Chapter 2: Time-Domain Representations of Linear Time-Invariant Systems

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Outline

- Introduction
- The Convolution Sum
- Convolution Sum Evaluation Procedure
- The Convolution Integral
- Convolution Integral Evaluation Procedure
- Interconnections of LTI Systems
- Relations between LTI System Properties and the Impulse Response
- Step Response
- Differential and Difference Equation Representations
- Solving Differential and Difference Equations

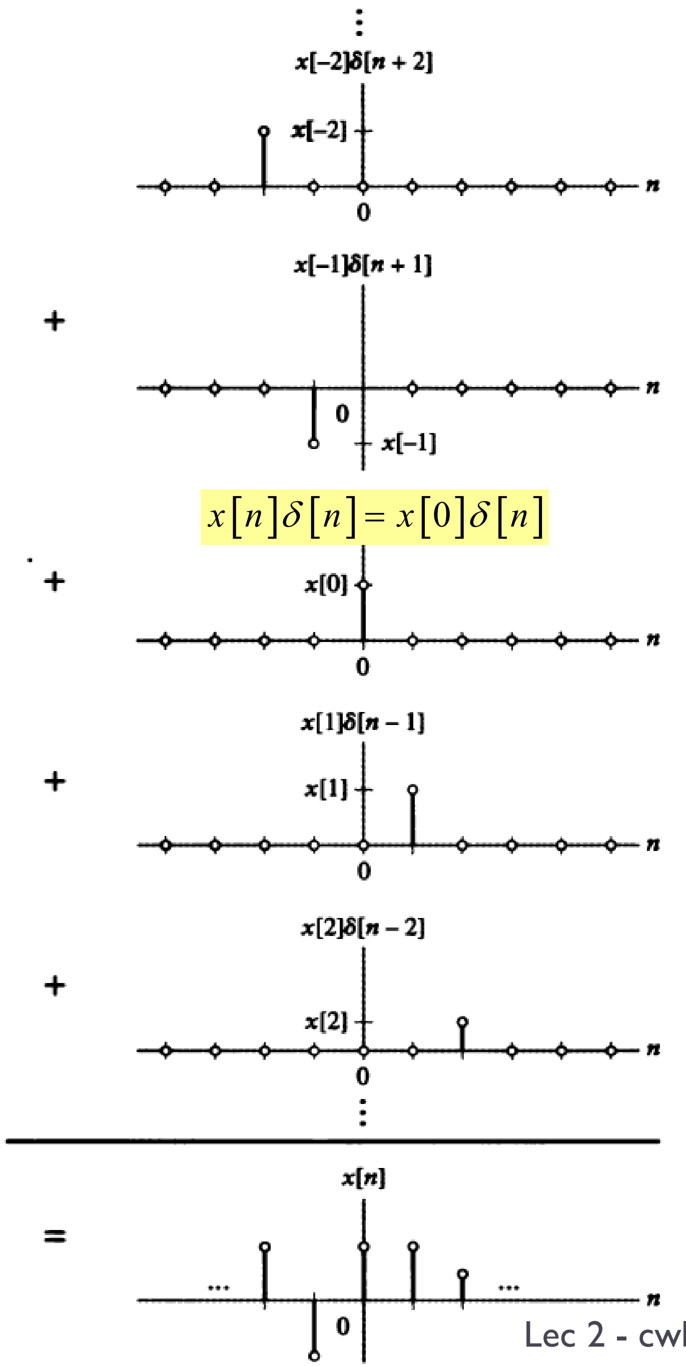
Outline

- Characteristics of Systems Described by Differential and Difference Equations
- Block Diagram Representations
- State-Variable Descriptions of LTI Systems
- Exploring Concepts with MATLAB
- Summary

Introduction

- Methods of time-domain characterizing an LTI system
  - An IO-relationship that both output signal and input signal are represented as functions of time
  - Impulse Response
    - The output of an LTI system due to a unit impulse signal input applied at time t=0 or n=0
  - Linear constant-coefficient differential or difference equation
  - Block Diagram
    - Graphical representation of an LTI system by scalar multiplication, addition, and a time shift (for discrete-time systems) or integration (for continuous-time systems)
  - State-Variable Description
    - A series of coupled equations representing the behavior of the system's states and relating states to the output of the system

An arbitrary signal is expressed as a weighted superposition of shifted impulses



$$x[n]\delta[n]=x[0]\delta[n]$$

$$x[n]\delta[n-k]=x[k]\delta[n-k]$$

$x[n]$ = entire signal;

$x[k]$ = specific value of the signal $x[n]$ at time k.

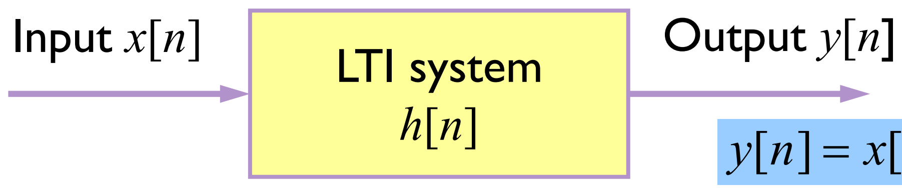
A series of time-shifted versions of the weighted impulse signal

$$x[n]=\cdots+x[-2]\delta[n+2]+x[-1]\delta[n+1]+x[0]\delta[n]+x[1]\delta[n-1]+x[2]\delta[n-2]+\cdots$$

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

The Convolution Sum and The Impulse Response

- An arbitrary signal can be expressed as a weighted superposition of shifted impulses
  - The weights are just the input sample values at the corresponding time shifts
- Impulse response of LTI system $H\{\cdot\}$: $h[n] \equiv H\{\delta[n]\}$



$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$y[n] = H\{x[n]\} = H\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\}$$

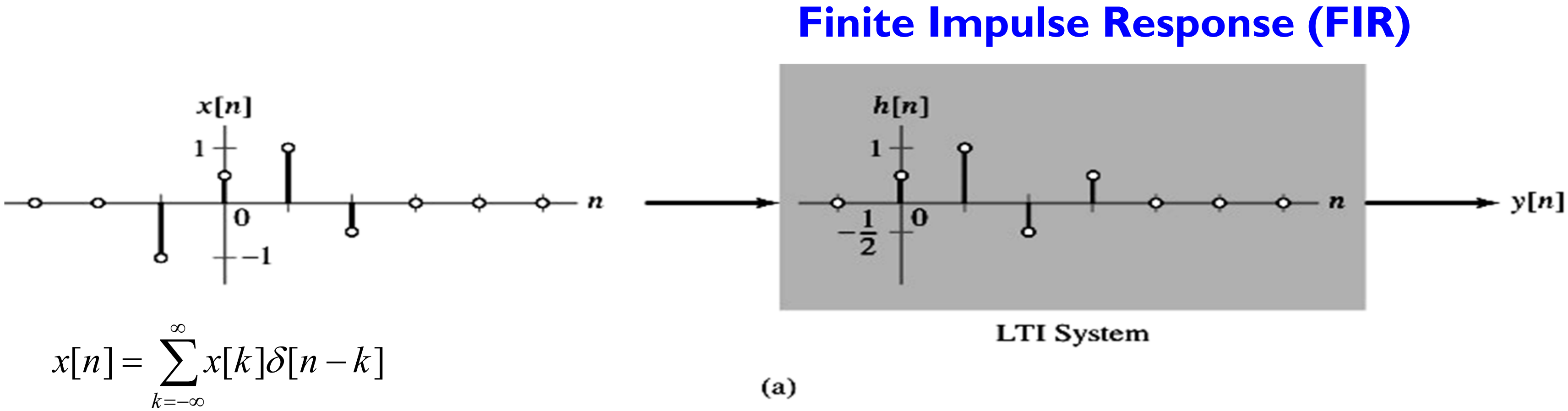
linear, time-invariant

$$= \sum_{k=-\infty}^{\infty} x[k] H\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

Convolution sum $\equiv x[n] * h[n]$

Convolution Sum

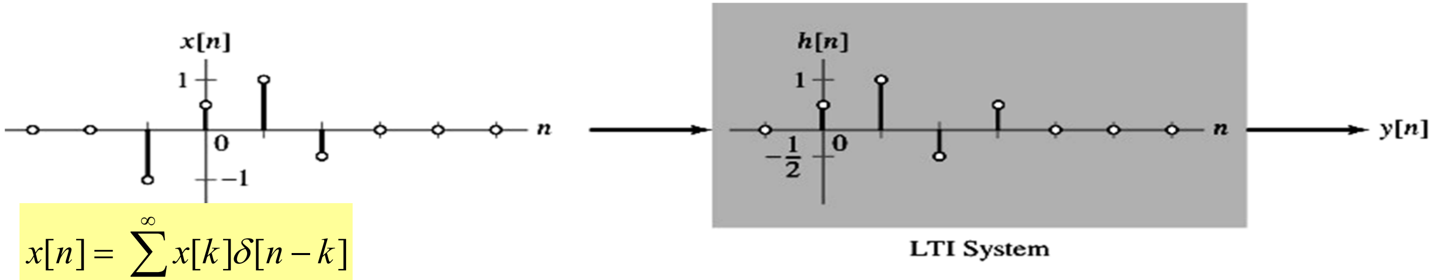
- Example



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

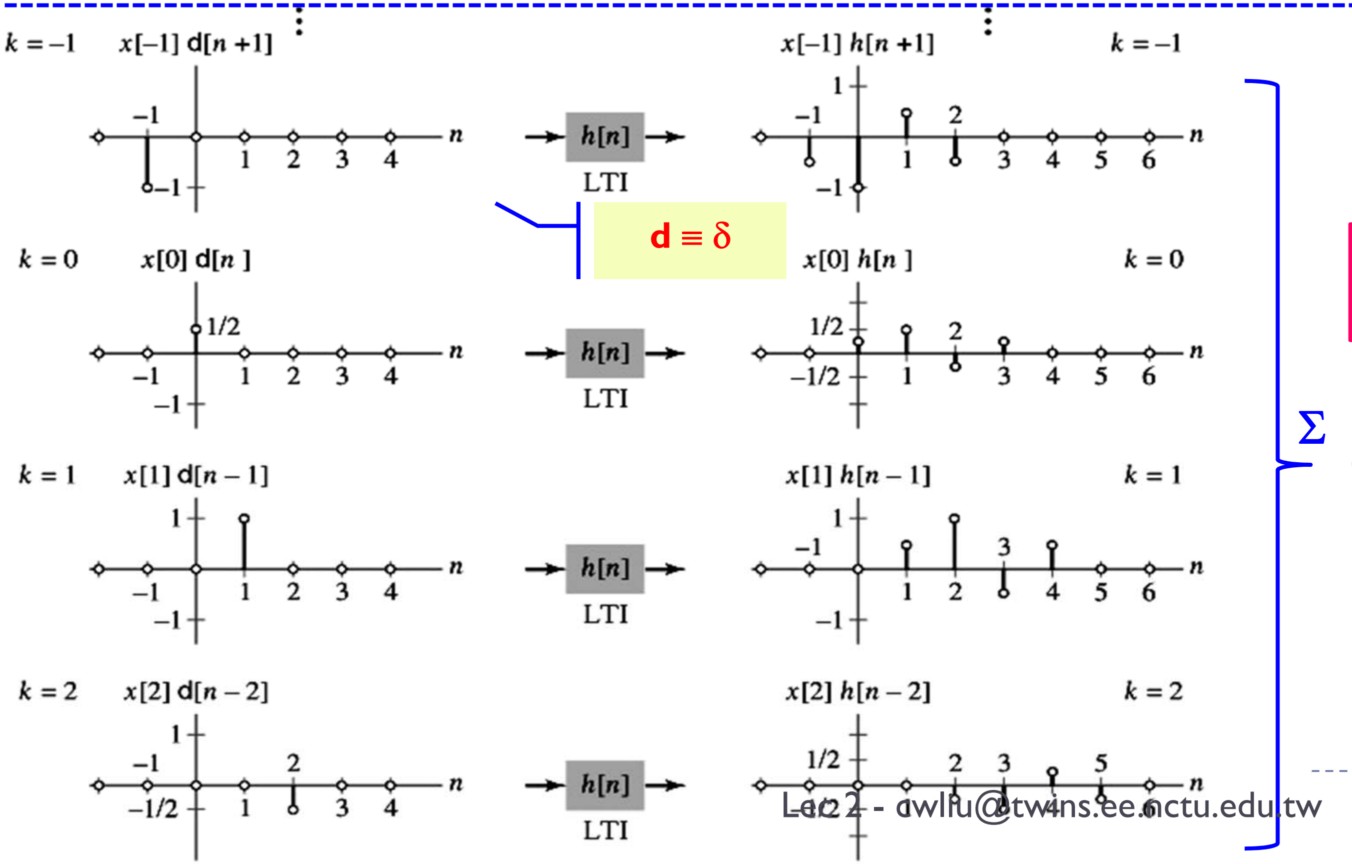
$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$???

Convolution Sum?



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

(a)



$$x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

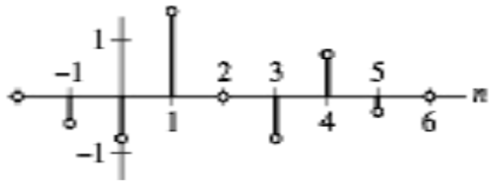


Illustration of Convolution Sum/Integral

- Continuous-time signals

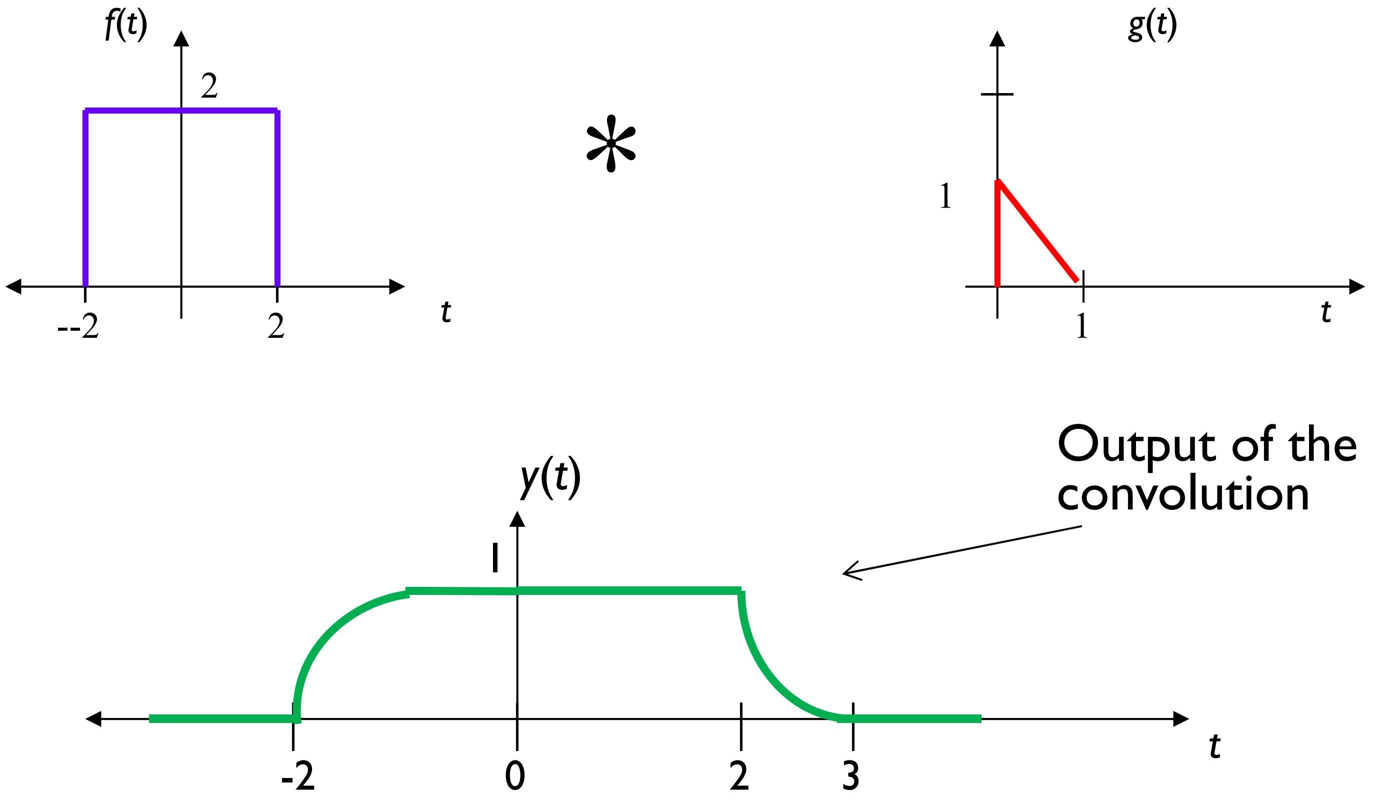
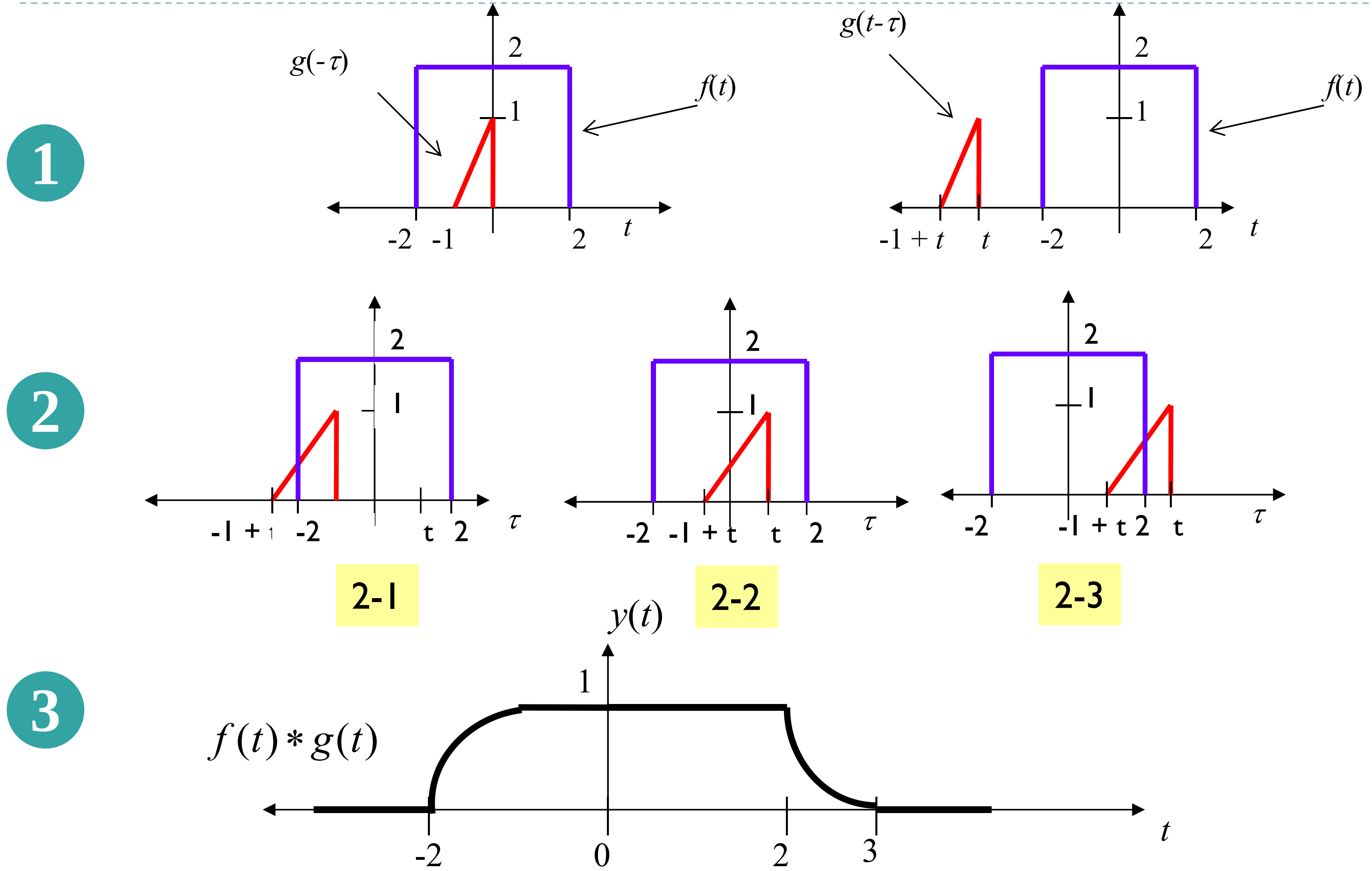


Illustration of Convolution Sum



Convolution Sum Evaluation Procedure

- Reflect and shift convolution sum evaluation

1. Graph both $x[k]$ and $h[n-k]$ as a function of the independent variable k. To determine $h[n-k]$, first reflect $h[k]$ about $k=0$ to obtain $h[-k]$. Then shift by $-n$.
2. Begin with n large and negative. That is, shift $h[-k]$ to the far left on the time axis.
3. Write the mathematical representation for the intermediate signal $\omega_n[k]=x[k]h[n-k]$.
4. Increase the shift n (i.e., move $h[n-k]$ toward the right) until the mathematical representation for $\omega_n[k]$ changes. The value of n at which the change occurs defines the end of the current interval and the beginning of a new interval.
5. Let n be in the new interval. Repeat step 3 and 4 until all intervals of times shifts and the corresponding mathematical representations for $\omega_n[k]$ are identified. This usually implies increasing n to a very large positive number.
6. For each interval of time shifts, sum all the values of the corresponding $\omega_n[k]$ to obtain $y[n]$ on that interval

$$y[n] = \sum_{k=-\infty}^{\infty} w_n[k] \quad (2.6)$$

Example 2.2

Consider a system with impulse response $h[n]=\left(\frac{3}{4}\right)^n u[n]$. Use Eq. (2.6) to determine the output of the system at time n=– 5, 5, and 10 when the input is $x[n]=u[n]$.

Sol:

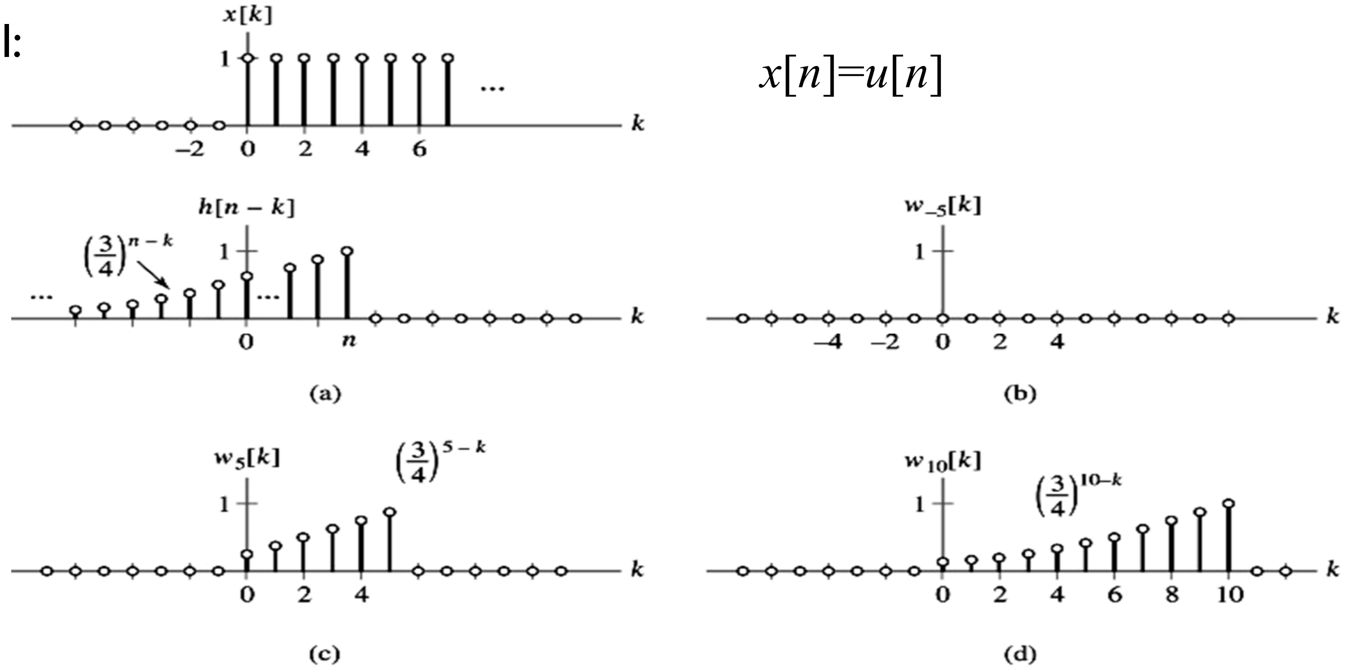


Figure 2.3 (a) The input signal $x[k]$ above the reflected and time-shifted impulse response $h[n – k]$, depicted as a function of k. (b) The product signal $w_5[k]$ used to evaluate y [–5]. (c) The product signal $w_5[k]$ to evaluate $y[5]$. (d) The product signal $w_{10}[k]$ to evaluate $y[10]$.

Example 2.2 (conti.)

1. $h[n-k]$:

$$h[n-k]=\begin{cases}\left(\frac{3}{4}\right)^{n-k}, & k \le n \\ 0, & \text{otherwise}\end{cases}$$

2. Intermediate signal $w_n[k]$:

For $n = -5$: $w_{-5}[k]=0$

Eq. (2.6) → $y[-5] = 0$

For $n = 5$:

$$w_5[k]=\begin{cases}\left(\frac{3}{4}\right)^{5-k}, & 0 \le k \le 5 \\ 0, & \text{otherwise}\end{cases}$$

Eq. (2.6) → $$y[5]=\sum_{k=0}^{5}\left(\frac{3}{4}\right)^{5-k}$$

$$y[5]=\left(\frac{3}{4}\right)^{5}\sum_{k=0}^{5}\left(\frac{4}{3}\right)^{k}=\left(\frac{3}{4}\right)^{5}\frac{1-\left(\frac{4}{3}\right)^{6}}{1-\left(\frac{4}{3}\right)}=3.288$$

For $n = 10$:

$$w_{10}[k]=\begin{cases}\left(\frac{3}{4}\right)^{10-k}, & 0 \le k \le 10 \\ 0, & \text{otherwise}\end{cases}$$

Eq. (2.6) →

$$y[10]=\sum_{k=0}^{10}\left(\frac{3}{4}\right)^{10-k}=\left(\frac{3}{4}\right)^{10}\sum_{k=0}^{10}\left(\frac{4}{3}\right)^{k}=\left(\frac{3}{4}\right)^{10}\frac{1-\left(\frac{4}{3}\right)^{11}}{1-\left(\frac{4}{3}\right)}$$

$$=3.831$$

Example 2.3 Moving-Average System

The output $y[n]$ of the four-point moving-average system is related to the input $x[n]$ by

$$y[n]=\frac{1}{4}\sum_{k=0}^{3}x[n-k]$$

Determine the output of the system when the input is $x[n]=u[n]-u[n-10]$

<Sol.>

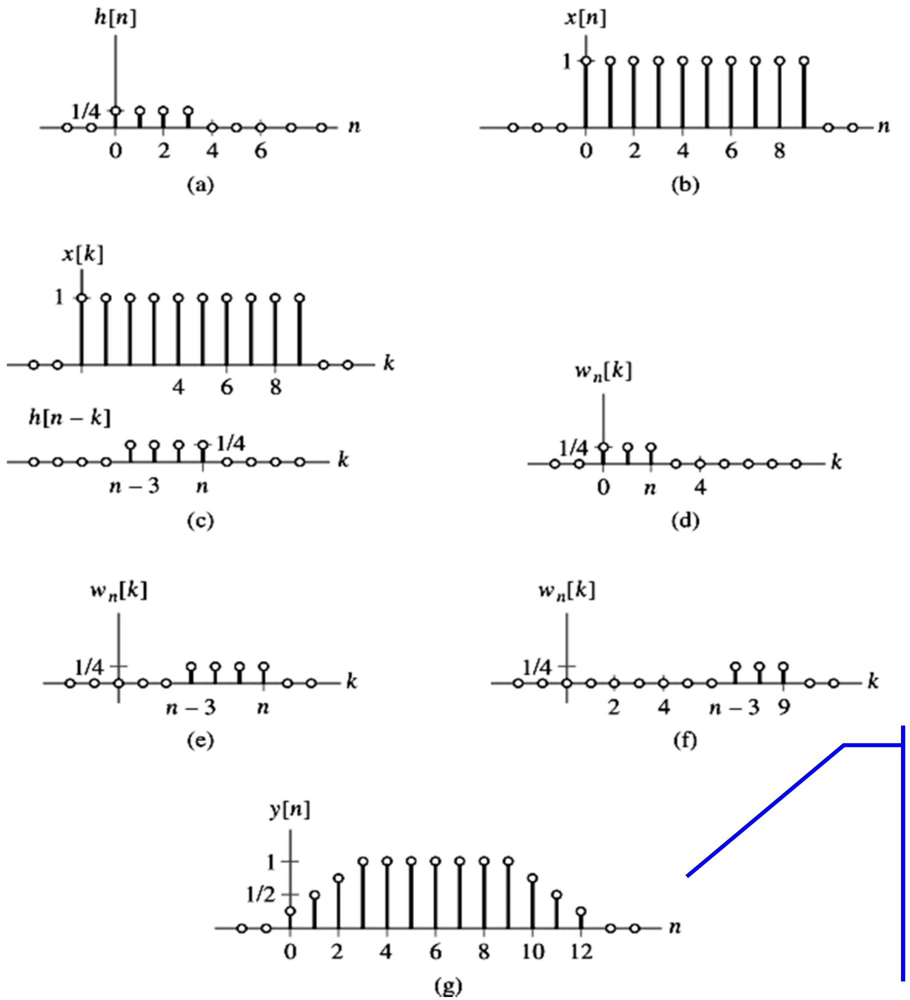
1. First find the impulse response $h[n]$ of this system by letting $x[n] = \delta[n]$, which yields

$$h[n]=\frac{1}{4}\left(u[n]-u[n-4]\right)$$

2. Reflect and shift convolution sum evaluation

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}w_n[k] \qquad (2.6)$$

Example 2.3 (conti.)



For $n < 0$ and $n > 12$: $y[n] = 0$.

d) For $0 \le n \le 3$:

$$y[n] = \sum_{k=0}^{n} 1/4 = \frac{n+1}{4}$$

e) For $3 < n \le 9$:

$$y[n] = \sum_{k=n-3}^{n} 1/4 = \frac{1}{4}\left(n - (n-3) + 1\right) = 1$$

f) For $9 < n \le 12$:

$$y[n] = \sum_{k=n-3}^{9} 1/4 = \frac{1}{4}\left(9 - (n-3) + 1\right) = \frac{13-n}{4}$$

1'st interval: $n < 0$
2'nd interval: $0 \le n \le 3$
3'rd interval: $3 < n \le 9$
4th interval: $9 < n \le 12$
5th interval: $n > 12$

Example 2.4
Infinite Impulse Response (IIR) System

The input-output relationship for the first-order recursive system is given by

$$y[n]-\rho y[n-1]=x[n]$$

Determine the output of the system when the input is $x[n]=b^n u[n+4]$, assuming that $b \neq \rho$ and that the system is causal.

<Sol.>

1. First find the impulse response $h[n]$ of this system by letting $x[n] = \delta[n]$, which yields

$$h[n]=\rho h[n-1]+\delta[n]$$

Since the system is causal, we have $h[n] = 0$ for $n < 0$. For $n = 0, 1, 2, \ldots,$ we find that $h[0] = 1$, $h[1] = \rho$, $h[2] = \rho^2, \ldots,$ or $\boxed{h[n]=\rho^n u[n]}$ **Infinite impulse response (IIR)**

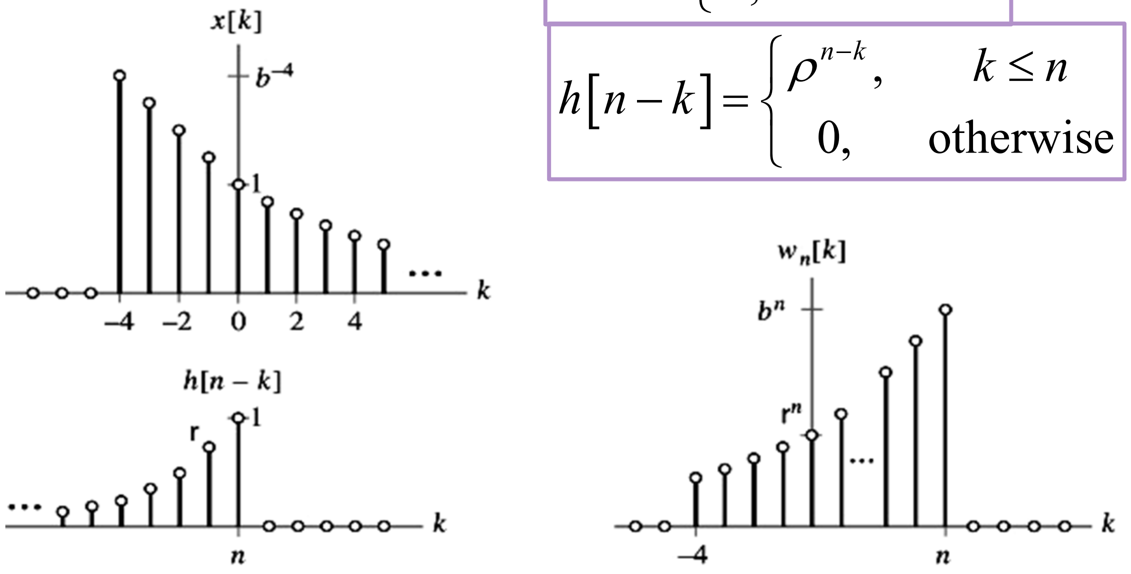
2. Reflect and shift convolution sum evaluation

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=\sum_{k=-\infty}^{\infty} w_n[k] \qquad (2.6)$$

Example 2.4 (conti.)

$$x[k]=\begin{cases} b^k, & -4\le k \\ 0, & \text{otherwise}\end{cases}$$

$$h[n-k]=\begin{cases} \rho^{n-k}, & k\le n \\ 0, & \text{otherwise}\end{cases}$$



(a) (b)

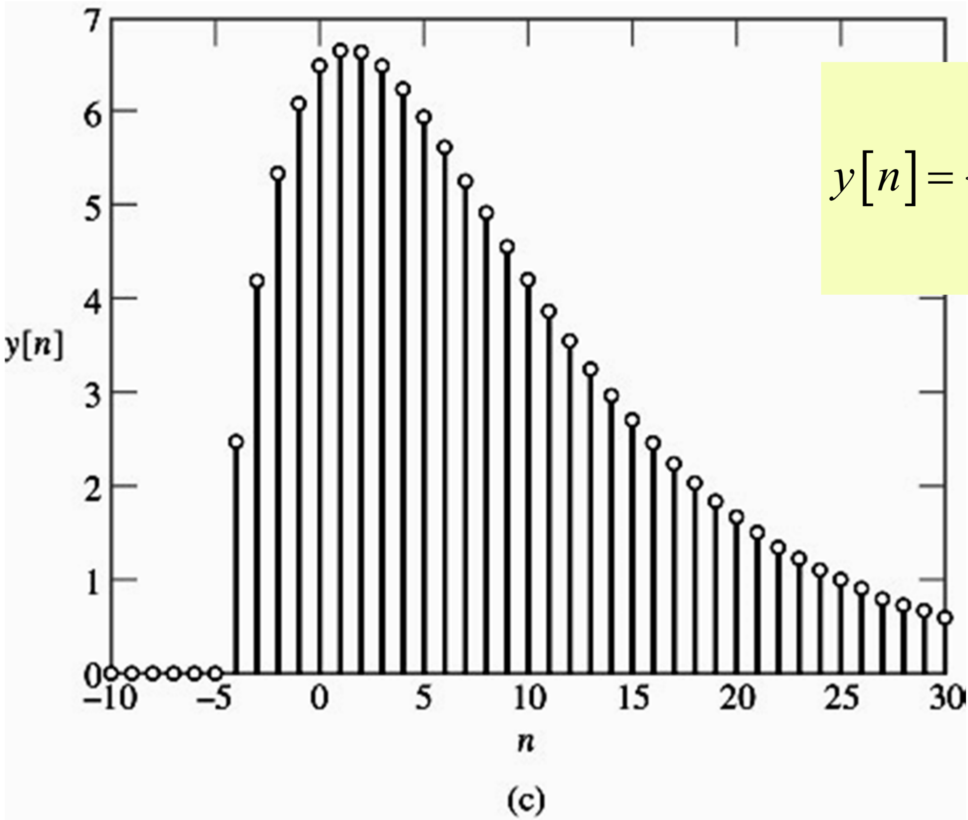
$$w_n[k]=\begin{cases} b^k\rho^{n-k}, & -4\le k\le n \\ 0, & \text{otherwise}\end{cases}$$

1) For $n < -4$: $y[n]=0$.

2) For $n \ge -4$:

$$y[n]=\sum_{k=-4}^{n} b^k\rho^{n-k}=\rho^n\sum_{k=-4}^{n}\left(\frac{b}{\rho}\right)^k=\rho^n\frac{\left(\frac{b}{\rho}\right)^{-4}\left(1-\left(\frac{b}{\rho}\right)^{n+5}\right)}{1-\frac{b}{\rho}}$$

Example 2.4 (conti.)



$$y[n]=\begin{cases} 0, & n<-4 \\ b^{-4}\left(\dfrac{\rho^{n+5}-b^{n+5}}{\rho-b}\right), & -4\le n \end{cases}$$

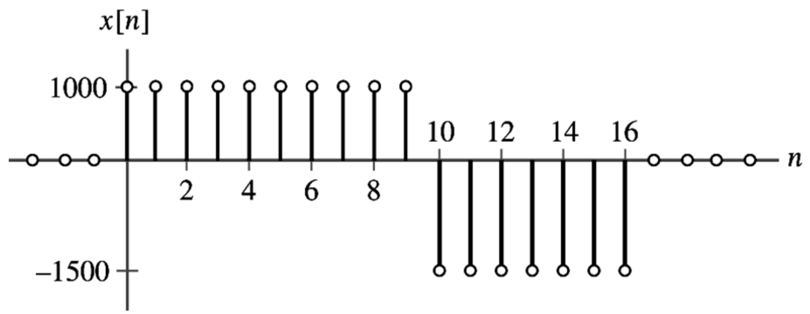
Figure 2.5c (p. 110)

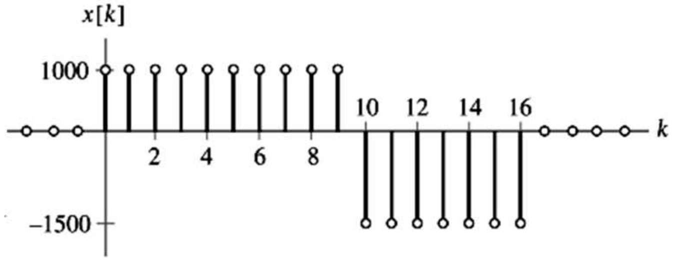
(c) The output *y*[*n*] assuming that *p* = 0.9 and *b* = 0.8.

Example 2.5 Investment Computation

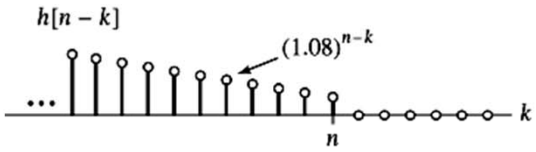
- $y[n]$: the investment at the start of period n
- Suppose an interest at a fixed rate per period $r\%$, then the investment compound growth rate is $\rho=1+r\%$
- If there is no deposits or withdrawals, then $y[n]=\rho y[n-1]$
- If a there is deposits or withdrawals occurred at the start of period *n*, says *x*[*n*], then $y[n]=\rho y[n-1]+x[n]$

- Please find the value of an investment earing 8% per year if $1000 is deposited at the start of each year for 10 years and then $1500 is withdrawn at the start of each year for 7 years.

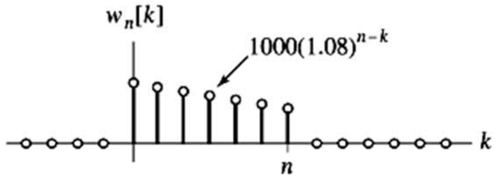




$$w_n[k]=\begin{cases}1000(1.08)^{n-k}, & 0\le k\le n\\ 0, & \text{otherwise}\end{cases}$$

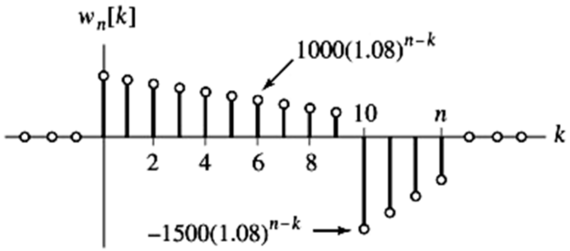


(a)

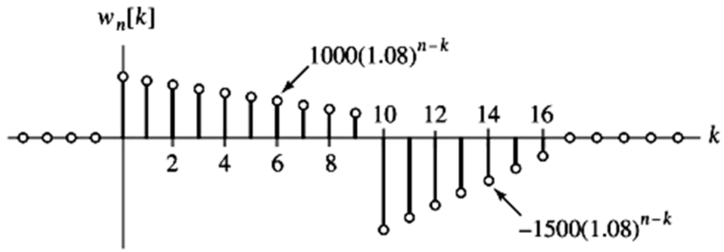


(b)

$0 \le n \le 9.$



(c)



(d)

$10 \le n \le 16$

$$w_n[k]=\begin{cases}1000(1.08)^{n-k}, & 0\le k\le 9\\ -1500(1.08)^{n-k}, & 10\le k\le n\\ 0, & \text{otherwise}\end{cases}$$

$17 \le n$

$$w_n[k]=\begin{cases}1000(1.08)^{n-k}, & 0\le k\le 9\\ -1500(1.08)^{n-k}, & 10\le k\le 16\\ 0, & \text{otherwise}\end{cases}$$

Example 2.5 (conti.)

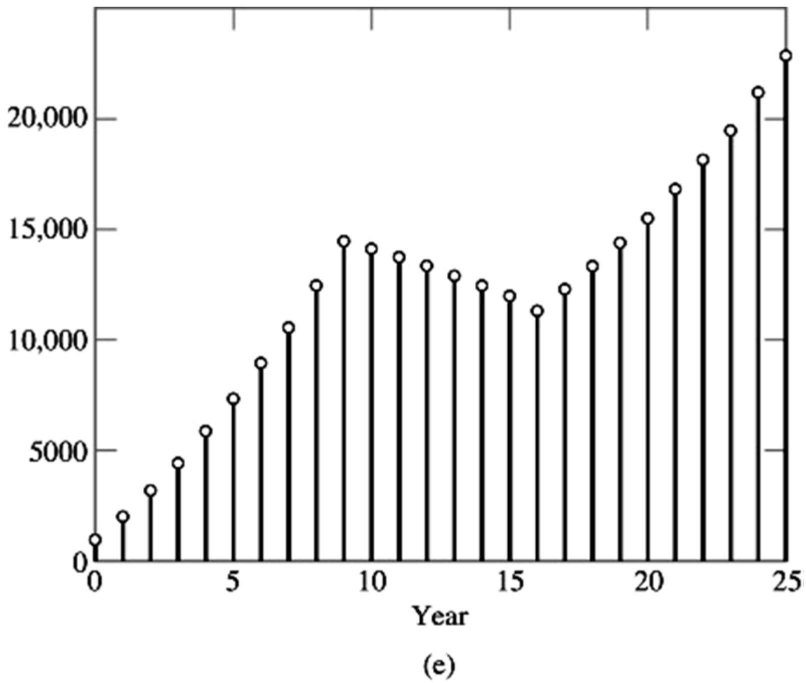


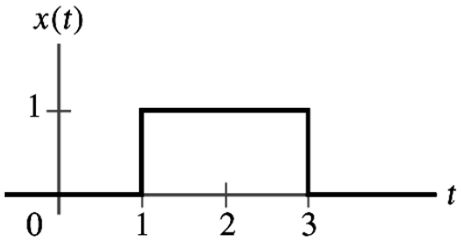
Figure 2.7
(e) The output *y*[*n*] representing the value of the investment immediately after the deposit or withdrawal at the start of year *n*.

Continuous Integral $x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$

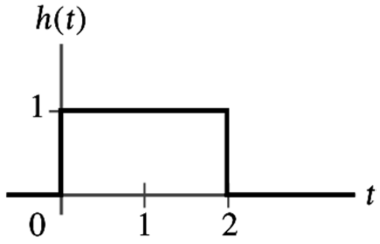
- Reflect-and-shift continuous integral evaluation (analogous to the continuous sum)
  1. Graph both $x(\tau)$ and $h(t-\tau)$ as a function of the independent variable τ
  2. Begin with the shift t large and negative, i.e. shift $h(-\tau)$ to the far left on the time axis to obtain $h(t-\tau)$
  3. Write the mathematical representation for the intermediate signal $w_t(\tau)=x(\tau)h(t-\tau)$.
  4. Increase the shift t (i.e. move $h(t-\tau)$ toward the right) until the mathematical representation of $w_t(\tau)$ changes. The value t at which the change occurs defines the end of the current set and the beginning of a new set.
  5. Let t be in the new set. Repeat step 3 and 4 until all sets of shifts t and the corresponding $w_t(\tau)$ are identified. This usually implies increasing t to a very large positive number.
  6. For each sets of shifts t, integrate $w_t(\tau)$ from $\tau=-\infty$ to $\tau=\infty$, $\int_{-\infty}^{\infty}w_t(\tau)d\tau$ to obtain $x(t)*h(t)$.

Example 2.6

Given



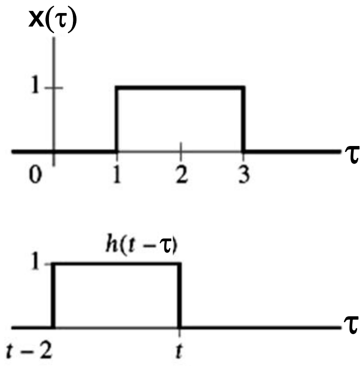
and



$$x(t) = u(t-1) - u(t-3) \qquad h(t) = u(t) - u(t-2)$$

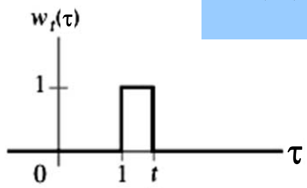
Evaluate the convolution integral $y(t) = x(t) * h(t)$.

<Sol.>



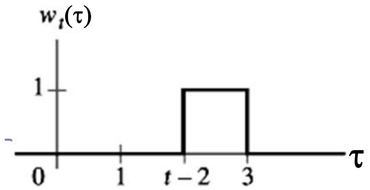
(a)

$$w_t(\tau) = \begin{cases} 1, & 1 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

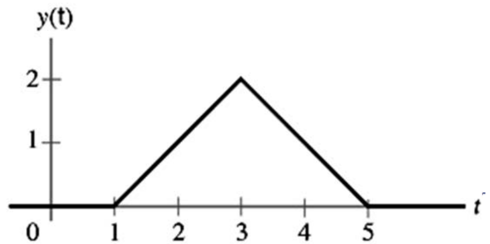


(b)

$$w_t(\tau) = \begin{cases} 1, & t-2 < \tau < 3 \\ 0, & \text{otherwise} \end{cases}$$



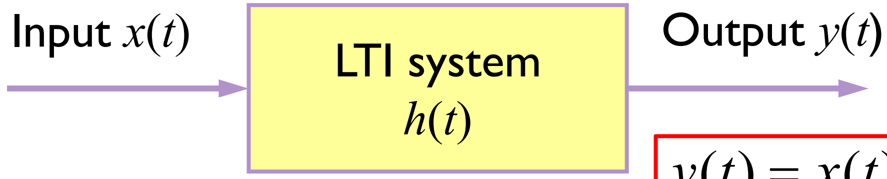
(c)



(d)

$$y(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 3 \\ 5-t, & 3 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

Impulse Response of Continuous-Time LTI System $h(t) \equiv H\{\delta(t)\}$



$$y(t) = x(t) * h(t)$$

- Recall that $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$

Linear $= \int_{-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}d\tau$

Time-invariant $= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

$$\equiv x(t) * h(t)$$

The output is a weighted superposition of impulse responses time shifted by τ

Example 1.21&2.7 RC Circuit System

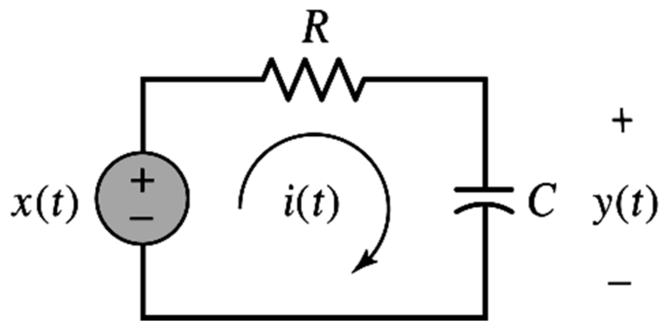
According to KVL, we have $Ri(t) + y(t) = x(t)$, i.e. $RC\dfrac{dy(t)}{dt} + y(t) = x(t)$

If $x(t) = u(t)$, i.e. the step response, the solution is

$$y(t) = (1 - e^{-(t/RC)})u(t)$$

Please find the impulse response of the RC circuit

<Sol.>



- RC circuit is LTI system.

$$\begin{cases} x_1(t) = \dfrac{1}{\Delta}u(t + \dfrac{\Delta}{2}) \\ x_2(t) = \dfrac{1}{\Delta}u(t - \dfrac{\Delta}{2}) \end{cases}$$

$$y_1 = \frac{1}{\Delta}\left[1 - e^{-\left(t+\frac{\Delta}{2}\right)/(RC)}\right]u\left(t + \frac{\Delta}{2}\right), \quad x(t) = x_1(t)$$

$$y_2 = \frac{1}{\Delta}\left[1 - e^{-\left(t-\frac{\Delta}{2}\right)/(RC)}\right]u\left(t - \frac{\Delta}{2}\right), \quad x(t) = x_2(t)$$

$$x_\Delta(t) = x_1(t) - x_2(t)$$

$$y_\Delta(t) = \frac{1}{\Delta}(1 - e^{-((t+\Delta/2)/(RC))})u(t + \Delta/2) - \frac{1}{\Delta}(1 - e^{-((t-\Delta/2)/(RC))})u(t - \Delta/2)$$

$$= \frac{1}{\Delta}(u(t+\Delta/2) - u(t-\Delta/2)) - \frac{1}{\Delta}(e^{-((t+\Delta/2)/(RC))}u(t+\Delta/2) - e^{-((t-\Delta/2)/(RC))}u(t-\Delta/2))$$

$$\delta(t) = \lim_{\Delta \to 0} x_\Delta(t)$$

$$y(t) = \lim_{\Delta \to 0} y_\Delta(t)$$

$$= \delta(t) - \frac{d}{dt}\left(e^{-t/(RC)}u(t)\right)$$

$$= \delta(t) - e^{-t/(RC)}\frac{d}{dt}u(t) - u(t)\frac{d}{dt}\left(e^{-t/(RC)}\right)$$

$$= \delta(t) - e^{-t/(RC)}\delta(t) + \frac{1}{RC}e^{-t/(RC)}u(t), \quad x(t) = \delta(t)$$

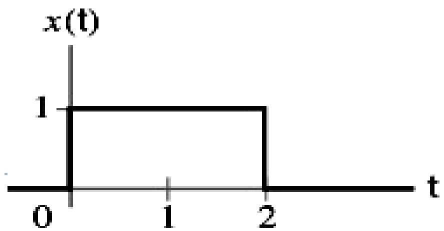
Cancel each other!

$$y(t) = \frac{1}{RC}e^{-t/(RC)}u(t), \quad x(t) = \delta(t) \quad \textbf{(1.93)}$$

i.e. the impulse response of the RC circuit system

Example 2.7 – RC Circuit Output

We now assume the time constant in the RC circuit system is RC = 1s. Use convolution to determine the voltage across the capacitor, y(t), resulting from an input voltage $x(t) = u(t) - u(t-2)$.



Example 2.7 (conti.)

<Sol.> RC circuit is LTI system, so $y(t) = x(t) * h(t)$.

1. Graph of $x(\tau)$ and $h(t-\tau)$:

$$x(\tau)=\begin{cases}1, & 0<\tau<2\\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad h(t-\tau)=e^{-(t-\tau)}u(t-\tau)=\begin{cases}e^{-(t-\tau)}, & \tau<t\\ 0, & \text{otherwise}\end{cases}$$

2. Intervals of time shifts:

(1). For $t<0$, $w_t(\tau)=0$

(2). For $0 \le t < 2$, $w_t(\tau)=\begin{cases}e^{-(t-\tau)}, & 0<\tau<t\\ 0, & \text{otherwise}\end{cases}$

(3). For $2 \le t$,

$$w_t(\tau)=\begin{cases}e^{-(t-\tau)}, & 0<\tau<2\\ 0, & \text{otherwise}\end{cases}$$

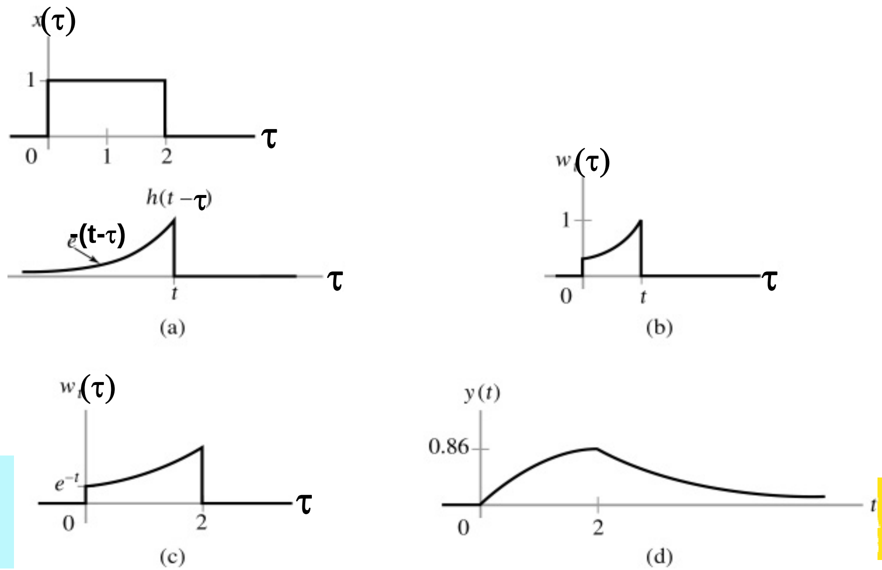
3. Convolution integral:

2) For second interval $0 \le t < 2$:

$$y(t)=\int_0^t e^{-(t-\tau)}d\tau = e^{-t}\left(e^{\tau}\Big|_0^t\right)=1-e^{-t}$$

3) For third interval $2 \le t$:

$$y(t)=\int_0^2 e^{-(t-\tau)}d\tau = e^{-t}\left(e^{\tau}\Big|_0^2\right)=\left(e^2-1\right)e^{-t}$$



Example 2.8

Suppose that the input $x(t)$ and impulse response $h(t)$ of an *LTI* system are

$$x(t)=(t-1)\left[u(t-1)-u(t-3)\right] \text{ and } h(t)=u(t+1)-2u(t-2)$$

Find the output of the system.

<Sol.>

There are five intervals

1'st interval: $t<0$
2'nd interval: $0\le t<2$
3'rd interval: $2\le t<3$
4th interval: $3\le t<5$
5th interval: $t\ge 5$

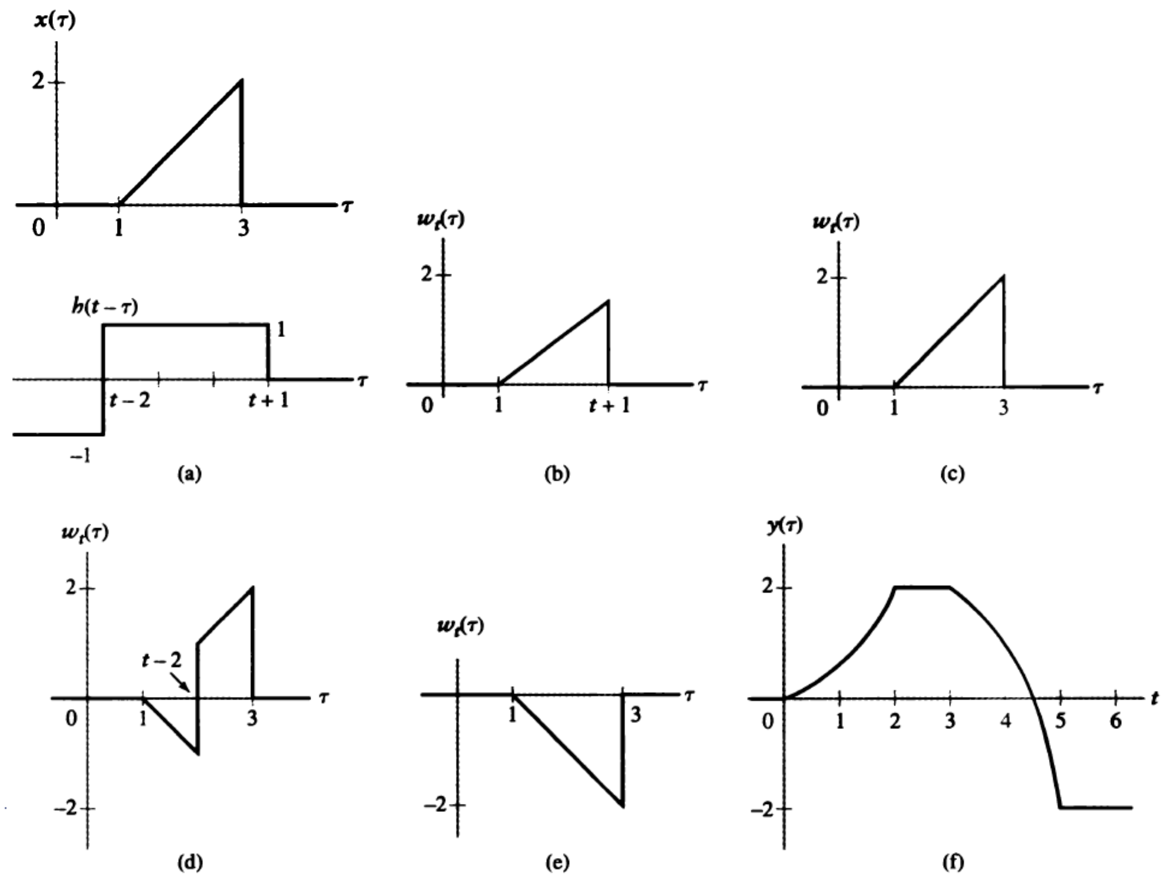


Figure 2.14
(a) The reflected and time-shifted impulse response $h(t-\tau)$, depicted as a function of τ.
(b) The product signal $w_t(\tau)$ for $0\le t<2$.
(c) The product signal $w_t(\tau)$ for $2\le t<3$.
(d) The product signal $w_t(\tau)$ for $3\le t<5$.
(e) The product signal $w_t(\tau)$ for $t\ge 5$.
(f) The system output $y(t)$.

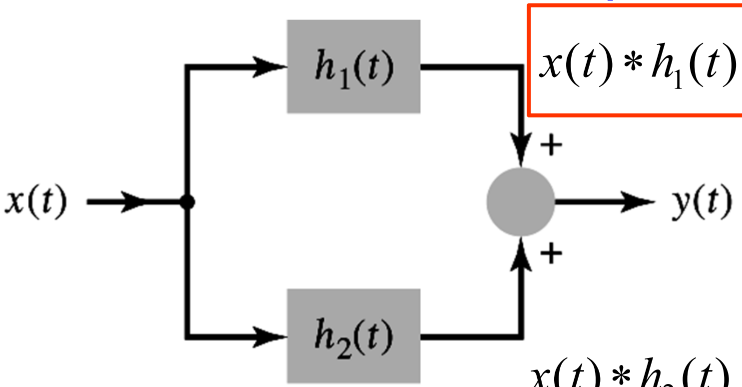
Outline

- Introduction
- The Convolution Sum
- Convolution Sum Evaluation Procedure
- The Convolution Integral
- Convolution Integral Evaluation Procedure
- Interconnections of LTI Systems
- Relations between LTI System Properties and the Impulse Response
- Step Response
- Differential and Difference Equation Representations
- Solving Differential and Difference Equations

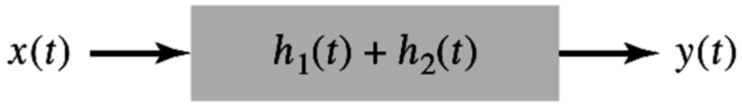
Interconnection of LTI systems

- The results for continuous- and discrete-time systems are nearly identical

1. Parallel Connection of LTI Systems



$$x(t)*h_1(t)=\int_{-\infty}^{\infty}x(\tau)h_1(t-\tau)d\tau$$



Equivalent system.

$$x(t)*h_2(t)=\int_{-\infty}^{\infty}x(\tau)h_2(t-\tau)d\tau$$

$$y(t)=x(t)*h_1(t)+x(t)*h_2(t)=\int_{-\infty}^{\infty}x(\tau)h_1(t-\tau)d\tau+\int_{-\infty}^{\infty}x(\tau)h_2(t-\tau)d\tau$$

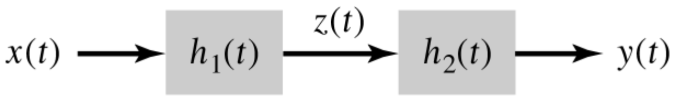
$$=\int_{-\infty}^{\infty}x(\tau)(h_1(t-\tau)+h_2(t-\tau))d\tau=x(t)*(h_1(t)+h_2(t))$$

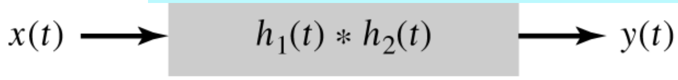
- Distributive property
  - Continuous-time case $x(t)*\{h_1(t)+h_2(t)\}=x(t)*h_1(t)+x(t)*h_2(t)$
  - Discrete-time case $x[n]*\{h_1[n]+h_2[n]\}=x[n]*h_1[n]+x[n]*h_2[n]$

Interconnection of LTI systems

2. Cascade Connection of LTI Systems

$$h(t)=h_1(t)*h_2(t)=\int_{-\infty}^{\infty}h_1(\tau)h_2(t-\tau)d\tau$$





Equivalent system

$$z(t)=x(t)*h_1(t)$$

$$y(t)=z(t)*h_2=\int_{-\infty}^{\infty}z(\tau)h_2(t-\tau)d\tau$$

$$=\int_{-\infty}^{\infty}(x(\tau)*h_1(\tau))h_2(t-\tau)d\tau=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x(\nu)h_1(\tau-\nu)d\nu\right)h_2(t-\tau)d\tau$$

$$=\int_{-\infty}^{\infty}x(\nu)\left(\int_{-\infty}^{\infty}h_1(\tau-\nu)h_2(t-\tau)d\tau\right)d\nu$$

Change variable by $\eta=\tau-\nu$ $$=\int_{-\infty}^{\infty}x(\nu)\left(\int_{-\infty}^{\infty}h_1(\eta)h_2(t-\nu-\eta)d\eta\right)d\nu$$

$$=\int_{-\infty}^{\infty}x(\nu)h(t-\nu)d\nu=x(t)*h(t)$$

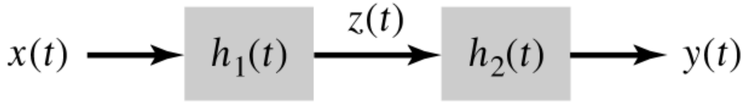
- Associative property

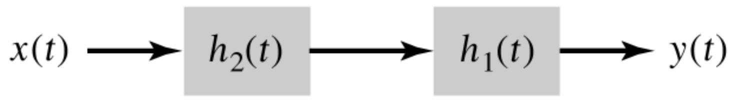
$$\{x(t)*h_1(t)\}*h_2(t)=x(t)*\{h_1(t)*h_2(t)\}$$

$$\{x[n]*h_1[n]\}*h_2[n]=x[n]*\{h_1[n]*h_2[n]\}$$

Interconnection of LTI systems

2. Cascade Connection of LTI Systems





$$h(t) = h_1(t) * h_2(t) = \int_{-\infty}^{\infty} h_1(\tau) h_2(t-\tau) d\tau$$

Change variable by $\nu = t - \tau$ $\quad = \int_{-\infty}^{\infty} h_1(t-\nu) h_2(\nu) d\nu = h_2(t) * h_1(t)$

- Commutative property

$$h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

$$h_1[n] * h_2[n] = h_2[n] * h_1[n]$$

Table 2.1 Interconnection Properties for LTI Systems

Property	Continuous-time system	Discrete-time system
Distributive	$x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}$	$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \{h_1[n] + h_2[n]\}$
Associative	$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$	$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$
Commutative	$h_1(t) * h_2(t) = h_2(t) * h_1(t)$	$h_1[n] * h_2[n] = h_2[n] * h_1[n]$

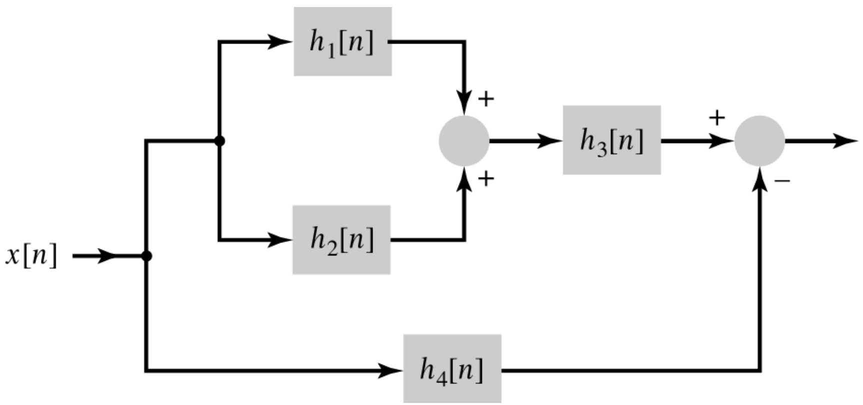
Example 2.11

Consider the interconnection of four *LTI* systems. The impulse responses of the systems are

$$h_1[n]=u[n],\quad h_2[n]=u[n+2]-u[n],\quad h_3[n]=\delta[n-2],\quad \text{and}\quad h_4[n]=\alpha^n u[n].$$

Find the impulse response $h[n]$ of the overall system.

<Sol.>

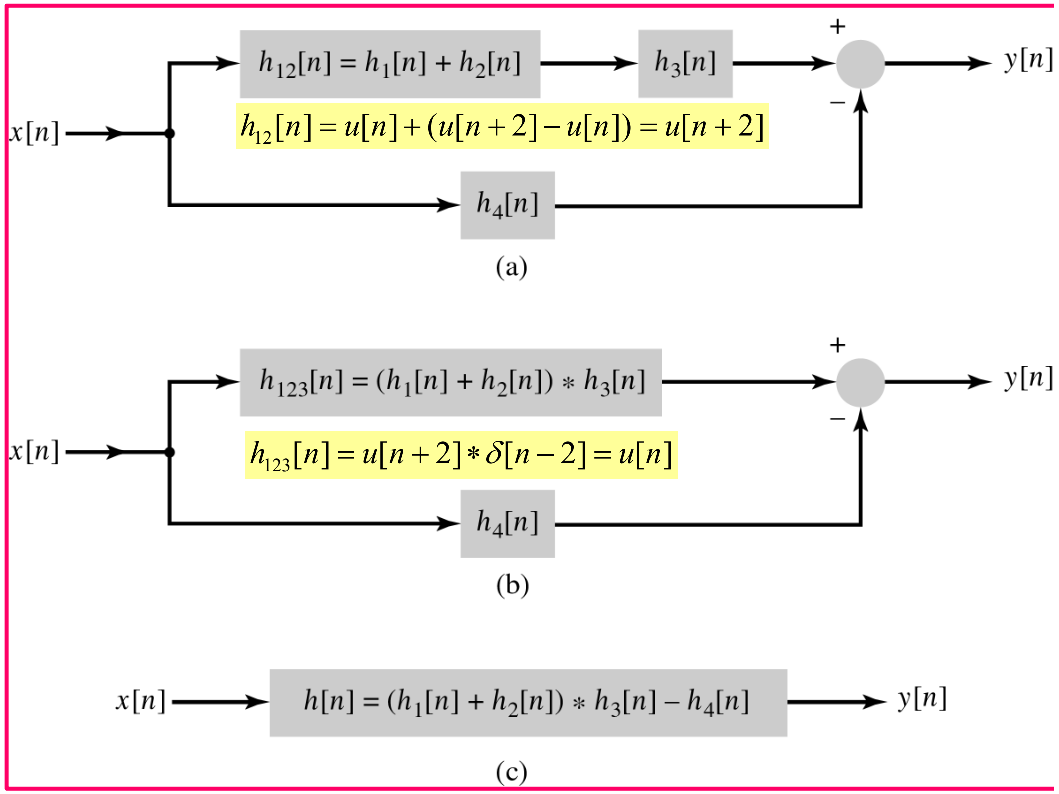


(a). Parallel combination of $h_1[n]$ and $h_2[n]$:

(b). $h_{12}[n]$ is in series with $h_3[n]$:

(c). $h_{123}[n]$ is in parallel with $h_4[n]$:

$$h_4[n]=u[n]-\alpha^n u[n]$$



Relation Between
LTI System Properties and the Impulse Response

- The impulse response completely characterizes the IO behavior of an LTI system.
- Using impulse response to check whether the LTI system is memory, causal, or stable.

1. Memoryless LTI Systems
   - The output depends only on the current input
   - Condition for memoryless LTI systems $h(\tau) = c\delta(\tau)$

     $h[k] = c\delta[k]$

     simply perform scalar multiplication on the input

2. Causal LTI System
   - The output depends only on past or present inputs

     $$y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots.$$

   - Condition for causal LTI systems $h[k] = 0 \quad \text{for} \quad k < 0$

     $h(\tau) = 0 \quad \text{for} \quad \tau < 0$

     cannot generate an output before the input is applied

Relation Between
LTI System Properties and the Impulse Response

3. BIBO stable LTI Systems

- The output is guaranteed to be bounded for every bounded input.

$$|y[n]| = |x[n] * h[n]|$$

$$= |h[n] * x[n]| = \left|\sum_{k=\infty}^{\infty} h[k]x[n-k]\right|$$

$$\leq \sum_{k=\infty}^{\infty} |h[k]||x[n-k]|$$

$|x[n]| \leq M_x \leq \infty$ ⟹ $$\leq \sum_{k=\infty}^{\infty} |h[k]| M_x = M_x \sum_{k=\infty}^{\infty} |h[k]|$$

- Condition for memoryless LTI systems $\sum_{k=\infty}^{\infty} |h[k]| < \infty$ / $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

The impulse response is absolutely summable/integrable

Example 2.12 First-Order Recursive System

The first-order system is described by the difference equation $y[n] = \rho y[n-1] + x[n]$ and has the impulse response $h[n] = \rho^n u[n]$

Is this system causal, memoryless, and BIBO stable?

<Sol.>

1. The system is causal, since $h[n] = 0$ for $n < 0$.
2. The system is not memoryless, since $h[n] \neq 0$ for $n > 0$.
3. Stability: Checking whether the impulse response is absolutely summable?

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} \left|\rho^k\right| = \sum_{k=0}^{\infty} |\rho|^k < \infty \quad \text{iff } |\rho| < 1$$

A system can be unstable even though the impulse response has a finite value for all *t*.

Eg: Ideal integrator:

$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ with the impulse response: $h(t) = u(t)$.

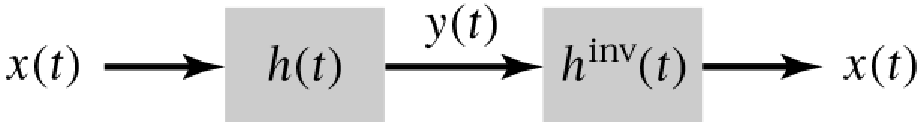
Ideal accumulator:

$y[n] = \sum_{k=-\infty}^{n} x[k]$ with the impulse response: $h[n] = u[n]$

Relation Between
LTI System Properties and the Impulse Response

4. Invertible Systems

- A system is invertible if the input to the system can be recovered from the output



$$x(t) = y(t) * h^{inv}(t)$$
$$= \{x(t) * h(t)\} * h^{inv}(t)$$

Associative law ➡ $$= x(t) * \{h(t) * h^{inv}(t)\}$$

- Condition for memoryless LTI systems $h(t) * h^{inv}(t) = \delta(t)$

- i.e. Deconvolution and Equalizer $h[n] * h^{inv}[n] = \delta[n]$

Easy condition, but difficult to find or implement

Summary

Table 2.2 *Properties of the Impulse Response Representation for LTI Systems*

Property	Continuous-time system	Discrete-time system
Memoryless	$h(t) = c\delta(t)$	$h[n] = c\delta[n]$
Causal	$h(t) = 0 \quad \text{for} \quad t < 0$	$h[n] = 0 \quad \text{for} \quad n < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(t) \rvert dt < \infty$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$
Invertibility	$h(t) * h^{inv} = \delta(t)$	$h[n] * h^{inv}[n] = \delta[n]$