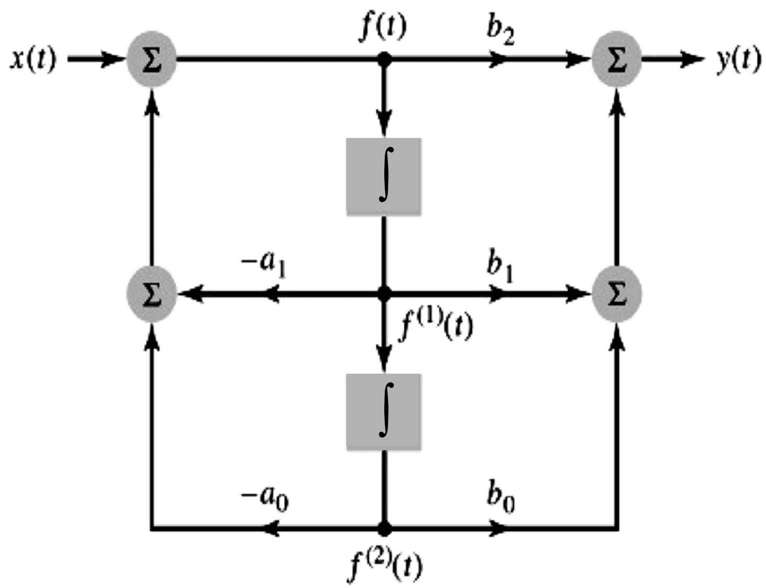
$$\frac{d}{dt}v^{(n)}(t)=v^{(n-1)}(t),\quad t>0 \quad \text{and} \quad n=1,\ 2,\ 3,\ \ldots \iff v^{(n)}(t)=\int_{-\infty}^{t}v^{(n-1)}(\tau)d\tau,\quad n=1,\ 2,\ 3,\ \ldots$$

Ex. Second-order system: $y(t)=-a_1y^{(1)}(t)-a_0y^{(2)}(t)+b_2x(t)+b_1x^{(1)}(t)+b_0x^{(2)}(t)$

Cascade Form (Direct Form I)

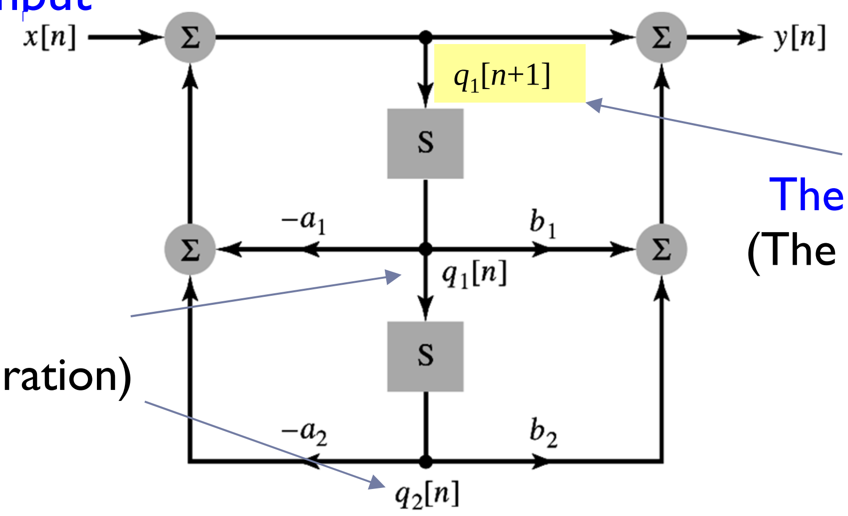


Canonical Form (Direct Form II)



State-Variable Description of LTI Systems

- The state-variable description consist of
  - A series of first-order differential or difference equations that describe how the state of the system evolves
  - An equation that relates the output of the system to the current state variables and input



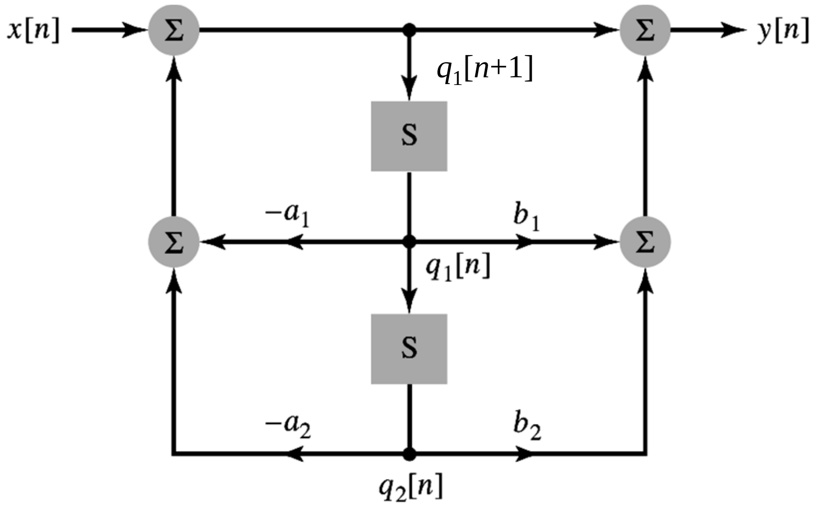
The next value of the state
(The input to the S operation)

State variables
(The output of the S operation)

- the state of a system may be defined as a minimal set of signals that represents the system's entire memory of the past

…. canonical Form

State-Variable Description Example



1. Start from the canonical form (i.e. direct form II) of the system
2. Choose state variables: $q_1[n]$ and $q_2[n]$.
3. State equation:

$$q_1[n+1] = -a_1 q_1[n] - a_2 q_2[n] + x[n]$$

$$q_2[n+1] = q_1[n]$$

3. Matrix Form of state equation:

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$$

4. Output equation:

$$y[n] = x[n] - a_1 q_1[n] - a_2 q_2[n] + b_1 q_1[n] + b_2 q_2[n],$$

$$y[n] = (b_1 - a_1) q_1[n] + (b_2 - a_2) q_2[n] + x[n]$$

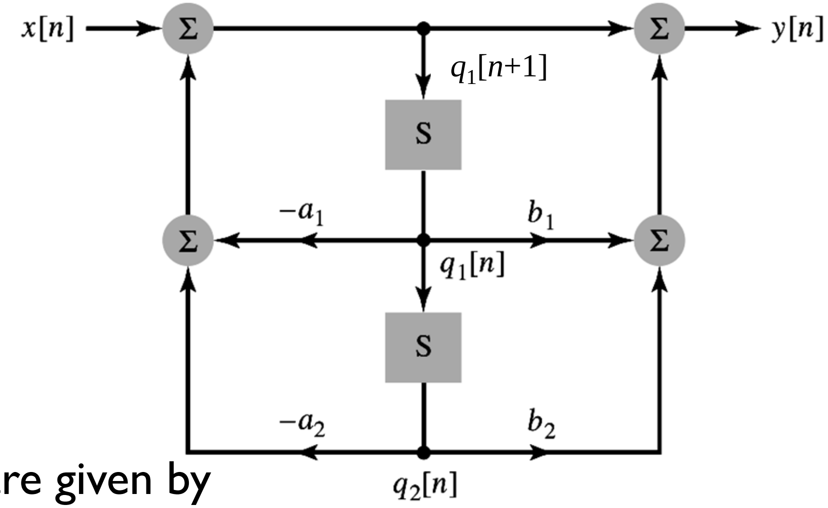
4. Matrix form of output equation:

$$y[n] = [b_1 - a_1 \quad b_2 - a_2] \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + [1] x[n]$$

State-Variable Description Example (conti.)

Define state vector as the column vector

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$



We can rewrite equations as

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + Dx[n]$$

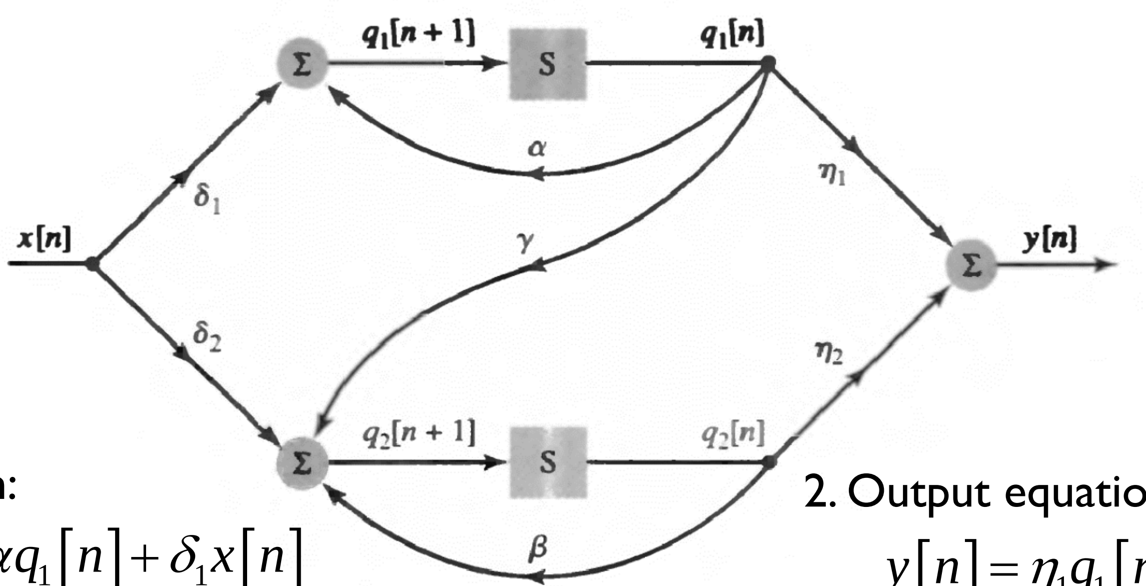
where matrix **A**, vectors **b** and **c**, and scalar *D* are given by

$$\mathbf{A} = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} b_1 - a_1 & b_2 - a_2 \end{bmatrix} \quad D = 1$$

1. Except block diagram, the state-variable describes the internal structure of the system
2. The state-variable description is used in any problem in which the internal system structure needs to be considered
3. For *N*-th order difference equation, the state vector ***q***[*n*] is $N\times 1$, and matrices **A** is $N\times N$, **b** is $N\times 1$, **c** is $1\times N$.

Example 2.28

Find the state-variable description corresponding to the system depicted in Fig. 2.40 by choosing the state variable to be the outputs of the unit delays.



<Sol.>

1. State equation:

$$q_1[n+1] = \alpha q_1[n] + \delta_1 x[n]$$

$$q_2[n+1] = \gamma q_1[n] + \beta q_2[n] + \delta_2 x[n]$$

2. Output equation:

$$y[n] = \eta_1 q_1[n] + \eta_2 q_2[n]$$

3. Define state vector as

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + Dx[n]$$

$$\mathbf{A} = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

$$\mathbf{c} = [\eta_1 \quad \eta_2] \quad D = [2]$$

State-Variable Description of Continuous-Time LTI System

- Recall discrete-time case

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$
$$y[n] = \mathbf{c}\mathbf{q}[n] + Dx[n]$$

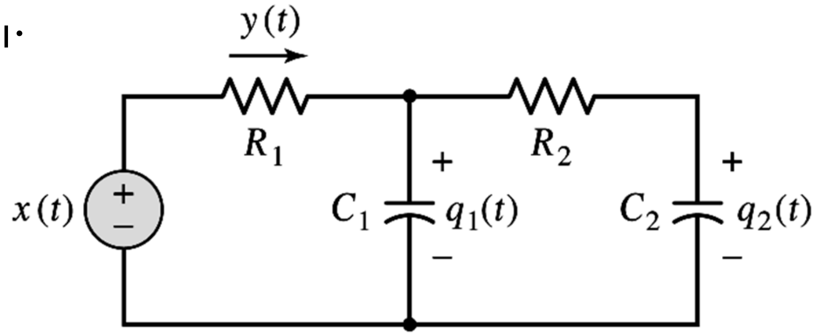
- For continuous-time LTI system

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$
$$y(t) = \mathbf{c}\mathbf{q}(t) + Dx(t)$$

  - States in continuous-time systems?
  - In electrical (continuous-time) systems, the (energy) storage devices are the capacitor and the inductor.

Example 2.29

Derive a state-variable description of the system if the input is the applied voltage $x(t)$ and the output is the current $y(t)$ through the resistor R_1.



<Sol.>

1. State variables: the voltage across each capacitor.
2. KVL Eq. for the loop involving $x(t)$, R_1, and C_1:

   $$x(t) = y(t)R_1 + q_1(t)$$

   KVL Eq. for the loop involving C_1, R_2, and C_2:

   $$q_1(t) = R_2 i_2(t) + q_2(t)$$

3. Current Eq:

   $$y(t) = i_1(t) + i_2(t)$$

   $$i_1(t) = C_1 \frac{d}{dt} q_1(t)$$

   $$i_2(t) = C_2 \frac{d}{dt} q_2(t)$$

$$y(t) = -\frac{1}{R_1} q_1(t) + \frac{1}{R_1} x(t)$$

Output eq.

$$\frac{d}{dt} q_2(t) = \frac{1}{C_2 R_2} q_1(t) - \frac{1}{C_2 R_2} q_2(t)$$

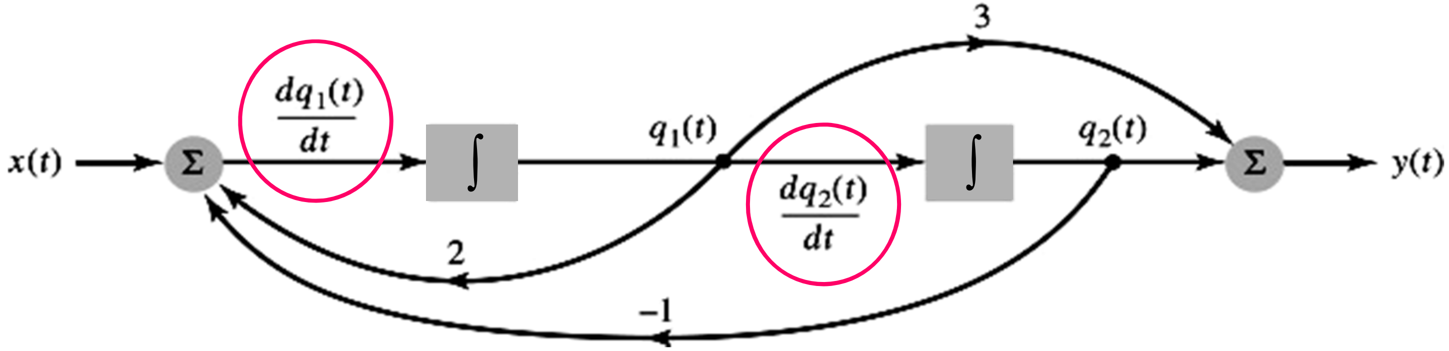
$$i_1(t) = y(t) - i_2(t)$$

$$= -\frac{1}{R_1} q_1(t) + \frac{1}{R_1} x(t) - \frac{1}{R_2} q_1(t) + \frac{1}{R_2} q_2(t)$$

$$\frac{d}{dt} q_1(t) = \frac{-1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) q_1(t) + \frac{1}{C_1 R_2} q_2(t) + \frac{1}{C_1 R_1} x(t)$$

Example 2.30

Determine the state-variable description corresponding to the block diagram.



<Sol.>

$$\frac{d}{dt}q_1(t) = 2q_1(t) - q_2(t) + x(t)$$

$$\frac{d}{dt}q_2(t) = q_1(t)$$

$$y(t) = 3q_1(t) + q_2(t)$$ Output eq.

State Transformation

- The transformation of states can be considered as a new set of state variables that are a weighted sum of the original ones.
- After state variables transformation, the IO characteristics of the system (or the system's behavior) is not changed.

1. Original state-variable description: $\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{b}x$

   $y = \mathbf{cq} + Dx$

2. Transformation: $\mathbf{q}' = \mathbf{Tq}$ ⇒ $\mathbf{q} = \mathbf{T}^{-1}\mathbf{q}'$

3. New state-variable description:

   1) State equation: $\dot{\mathbf{q}}' = \mathbf{TAq} + \mathbf{Tb}x.$ ⇒ $\dot{\mathbf{q}}' = \mathbf{TAT}^{-1}\mathbf{q}' + \mathbf{Tb}x.$

   2) Output equation: $y = \mathbf{cq} + Dx$ ⇒ $y = c\mathbf{T}^{-1}\mathbf{q}' + Dx$

   then

   $\dot{\mathbf{q}}' = \mathbf{A}'\mathbf{q}' + \mathbf{b}'x$ and $\mathbf{A}' = \mathbf{TAT}^{-1},\quad \mathbf{b}' = \mathbf{Tb},\quad \mathbf{c}' = \mathbf{cT}^{-1},\quad \text{and}\quad D' = D$

   $y = \mathbf{c}'\mathbf{q}' + D'x$ $\quad \mathbf{q}' = \mathbf{Tq}$