Filter Design

✧ Introduction

- Filter – An important class of LTI systems
- We discuss frequency-selective filters mostly: LP, HP, …
- We concentrate on the design of ***causal*** filters.
- Three stages in filter design:
  - Specification: application dependent
  - "Design": approximate the given spec using a causal discrete-time system
  - Realization: architectures and circuits (IC) implementation
- IIR filter design techniques
- FIR filter design techniques

Frequency domain specifications

Magnitude: $|H(e^{j\omega})|$, *Phase*: $\angle H(e^{j\omega})$

Ex., Low-pass filter: Passband , Transition, Stopband

Frequencies: Passband cutoff ω_p

Stopband cutoff ω_s

Transition bandwidth $\omega_s - \omega_p$

Error tolerance δ_1, δ_2

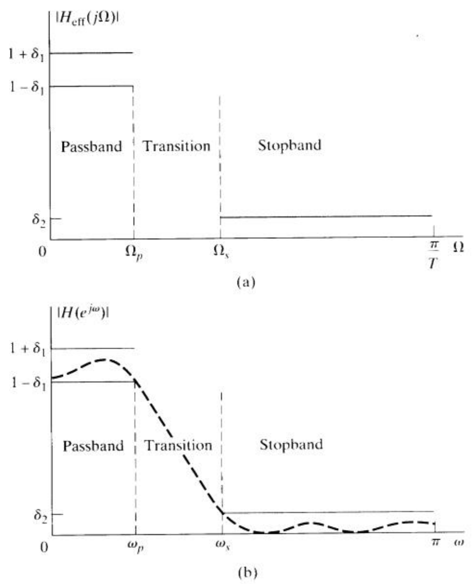


Figure 7.2 (a) Specifications for effective frequency response of overall system in Figure 7.1 for the case of a lowpass filter. (b) Corresponding specifications for the discrete-time system in Figure 7.1.

✧ Analog Filters

- **Butterworth Lowpass Filters**

  - Monotonic magnitude response in the passband and stopband
  - The magnitude response is maximally flat in the passband.

    For an Nth-order lowpass filter

    $\Rightarrow$ The first (2N-1) derivatives of $|H_c(j\Omega)|^2$ are zero at $\Omega = 0$.

    $$|H_c(j\Omega)|^2 = \frac{1}{1+(\frac{j\Omega}{j\Omega_c})^{2N}}$$

    N: filter order

    Ω_c: 3-dB cutoff frequency (magnitude = 0.707)

  - Properties

    (a) $|H_c(j\Omega)|_{\Omega=0} = 1$

    (b) $|H_c(j\Omega)|^2{}_{\Omega=\Omega_c} = 1/2$ or $|H_c(j\Omega)|_{\Omega=\Omega_c} = 0.707$

    (c) $|H_c(j\Omega)|^2$ is monotonically decreasing (of Ω)

    (d) $N \to \infty$ ➔ $|H_c(j\Omega)| \to$ ideal lowpass

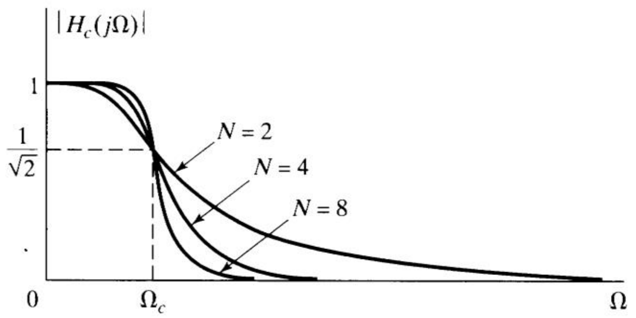


Figure B.2 Dependence of Butterworth magnitude characteristics on the order *N*.

  - Poles

    $$H_c(s)H_c(-s) = \frac{1}{1+(\frac{s}{j\Omega_c})^{2N}}$$

    Roots: $s_k = (-1)^{\frac{1}{2N}}(j\Omega_c) = \Omega_c e^{j\frac{\pi}{2N}(2k+N-1)}, \quad k = 0,1,\ldots,2N-1$

(a) $2N$ poles in pairs: $s_k, -s_k$ symmetric w.r.t. the imaginary axis; never on the imaginary axis. If N odd, poles on the real axis.

(b) Equally spaced on a circle of radius Ω_c

(c) $H_c(s)$ causal, stable ← all poles on the left half plane

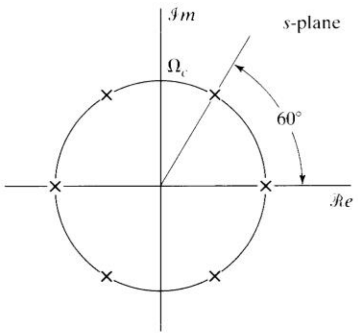


Figure B.3 s-plane pole locations for a third-order Butterworth filter.

- Usage (There are only two parameters N, Ω_c)

  Given specifications $\varepsilon, \Omega_p, \delta_2, \Omega_s$ → N, Ω_c

$$|H(j\Omega)|^2 = \frac{1}{1+(\frac{\Omega}{\Omega_c})^{2N}} = \frac{1}{1+\varepsilon^2(\frac{\Omega}{\Omega_p})^{2N}}$$

Thus, $|H(j\Omega)|^2 = \dfrac{1}{1+\varepsilon^2}$ at $\Omega = \Omega_p$ => $\Omega_c = \Omega_p \big/ \varepsilon^{\frac{1}{N}}$

At $\Omega = \Omega_s$, $|H(j\Omega)|^2_{\Omega_x} = \delta_2^2 = \dfrac{1}{1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}}$ $\quad N = \dfrac{\log[(\frac{1}{\delta_2})^2 - 1]}{2\log(\Omega_s/\Omega_c)}$

Chebyshev Filters

- **Type I:** Equiripple in the passband; monotonic in the stopband

  Type II: Equiripple in the stopband; monotonic in the passband
- Same N as the Butterworth filter, it would have a sharper transition band. (A smaller N would satisfy the spec.)
- **Type I:**

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)}$$

where $V_N(x)$ is the Nth-order Chebyshev polynominal

$$V_N(x) = \cos(N\cos^{-1}(x)),\ 0 < V_N(x) < 1 \ \ for\ 0 < x < 1$$

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x)$$

$$V_N(x)|_{x=1} = 1 \text{ for all } N$$

<The first several Chebyshev polynominals>

N	***V_N(x)***
0	*1*
1	*x*
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$

■ Properties (Type I)

(a) $$|H_c(j\Omega)|^2_{\Omega=0} = \begin{cases} 1, & \text{if N odd} \\ \dfrac{1}{1+\varepsilon^2}, & \text{if N even} \end{cases}$$

(b) The magnitude squared frequency response oscillates between 1 and $\dfrac{1}{1+\varepsilon^2}$ within the passband:

$$|H_c(j\Omega)|^2_{\Omega=\Omega_c} = \frac{1}{1+\varepsilon^2} \qquad \text{at} \quad \Omega = \Omega_c$$

(c) $|H_c(j\Omega)|^2$ is monotonic outside the passband.

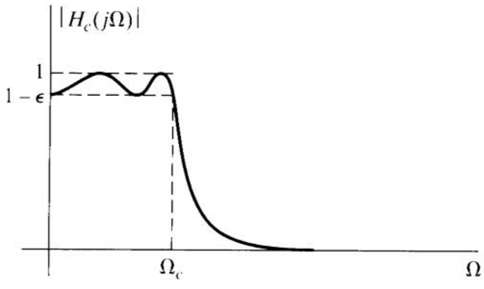


Figure B.4 Type I Chebyshev lowpass filter approximation.

- Poles (Type I)

  On the ellipse specified by the following:

  Length of minor axis = $2a\Omega_c$, $a = \frac{1}{2}\left(\alpha^{\frac{1}{N}} - \alpha^{-\frac{1}{N}}\right)$

  Length of major axis = $2b\Omega_c$, $b = \frac{1}{2}\left(\alpha^{\frac{1}{N}} + \alpha^{-\frac{1}{N}}\right)$

  and $\alpha = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$

  (a) Locate equal-spaced points on the major circle and minor circle with angle

  $$\Phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{N},\ k = 0,1,\cdots,N-1$$

  (b) The poles are (x_k, y_k): $x_k = a\Omega_c \cos\phi_k$, $y_k = b\Omega_c \sin\phi_k$

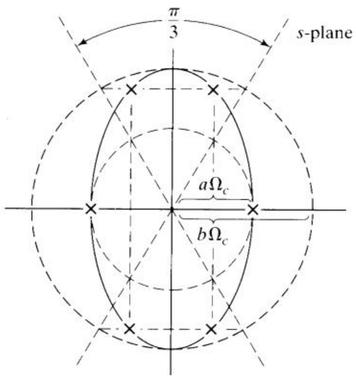


Figure B.5 Location of poles for a third-order type I lowpass Chebyshev filter.

- **Type II:**

  $$|H_a(j\Omega)|^2 = \frac{1}{1+[\varepsilon^2 V_N^2(\Omega_c/\Omega)]^{-1}}$$ has both poles and zeros.

- Usage (There are only two parameters N, Ω_c)

  Given specifications $\varepsilon, \Omega_p, \delta_2, \Omega_s$ ➔ N, Ω_c

  $\Omega_c = \Omega_p$

  $$N = \frac{\log[(\sqrt{1-\delta_2^2} + \sqrt{1-\delta_2^2(1+\varepsilon^2)})/\varepsilon\delta_2]}{\log[(\Omega_s/\Omega_p) + \sqrt{(\Omega_s/\Omega_p)^2 - 1}]}$$

  $$= \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \qquad \left(\delta_2 = \frac{1}{\sqrt{1+\delta^2}}\right)$$

- **Elliptic Filters**
  - Equiripple at both the passband and the stopband
  - Optimum: smallest $(\Omega_s - \Omega_p)$ at the same N

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N^2(\Omega/\Omega_p)}$$

where $U_N(x)$: Jacobian elliptic function (Very complicated! Skip!)

  - Usage (There are only two parameters N, Ω_c)

Given specifications $\varepsilon, \Omega_p, \delta_2, \Omega_s$ ➔ N, Ω_c

$$N = \frac{K(\Omega_p/\Omega_s)K(\sqrt{1-(\varepsilon^2/\delta^2)})}{K(\varepsilon/\delta)K(\sqrt{1-(\Omega_p/\Omega_s)^2})} \qquad \left(\delta_2 = \frac{1}{\sqrt{1+\delta^2}}\right)$$

where $K(x)$ is the complete elliptic integral of the first kind

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}$$

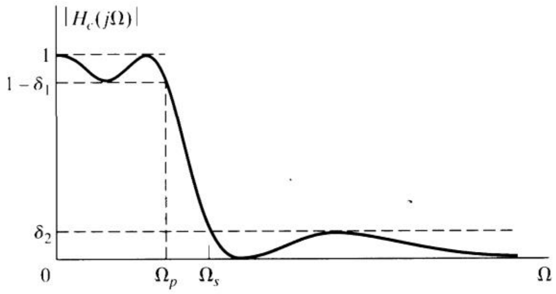


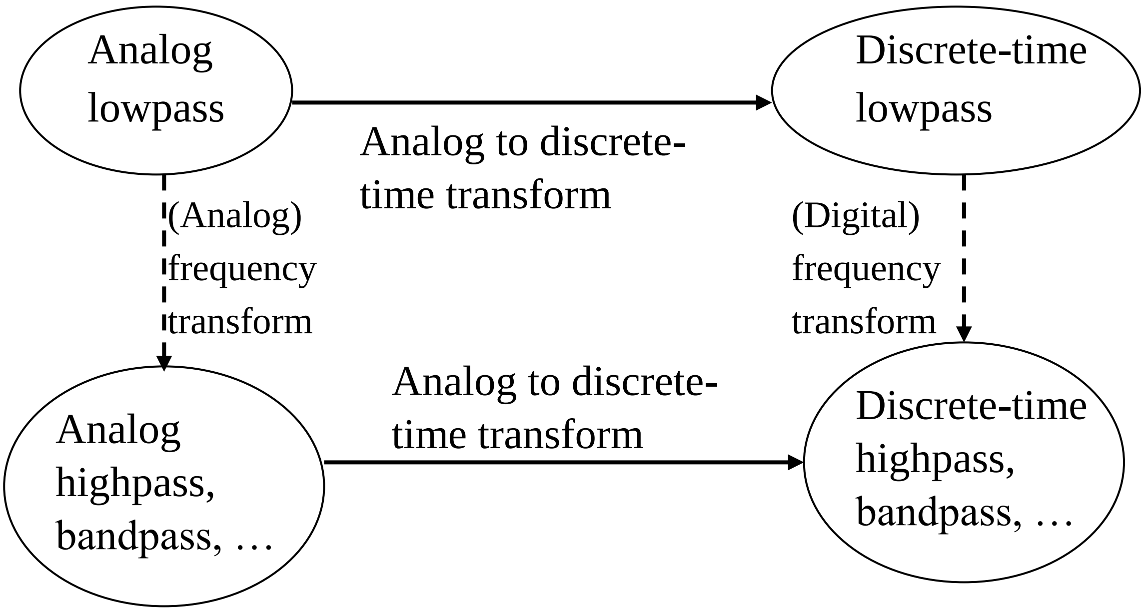
Figure B.6 Equiripple approximation in both passband and stopband.

Remark: The drawback of the elliptic filters: They have more nonlinear phase response in the passband than a comparable Butterworth filter or a Chebyshev filter, particularly, near the passband edge.

✧ Design Digital IIR Filters from Analog Filters

- Why based on analog filters?
  - Analog filter design methods have been well developed.
  - Analog filters often have simple *closed-from* design formulas.

    ← Direct digital filter design methods often don't have *closed-form* formulas.
- There are two types of transformations
  - Transformation from analog to discrete-time
  - Transformation from one type filter to another type (so called *frequency transformation*)



- Methods in analog to discrete-time transformation
  - Impulse invariance
  - Bilinear transformation
  - Matched-z transformation
- Desired properties of the transformations
  - Imaginary axis of the s-plane → The unit circle of the z-plane
  - Stable analog system → Stable discrete-time system

    (Poles in the left s-plane → Poles inside the unit circle)

- Steps in the design

  (1) Digital specifications → Analog specifications

  (2) Design the desired analog filter

  (3) Analog filter → Discrete-time filter

- **Impulse Invariance**

  -- Sampling the impulse of a continuous-time system

$$\begin{aligned} h[n] &= T_d h_c(nT_d) \\ &= T_d h_c(t)\,|_{t=nT_d} \end{aligned}$$

T_d : Sampling period

✓ *Important:* to avoid aliasing

✓ Does not show up in the final discrete formula if we start from the digital specifications, ...

■ Frequency response

Sampling in time → Sifted duplication in frequency

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$

If $H_c(j\Omega)$ is band-limited and $f_d = 1/T_d$ is higher than the Nyquist sampling frequency (no aliasing)

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \qquad |\omega| \le \pi$$

Remark: This is not possible because the IIR analog filter is typically not bandlimited.

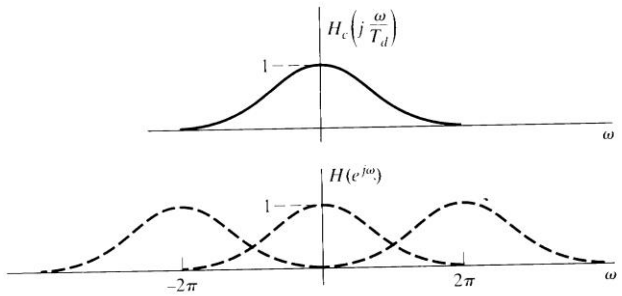


Figure 7.3 Illustration of aliasing in the impulse invariance design technique.

Approach 1: Sampling $h[n]$

Approach 2: Map $H_c(s)$ to $H(z)$ because we need $H(z)$ to implement a digital filter anyway.

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$\begin{aligned} h[n] &= T_d h_c(nT_d) \\ &= T_d \sum_{K=1}^{N} A_k e^{s_k n T_d} u[n] \\ &= \sum_{K=1}^{N} (T_d A_k)(e^{s_k T_d})^n u[n] \end{aligned}$$

$$H(z) = \sum_{K=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Essentially, **factorize and map:**

Analog pole

$\Downarrow$

Discrete-time pole

Remarks: (1) Stability is preserved:

LHS poles → poles inside the unit circle

(2) No simple correspondence for zeros

***Design Example:* Low-pass filter**

Using Butterworth continuous-time filter

Given specifications in the digital domain

"-1 dB in passband" and "-15 dB in stopband"

$$0.89125 \le |H(e^{j\omega})| \le 1, \qquad 0 \le |\omega| \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.17783, \qquad 0.3\pi \le |\omega| \le \pi$$

Step 1: Convert the above specifications to the analog domain

(Assume "negligible aliasing")

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T_d}) \qquad |\omega| \le \pi$$

$$0.89125 \le |H(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi / T_d$$

$$|H(j\Omega)| \le 0.17783, \qquad 0.3\pi / T_d \le |\Omega| \le \pi / T_d$$

Step 2: Design a Butterworth filter that satisfies the above specifications. That is, select proper N, Ω_c.

$$\begin{cases} |H_c(j\frac{0.2\pi}{T_d})| \ge 0.89125 \\ |H_c(j\frac{0.3\pi}{T_d})| \le 0.17783 \end{cases}$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega / \Omega_c)^{2N}}$$

Thus, $$\begin{cases} 1+\left(\frac{0.2\pi}{T_d\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \\ 1+\left(\frac{0.3\pi}{T_d\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2 \end{cases}$$

➔ $N = 5.8858, \quad T_d\Omega_c = 0.70474$

➔ (Taking integer) $N = 6, \quad T_d\Omega_c = 0.7032$

(Meet passband spec. exactly; overdesign at stopband)

<Case 1: Assume $T_d = 1 \Rightarrow s_k = \Omega_c e^{j\frac{\pi}{2N}(2k+N-1)}$

<Case 2: Assume $T_d \ne 1 \Rightarrow s_k = \left(\frac{0.7032}{T_d}\right) e^{j\frac{\pi}{2N}(2k+N-1)}$

$$H_c(s) = \frac{0.12093}{(s^2+0.365s+0.495)(s^2+0.995s+0.495)(s^2+1.359s+0.495)}$$

Step 3: Convert analog filter to discrete-time

Analog pole s_k

$$\Downarrow$$

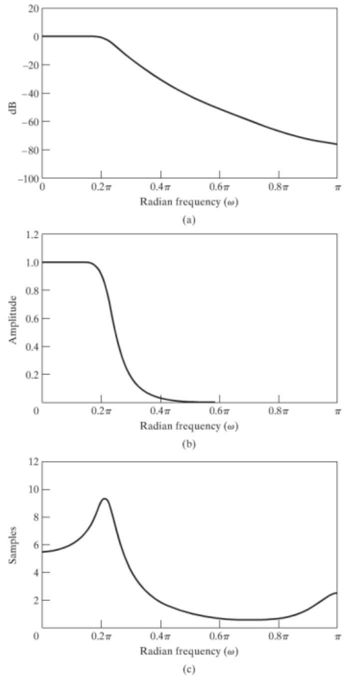
Discrete-time pole e^{s_k}

<Case 1: Assume $T_d = 1 \Rightarrow$ $z_k = \exp\left[0.7032\, e^{j\frac{\pi}{2N}(2k+N-1)}\right]$

<Case 2: Assume $T_d \neq 1 \Rightarrow$ $z_k = \exp\left[T_d\left(\frac{0.7032}{T_d}\right)e^{j\frac{\pi}{2N}(2k+N-1)}\right]$

They are identical! (In general, this is true.)

$$H(z) = \frac{0.287 - 0.447z^{-1}}{1 - 1.297z^{-1} + 0.695z^{-2}} + \frac{-2.143 + 1.145z^{-1}}{1 - 1.069z^{-1} + 0.370z^{-2}} + \frac{1.856 - 0.630z^{-1}}{1 - 0.997z^{-1} + 0.257z^{-2}}$$



Group Delay

Remarks: (1) In some filter design problems, a primary objective maybe to control some aspect of the time response. ⇒ design the discrete-time filter by impulse invariance or by step invariance.

(Note: Designs by impulse invariance and by step invariance don't lead to the same discrete-time filter!)

(2) Impulse invariance method has a precise control on the shape of the time signal. Except for aliasing, the shape of the frequency response is preserved.

(3) Impulse invariance technique is appropriate only for bandlimited filters.

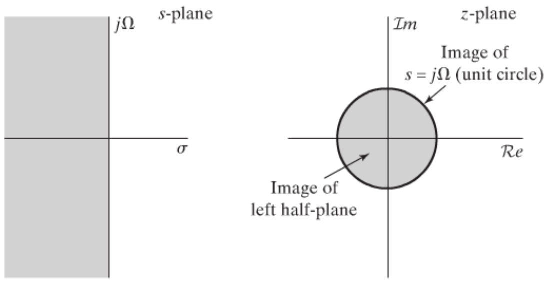
- **Bilinear Transform**

  - Avoid aliasing but distort the frequency response – uneven stretch of the frequency axis.
  - $s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ or $z = \frac{1+sT_d/2}{1-sT_d/2}$

    $H_c(s) \rightarrow H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$

    Note: $j\Omega$ axis on the s-plane → unit circle on the z-plane

    LHS of the s-plane → Interior of the unit circle on the z-plane



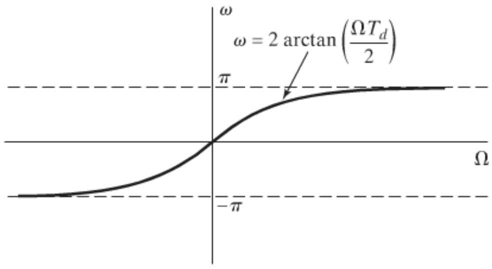
  - How the $j\Omega$ axis is mapped to the unit circle?

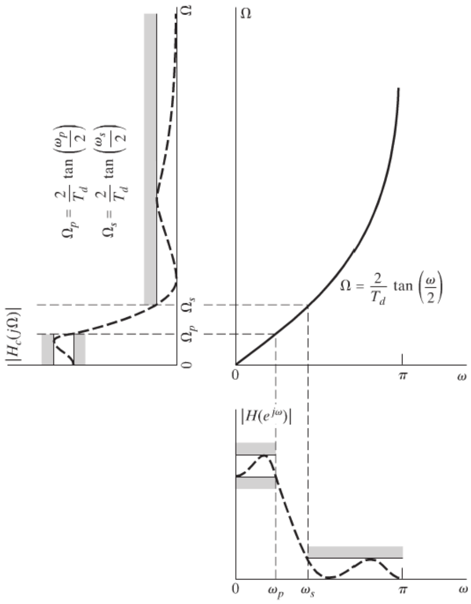
$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\Bigg|_{Z=e^{j\omega}} = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right)$$

$$= \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}\left(j\sin\frac{\omega}{2}\right)}{2e^{-j\omega/w}\left(\cos\frac{\omega}{2}\right)}\right]$$

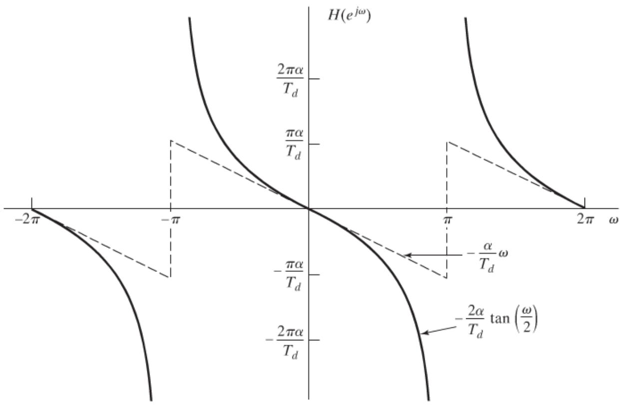
$$= \frac{2j}{T_d}\tan\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \text{ or } \omega = 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right)$$





Problem in design – nonlinear distortion in magnitude and phase



- Steps in the design

  (1) Digital specifications to analog specifications: prewarp

  (2) Design the desired analog filter

  (3) Analog filter to discrete-time filter: bilinear transform

Design Example: **Lowpass filter**

Using Butterworth continuous-time filter

Given specifications in the digital domain (same as the previous ex.)

$$0.89125 \le |H(e^{j\omega})| \le 1, \qquad 0 \le |\omega| \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.17783, \qquad 0.3\pi \le |\omega| \le \pi$$

Step 1: Prewarp $\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right)$

Passband freq. $\Omega_p = \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$

Stopband freq. $\Omega_s = \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right)$

Let $T_d = 1$ since T_d will disappear after “analog to discrete”.

Step 2: Design a Butterworth filter -- select proper N, Ω_c.

$$\begin{cases} |H_c(j2\tan(0.1\pi))| \ge 0.89125 \\ |H_c(j2\tan(0.15\pi))| \le 0.17783 \end{cases}$$

Because $|H_c(j\Omega)|^2 = \dfrac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$

➔ $$\begin{cases} 1+\left(\dfrac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.89125}\right)^2 \\ 1+\left(\dfrac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.17783}\right)^2 \end{cases}$$

➔ $N = 5.30466,$

➔ $N = 6, \quad T_d\Omega_c = 0.76622$

(Meet stopband spec. exactly; exceed passband spec.)

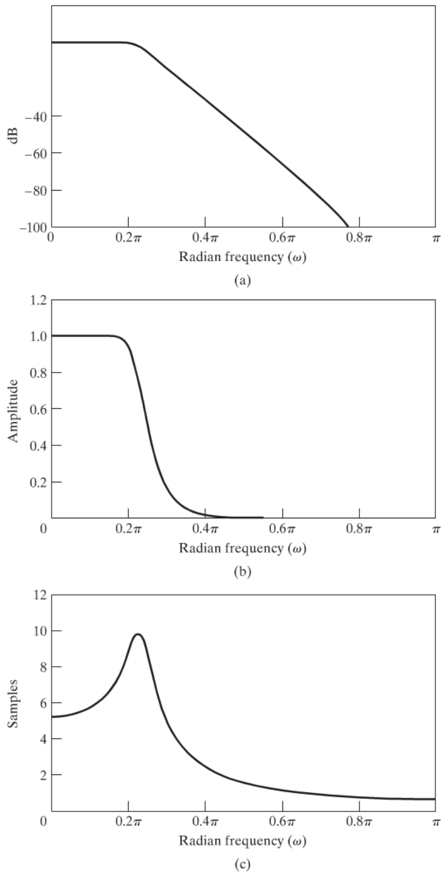
$$H_c(s) = \frac{0.20238}{(s^2+0.3996s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)}$$

Step 3: Convert analog filter to discrete-time

$$H_c(s) \to H(z) = H_c\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})}$$

$$\times \frac{1}{(1-0.9044z^{-1}+0.2155z^{-2})}$$



Remarks: (1) Bilinear transforms warps frequency values but preserves the magnitude. Therefore, the discrete-time Butterworth filter still has the maximal flat property; Chebyshev and Ellipic filters have equal ripple property.

(2) Although we may obtain $H_c(s)$ in closed form, it is often difficult to find the locations of poles and zeros of $H(z)$ from $H_c(s)$ directly.

Bilinear Transform Design Example using 4 analog filters:

passband edge frequency $\omega_p = 0.5\pi$

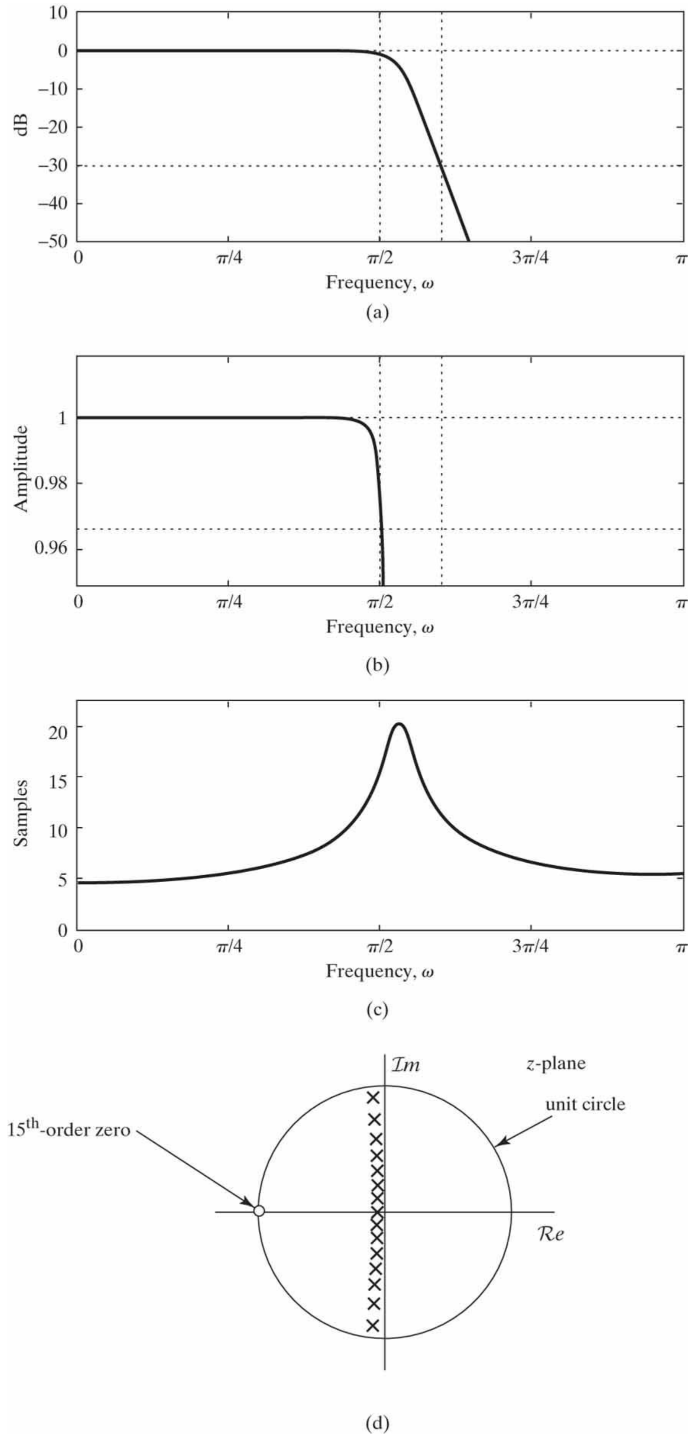
stopband edge frequency $\omega_s = 0.6\pi$

maximum passband gain $= 0\text{dB}$

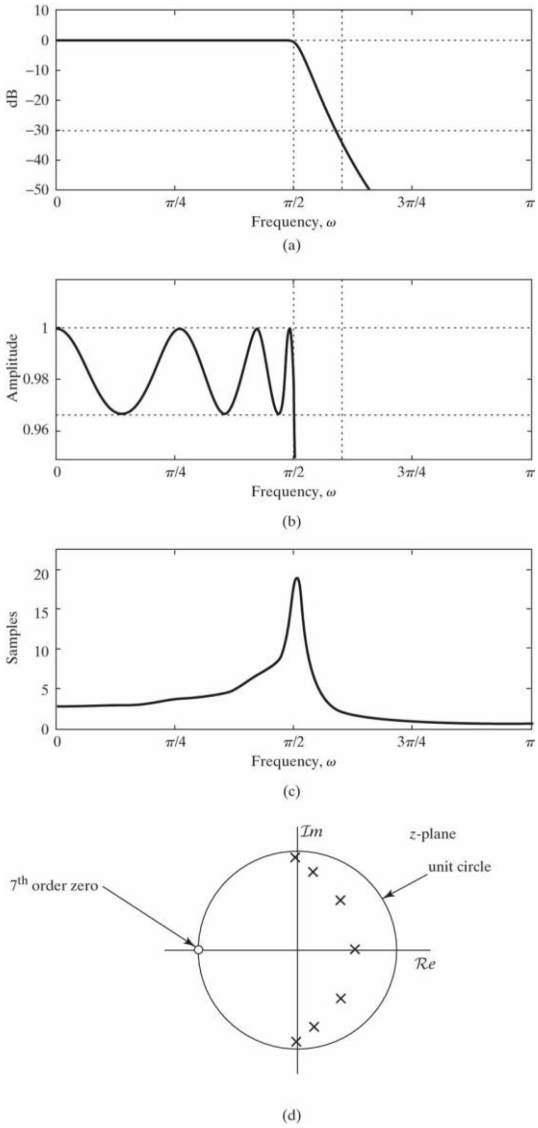
minimum passband gain $= -0.3\text{dB}$

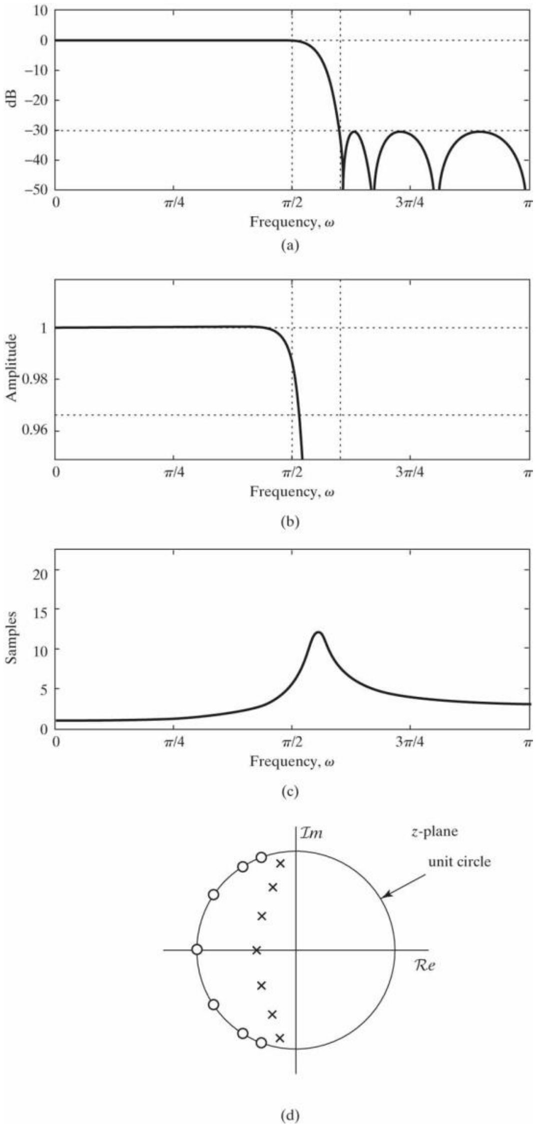
maximum stopband gain $= -30\text{dB}$

Butterworth: 15^{th} order

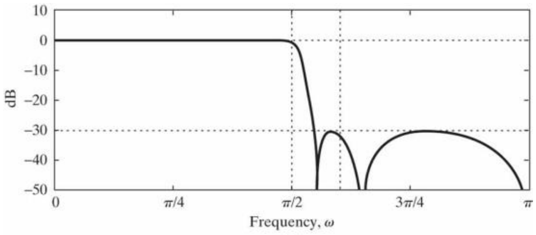


Chebyshev I and II: 7th order

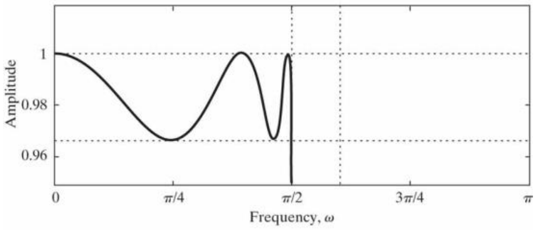




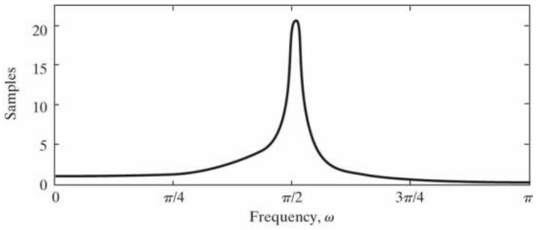
Elliptic: 5th order



(a)



(b)



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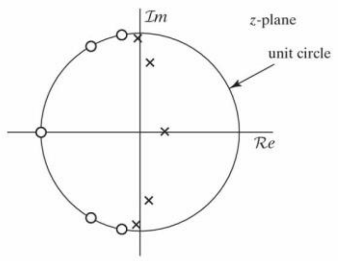


(d)

- **Frequency Transformation**

-- Transform one-type (often lowpass) filter to another type.

Typically, we first design a *frequency-normalized prototype lowpass* filter. Then, use an algebraic transformation to derive the desired lowpass, high pass , …, filters from the prototype lowpass filter.

<Prototype filter> → <Desired filter>

Z → z

$$Z^{-1} = G(z^{-1})$$

$$H_{lp}(Z)\Big|_{Z^{-1}=G(z^{-1})} \rightarrow H(z)$$

Typically, this transform is made of all-pass like factors

$$G(z^{-1}) = \pm\prod_{K=1}^{N}\left(\frac{z^{-1}-\alpha_k}{1-\alpha_k z^{-1}}\right)$$

Remarks: The desired properties of $G(.)$ are

(1) transforms the unit circle in Z to the unit circle in z,

(2) transforms the interior of the unit circle in Z to the interior of the unit circle in z,

(3) $G(.)$ is rational.

Example: **Lowpass to lowpass** (with different passband and stopband frequency, but magnitude is not changed)

$$Z^{-1} = \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$$

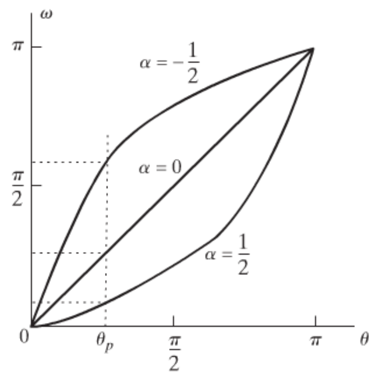
Check the relationship between θ (the Z filter) and ω (the z filter). α is a parameter. Different α offers different "shapes" of the transformed filters in ω.

$$e^{-j\theta} = \frac{e^{-j\omega}-\alpha}{1-\alpha e^{-j\omega}}$$

$$\omega = \tan^{-1}\left[\frac{(1-\alpha^2)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right]$$

If θ_p is to be mapped to ω_p, then

$$\alpha = \frac{\sin[(\theta_p - \omega_p)/2]}{\sin[(\theta_p + \omega_p)/2]}$$



■ Various Digital to Digital Transformations

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - az^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

✧ Design of FIR Filters by Windowing

- Why FIR filters?

  -- Always stable

  -- *Exact* linear phase

  -- Less sensitive to inaccurate coefficients

  <Disadvantage> Higher complexity (storage, multiplication) due to higher orders

- Design Methods

  (1) Windowing

  (2) Frequency sampling

  (3) Computer-aided design

  Remark: No meaningful analog FIR filters

- Windowing technique advantages

  -- Simple

  -- Pick up a "segment" (window) of the ideal (infinite) $h_d[n]$

  -- Filter order = window length = (M+1)

General form: $h[n] = h_{\mathrm{d}}[n]w[n]$

Filter impulse response = Desired response x Window

Example: **Rectangular window**

Window shape: $w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$

➔ $h[n] = \begin{cases} h_{\mathrm{d}}[n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$

- Because the filter specifications are (often) given in the frequency domain $H_d(e^{j\omega})$.

  We take the inverse DTFT to obtain $h_d[n]$.

$$h_{\mathrm{d}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{d}}\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

or, $H_{\mathrm{d}}\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_{\mathrm{d}}[n] e^{-j\omega n}$

Now, because of the inclusion of $w[n]$,

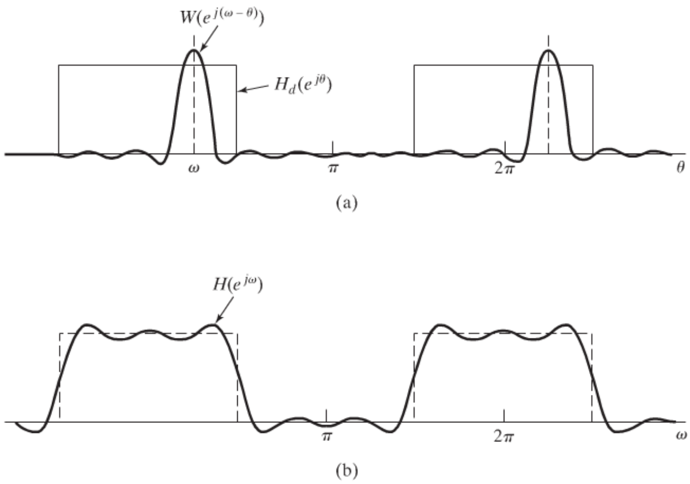
$$H\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_{\mathrm{d}}\left(e^{j\theta}\right)\cdot W\left(e^{j(\omega-\theta)}\right)d\theta \quad \text{(A periodic convolution)}$$

That is, $H(e^{j\omega})$ is "smeared" version of $H_d(e^{j\omega})$.

Why $W(e^{j\omega})$ cannot be $\delta(e^{j\omega})$? (If so, $H(e^{j\omega})=H_d(e^{j\omega})$!)

Parameters (to choose): (1) Window size (order of filter)

(2) Window shape



- **Rectangular Window**: $w[n]=\begin{cases}1, & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$

-- Narrow mainlobe

-- High sidelobe (Gibbs phenomenon)

-- Frequency response

$$\begin{aligned}W\left(e^{j\omega}\right)&=\sum_{n=0}^{M}1\cdot e^{-j\omega n}\\ &=\frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}\\ &=e^{-j\omega\frac{M}{2}}\frac{\sin\left[\omega\frac{(M+1)}{2}\right]}{\sin\left(\frac{\omega}{2}\right)}\end{aligned}$$

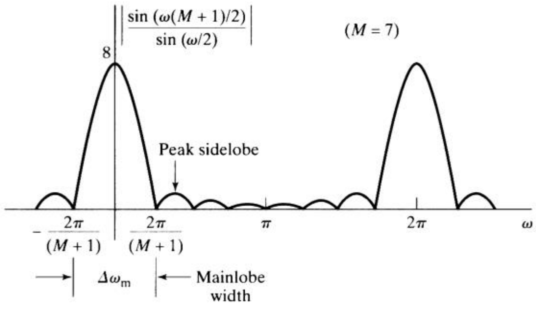


Figure 7.20 Magnitude of the Fourier transform of a rectangular window ($M = 7$).

-- Mainlobe ~ $\dfrac{4\pi}{M+1}$, $M\uparrow$, $W(e^{j\omega}) \to \delta(e^{j\omega})$

-- Peak sidelobe ~ -13 dB (lower than the mainlobe)

Area under each lobe remains constant with increasing *M*

→ Increasing M does not lower the (relative) amplitude of the sidelobe.

(Gibbs phenomemnon)

Remarks: For frequency selective filters (ideal lowpass, highpass, …),

narrow mainlobe → sharp transition

lower sidelobe → oscillation reduction

Commonly Used Windows

-- Sidelobe amplitude (area) vs. mainlobe width

-- Closed form, easy to compute

Bartlett (triangular) Window:

$$w[n] = \begin{cases} \dfrac{2n}{M}, & 0 \le n \le \dfrac{M}{2} \\ 2 - \dfrac{2n}{M}, & \dfrac{M}{2} < n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hanning Window:

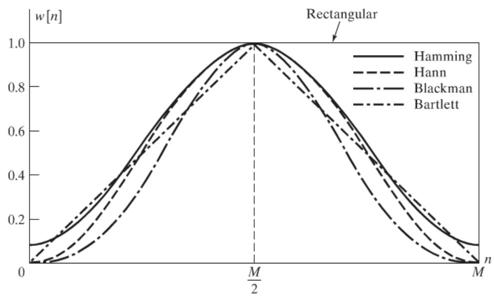
$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\dfrac{2\text{n}}{\text{M}}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

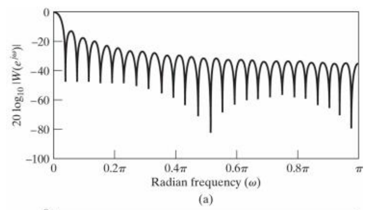
Hamming Window:

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2\,n}{M}\right),, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

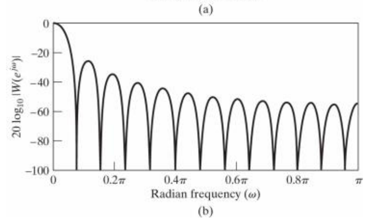
Blackman Window:

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\dfrac{2\,n}{M}\right) + 0.08\cos\left(\dfrac{2\,n}{M}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

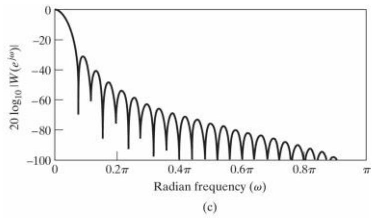




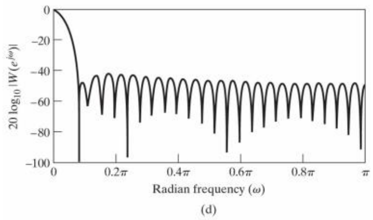
Rectangular



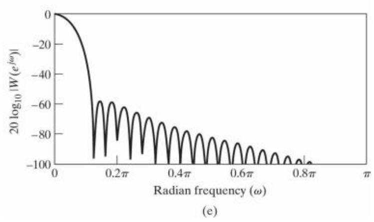
Barlett



Hanning



Hamming



Blackman

TABLE 7.2 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

- **Generalized Linear Phase Filters**

-- We wish $H(e^{j\omega})$ be (general) linear phase.

<Window> Choose windows such that

$$w[n] = w[M-n], \quad 0 \le n \le M$$

That is, symmetric about M/2 (samples)

$$W\left(e^{j\omega}\right) = W_e\left(e^{j\omega}\right)\cdot e^{-j\omega\frac{M}{2}}$$, where $W_e\left(e^{j\omega}\right)$ is real.

<Desired filter> Suppose the desired filter is also generalized linear phase

$$H_d\left(e^{j\omega}\right) = H_e\left(e^{j\omega}\right)\cdot e^{-j\omega\frac{M}{2}}$$

<Filter> $H(e^{j\omega})$ is a periodic convolution of $H_d(e^{j\omega})$ and $W(e^{j\omega})$

$$\begin{aligned} H\left(e^{j\omega}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right)\cdot W_e\left(e^{j(\omega-\theta)}\right)\cdot e^{-j\theta\frac{M}{2}} e^{-j\frac{(\omega-\theta)M}{2}}\, d\theta \\ &= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right)\cdot W_e\left(e^{j(\omega-\theta)}\right) d\theta}_{A_e\left(e^{j\omega}\right)} \cdot e^{-j\omega\frac{M}{2}} \end{aligned}$$

$A_e\left(e^{j\omega}\right)$ is real.

Thus, $H(e^{j\omega})$ is also generalized linear phase.

Example: **Linear phase lowpass filter**

Ideal lowpass: $H_{\text{lp}}\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega\frac{M}{2}}, & |\omega|<\omega_c \\ 0, & \omega_c<|\omega|\le\pi \end{cases}$

Impulse response: $h_{\text{lp}}[n]=\dfrac{\sin\left[\omega_c\left(n-\frac{M}{2}\right)\right]}{\pi\left(n-\frac{M}{2}\right)}$

Designed filter: $h\,[n]=\dfrac{\sin\left[\omega_c\left(n-\frac{M}{2}\right)\right]}{\pi\left(n-\frac{M}{2}\right)}\cdot w[n]$

ω_c : 1/2 amplitude of $H\left(e^{j\omega}\right)$ = cutoff frequency of the dieal lowpass filter

Peak to the left of ω_c occurs at ~ 1/2 mainlobe width

-Peak to the right of ω_c occurs at ~ 1/2 mainlobe width

Transition bandwidth $\Delta\omega$ ~ mainlobe width- (smaller)

Peak approximation error: proportional to sidelobe area

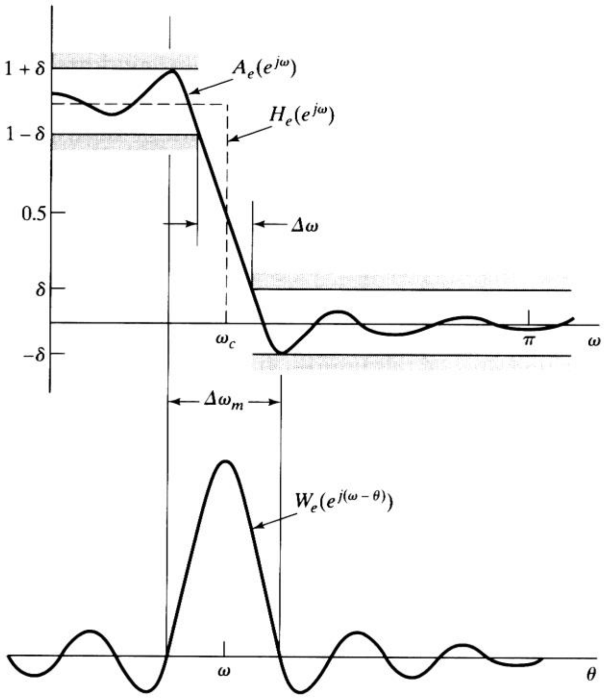


Figure 7.23 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

Kaiser Window

-- Nearly optimal trade-off between mainlobe width and sidelobe area

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\left(1-\left[(n-\alpha)/\alpha\right]^2\right)^{1/2}\right]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

where $I_0(\cdot)$: zeroth-order modified Bessel function of the first kind

α : $M/2$

β : shape parameter; $\beta = 0$, rectangular window

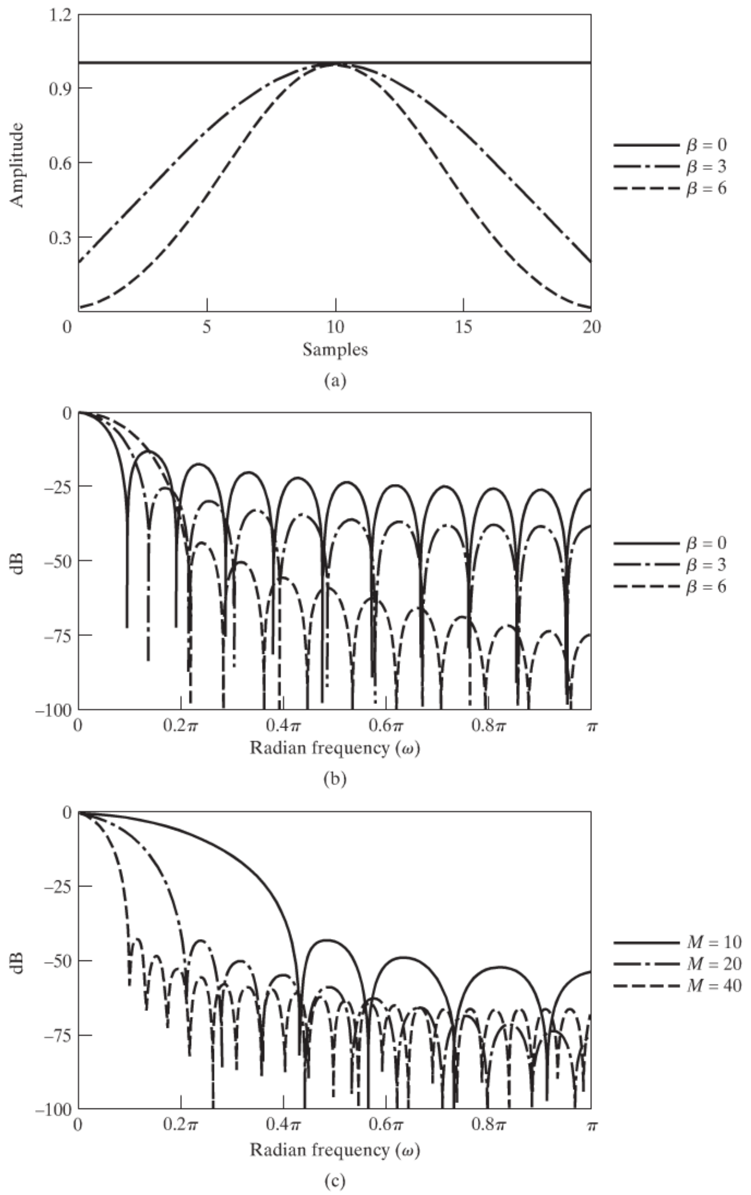
$\beta \uparrow$, mainlobe width $\uparrow$, sidelobe area $\downarrow$

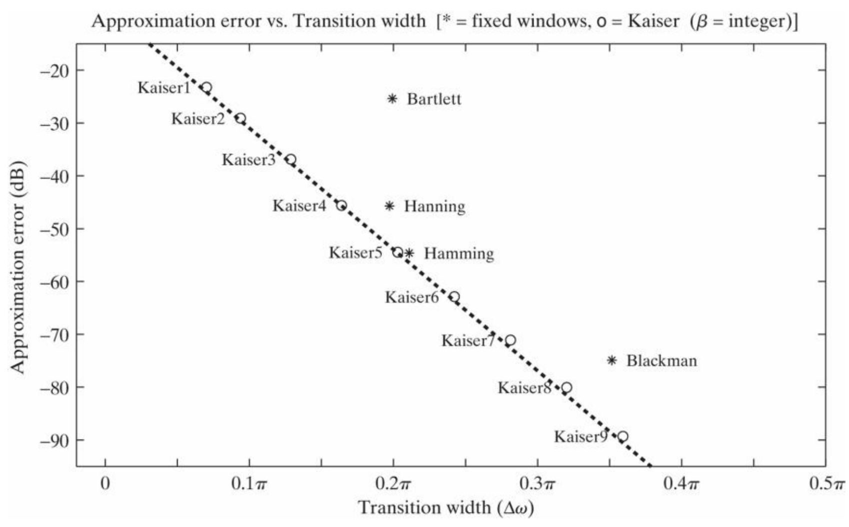
-- $A \equiv -20 \cdot \log_{10} \delta$

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.0 & A < 21 \end{cases}$$

-- $\Delta\omega = \omega_s - \omega_p$ (stopband – passband)

$M = \dfrac{A-8}{2.285 \cdot \Delta\omega}$ (within +-2 over a wide range of $\Delta\omega$ and A)


Amplitude
Samples
β = 0
β = 3
β = 6
(a)
dB
Radian frequency (ω)
β = 0
β = 3
β = 6
(b)
dB
Radian frequency (ω)
M = 10
M = 20
M = 40
(c)


Approximation error vs. Transition width [* = fixed windows, o = Kaiser (β = integer)]
Approximation error (dB)
Transition width (Δω)
Kaiser1
Kaiser2
Kaiser3
Kaiser4
Kaiser5
Kaiser6
Kaiser7
Kaiser8
Kaiser9
Bartlett
Hanning
Hamming
Blackman

***Kaiser window example* – lowpass**

Specifications: $\delta_1 = \delta_2 = 0.001$

Ideal lowpass cutoff: $\omega_c = \dfrac{\omega_s + \omega_p}{2} = 0.5\pi$

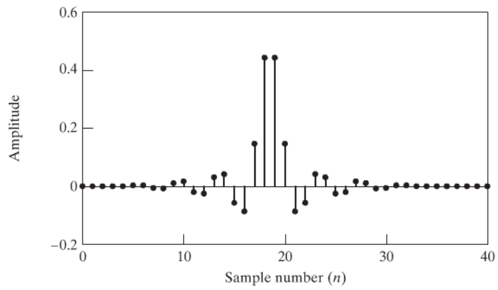
Select parameters: $\begin{cases} \Delta\omega = \omega_s - \omega_p = 0.2\pi \\ A = -20\log_{10}\delta = 60 \end{cases} \rightarrow \begin{cases} \beta = 5.653 \\ M = 37 \end{cases}$

$$\alpha = M/2 = 18.5$$

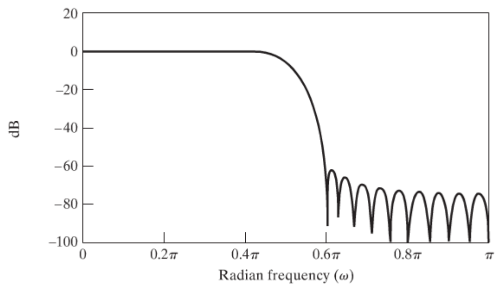
This is a type II, linear phase (odd M, even symmetry) filter.

Approximation error: $|H_d(e^{j\omega})| - |H(e^{j\omega})|$

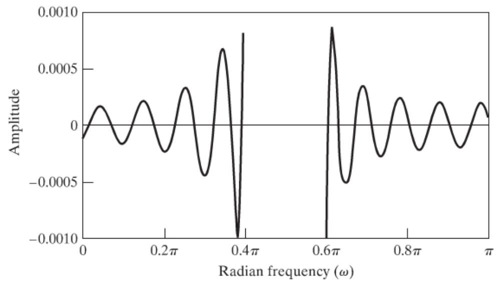
$$E_A(e^{j\omega}) = \begin{cases} 1 - A_e(e^{j\omega}), & 0 \le \omega < \omega_p \\ 0 - A_e(e^{j\omega}), & \omega_s < \omega \le \pi \end{cases}$$



(a)



(b)



(c)

***Kaiser window example* – highpass**

Ideal highpass: $H_{\text{hp}}\left(e^{j\omega}\right)=\begin{cases}0, & 0\le|\omega|<\omega_c\\ e^{-j\omega\frac{M}{2}}, & \omega_c<|\omega|\le\pi\end{cases}$

$$h_{\text{hp}}[n]=\frac{\sin\pi\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)}-\frac{\sin\omega_c\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)}$$

Specifications: $\delta_1=\delta_2=0.021$

Highpass cutoff: $\omega_c=\dfrac{\omega_s+\omega_p}{2}=\dfrac{0.35\pi+0.5\pi}{2}$

Select parameters: $\begin{cases}\Delta\omega\\ A\end{cases}\rightarrow\begin{cases}\beta=2.6\\ M=24\end{cases}$

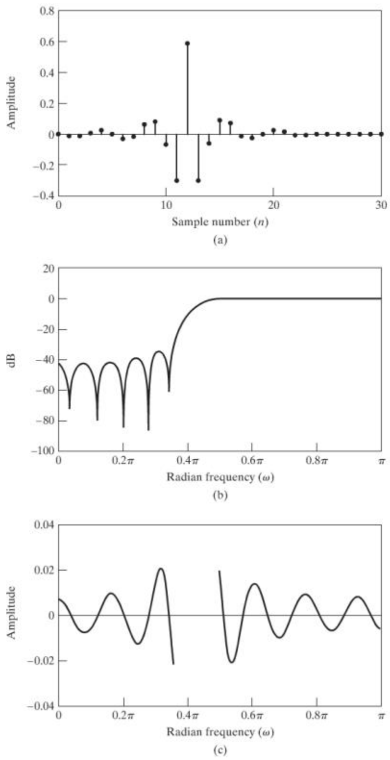
This is a Type I filter.

Check! Approximation error = 0.0213 > 0.021!!

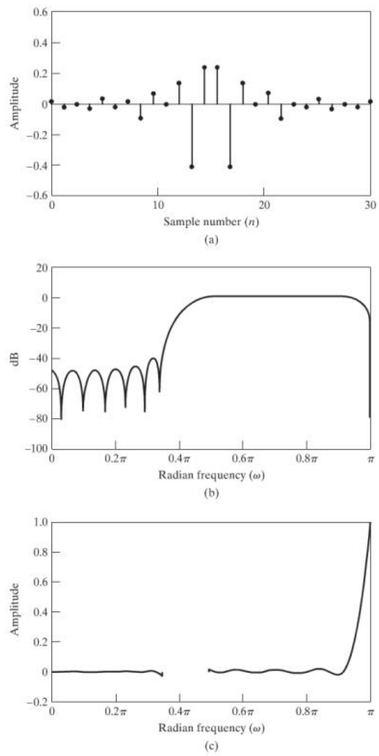
Increase M to 25 → Not good! This is a Type II filter: a zero at –1.→ $H_d\left(e^{j\pi}\right)=0$

But we want it to be 1 because this is a highpass filter.

Increase M to 26. Okay!



M = 24



M = 25

***Kaiser window example* – differentiator**

Ideal differentiator: $\sim \dfrac{d}{dt}$

$$H_{\text{diff}}\left(e^{j\omega}\right) = (j\omega)\cdot e^{-j\omega\frac{M}{2}}, \quad -\pi < \omega < \pi$$

$$h_{\text{diff}}[n] = \frac{\cos\pi\left(n-\frac{M}{2}\right)}{\left(n-\frac{M}{2}\right)} - \frac{\sin\pi\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)^2}$$

Note that both terms in $h_{diff}[n]$ are odd symmetric.

Hence, $h[n] = -h[M-n]$.

This must be a Type III or Type IV system.

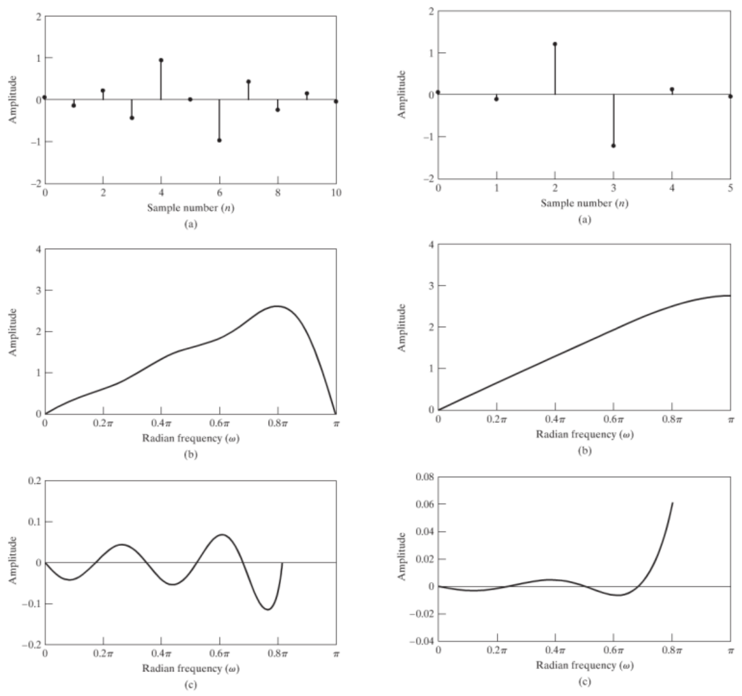
<Comparison>

Case 1: M=10, $\beta = 2.4$ → Type III

Zeros at 0 and –1. Approximation is not good at $\omega = \pi$.

Case 2: M=5, $\beta = 2.4$ → Type IV

Zeros at 0. Approximation error is smaller.



M = 10 M = 5

✧ Optimum Approximation of FIR Filters

- Why computer-aided design?

  -- Optimum: minimize an error criterion

  -- More freedom in selecting constraints.

  (In windowing method: must $\delta_1 = \delta_2 = \delta$)

- Several algorithms – ***Parks-McClellan algorithm*** (1972)

  Type I linear phase FIR filter

  Its symmetry property: $h_{\mathrm{e}}[n] = h_{\mathrm{e}}[-n]$ (omit delay)

  Check its frequency response:

$$
\begin{aligned}
A_{\mathrm{e}}\left(e^{j\omega}\right) &= \sum_{n=-L}^{L} h_{\mathrm{e}}[n] \cdot e^{-j\omega n} \\
&= h_{\mathrm{e}}[0] + \sum_{n=1}^{L} 2h_{\mathrm{e}}[n] \cdot \cos(\omega n) \\
&= a_0 + \sum_{n=1}^{L} a_k \cdot (\cos(\omega))^k \\
&= \sum_{n=0}^{L} a_k \cdot (\cos(\omega))^k \\
&= P(x)\big|_{x=\cos\omega}
\end{aligned}
$$

Note that $P(x) = \sum a_k x^k$ is an Lth-order polynominal. In the above process, we use a polynominal expression of cos(.), $\cos(\omega n) = T_n(\cos\omega)$, where $T_n(\cdot)$ is the nth-order Chebyshev polynominal. Thus, we shift our goal from finding (L+1) values of $\{h_e[n]\}$ to finding (L+1) values of $\{a_k\}$.

(want to use the polynominal approximation algorithms.)

<Our Problem now>

Adjustable parameters: $\{a_k\}$, (L+1) values

Specifications: $\omega_p, \omega_p, \delta_1/\delta_2 = K$, and L (L is often preselected)

Error criterion: $E(\omega) = W(\omega) \cdot \left[H_{\mathrm{d}}\left(e^{j\omega}\right) - A_{\mathrm{e}}\left(e^{j\omega}\right)\right]$

Goal: minimize the maximum error

$$\min_{\{h_e[n]\}^L}\left(\max_{\omega\in F}|E(\omega)|\right),$$ *F*: passband and stopband

(*Note:* Often, no constraint on the transition band)

(Why choose this minimization target? Even error values!

Recall: In the rectangular windowing method, we actually minimize $\varepsilon^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right)-H\left(e^{j\omega}\right)\right|^2 d\omega$. Although the total squared error can be small but errors at some frequencies may be large.)

<Alternation Theorem>

F_P:	closed subset consists of (the union) of disjoint closed subsets of the real axis x	*Example,* lowpass: $[0,\omega_p],[\omega_s,\pi]$ → $x=\cos\omega$ → $[1,\cos\omega_p],[\cos\omega_s,1]$
$P(x)$:	rth-order polynominal $P(x)=\sum_{k=0}^{r}a_k x^k$	$P(\cos\omega)=\sum_{k=0}^{L}a_k(\cos\omega)^k$
$D_P(x)$:	desired function of x continuous on F_P	$D_P(x)=\begin{cases}1, & x_p\le x\le 1\\ 0, & -1\le x\le x_s\end{cases}$ $x=\cos\omega$
$W_P(x)$:	weighting: positive, continuous on F_P	$W_P(x)=\begin{cases}1/K, & x_p\le x\le 1\\ 1, & -1\le x\le x_s\end{cases}$
$E_P(x)$:	weighted error $E_P(x)=W_P(x)[D_P(x)-P(x)]$	$E_P(x)=W_P(x)[D_P(x)-P(x)]$
$\|E\|$:	maximum error $\|E\|=\max_{x\in F_P}E_P(x)$	$\|E\|=\delta_2$

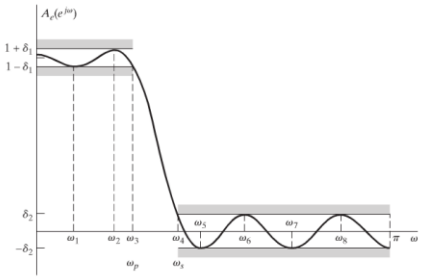
$P(x)$ is the *unique* rth-order polynominal that minimizes $\|E\|$

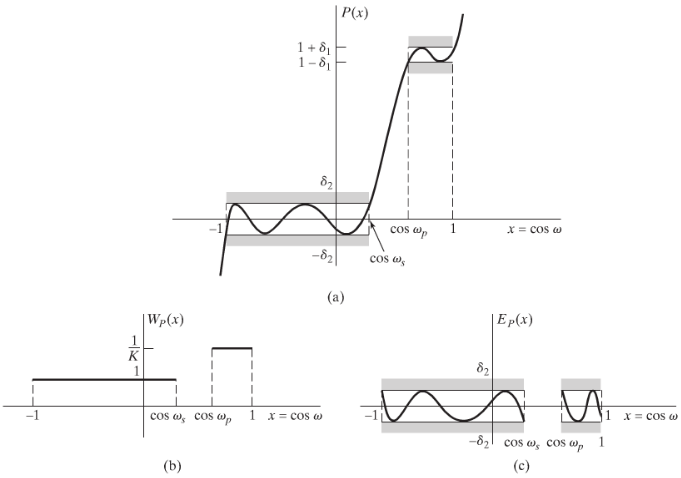
if and only if $E_P(x)$ exhibits *at least* (r+2) ***alternations***

Alternation: There exist (r+2) values x_i in F_P such that

$$E_P(x_i)=-E_P(x_{i+1})=\pm\|E\|,\ i=1,2,\cdots,(r+1),\text{ where } x_1<x_2<\cdots<x_{r+2}.$$

Remark: Two conditions here for alternation: value and sign.





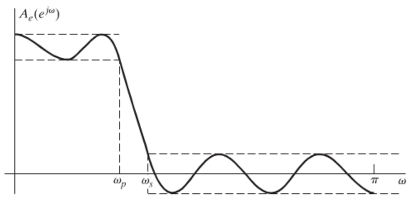
Type I linear phase FIR filter

(1) Maximum number of alternations of errors = (L+3)

(2) Alternations always occur at ω_p and ω_s

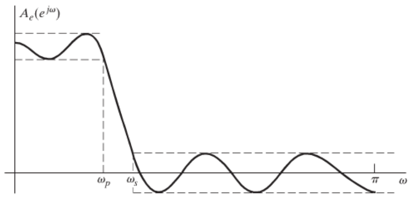
(3) Equiripple except possibly at $\omega = 0$ and $\omega = \pi$

L=7



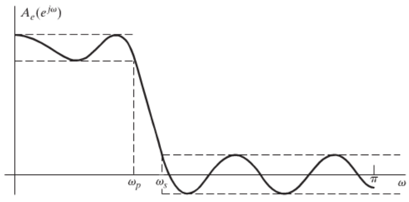
(a)

L+3



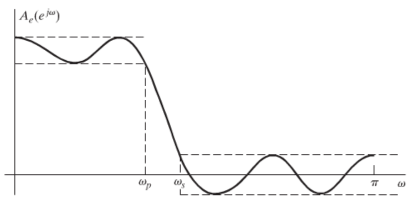
(b)

L+2



(c)

L+2



(d)

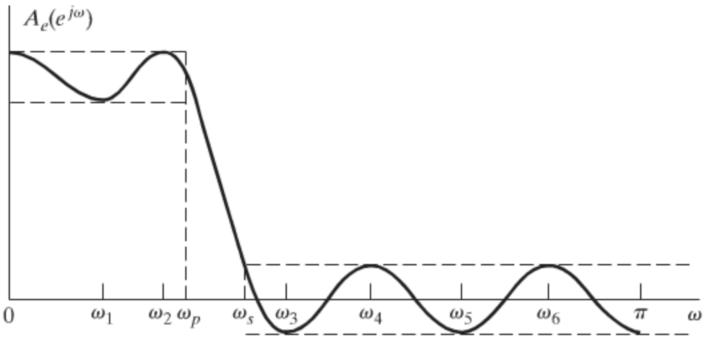
L+2

(Reasons)

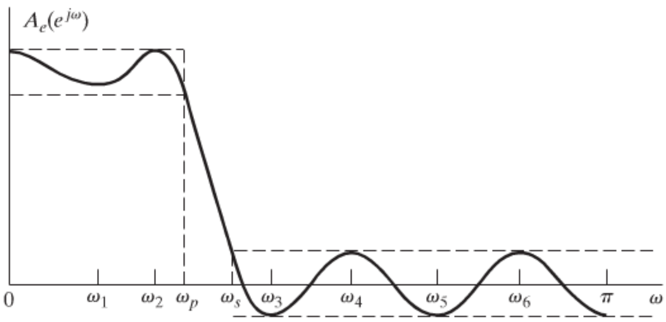
(a) Locations of extrema: *L*th-order polynominal has at most *L*-1 extrema. Now, in addition, the local extrema may locate at band edges $\omega = 0, \pi, \omega_p, \omega_s$. Hence, at most, there are (*L*+3) extrema or alternations.

(Note: Because $x = \cos\omega$, $\frac{dP(\cos\omega)}{d\omega} = 0$, at $\omega = 0$ and $\omega = \pi$.)

(b) If ω_p is not an alternation, for example, then because of the +- sign sequence, we loose two alternations → (*L*+1) alternations ← violates the (*L*+2) alternation theorem.



(c) The only possibility that the extrema can be a non-alternation is that it locates at $\omega = 0$ or $\omega = \pi$. In either case, we have (*L*+2) alternations – minimum requirement.



Type II linear phase FIR filter

Its symmetry property: $h_e[n] = h_e[M-n]$, M odd

Frequency response:

$$H\left(e^{j\omega}\right) = e^{-j\omega\frac{M}{2}}\left\{\sum_{n=1}^{(M+1)/2} b[n]\cdot\cos(\omega(n-1/2))\right\}$$

$$= e^{-j\omega\frac{M}{2}}\cos\left(\frac{\omega}{2}\right)\left\{\sum_{n=1}^{(M+1)/2}\tilde{b}[n]\cdot\cos(\omega n)\right\}$$

$\rightarrow$ $H\left(e^{j\omega}\right) = e^{-j\omega\frac{M}{2}}\cos\left(\frac{\omega}{2}\right)P(\cos\omega)$,

where $P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k$

Problem: How to handle $\cos\left(\frac{\omega}{2}\right)$?

Transfer specifications!

Let
$$H_d\left(e^{j\omega}\right) = D_p(\cos\omega) = \begin{cases} \dfrac{1}{\cos\left(\frac{\omega}{2}\right)}, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi \end{cases}$$

Original	New
Ideal: $D(\cos\omega) \Leftarrow \cos\left(\frac{\omega}{2}\right)P(\cos\omega)$	Ideal: $\dfrac{D(\cos\omega)}{\cos\left(\frac{\omega}{2}\right)} \Leftarrow P(\cos\omega)$

Thus,

$$W(\omega) = W_p(\cos\omega) = \begin{cases} \dfrac{\cos\left(\frac{\omega}{2}\right)}{K}, & 0 \le \omega \le \omega_p \\ \cos\left(\frac{\omega}{2}\right), & \omega_s \le \omega \le \pi \end{cases}$$

● Parks-McClellan Algorithm

<Type I Lowpass>

According to the preceding theorems, errors

$E(\omega) = W(\omega)\cdot\left[H_\mathrm{d}\left(e^{j\omega}\right) - A_\mathrm{e}\left(e^{j\omega}\right)\right]$ has alternations at $\omega_i, i = 1,\ldots, L+2$, if $A_\mathrm{e}\left(e^{j\omega}\right)$

is the *optimum* solution.

That is, let $\delta = \|E\|$, the maximum error,

$$W(\omega_i)\cdot\left[H_\mathrm{d}\left(e^{j\omega_i}\right) - A_\mathrm{e}\left(e^{j\omega_i}\right)\right] = (-1)^{i+1}\delta, \quad i = 1,2,..,L+2.$$

Because $A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos\omega)^k = a_0 1 + a_1\cos\omega + a_2(\cos\omega)^2 + \cdots$,

at ω_1: $a_0 1 + a_1\cos\omega_1 + a_2(\cos\omega_1)^2 + \cdots \leftrightarrow a_0 1 + a_1 x_1 + a_2(x_1)^2 + \cdots$

at ω_2: $a_0 1 + a_1\cos\omega_2 + a_2(\cos\omega_2)^2 + \cdots \leftrightarrow a_0 1 + a_1 x_2 + a_2(x_2)^2 + \cdots$

…

Hence,

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & \dfrac{1}{W(\omega_1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & \dfrac{-1}{W(\omega_2)} \\ \vdots & \ddots & & \ddots & & \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & \dfrac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \delta \end{bmatrix} = \begin{bmatrix} H_\mathrm{d}\left(e^{j\omega_1}\right) \\ H_\mathrm{d}\left(e^{j\omega_2}\right) \\ \vdots \\ H_\mathrm{d}\left(e^{j\omega_{L+2}}\right) \end{bmatrix}$$

Remark: For Type I lowpass filter, ω_p and ω_s must be two of the alternation frequencies $\{\omega_i\}$.

Now, we have *L*+2 simultaneous equations and *L*+2 unknowns, $\{a_i\}$ and δ.

The solutions are

$$\delta = \frac{\sum_{k=1}^{L+2} b_k H_\mathrm{d}\left(e^{j\omega_k}\right)}{\sum_{k=1}^{L+2} \dfrac{b_k(-1)^{k+1}}{W(\omega_k)}}, \quad b_k = \prod_{\substack{i=1 \\ i\neq k}}^{L+2} \frac{1}{(x_k - x_i)}.$$

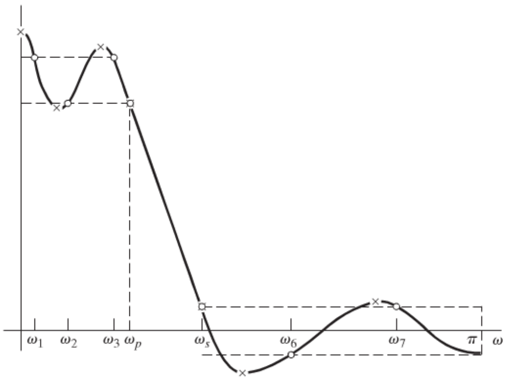
Once we know $\{a_i\}$, we can calculate $A_\mathrm{e}\left(e^{j\omega}\right)$ for all ω.

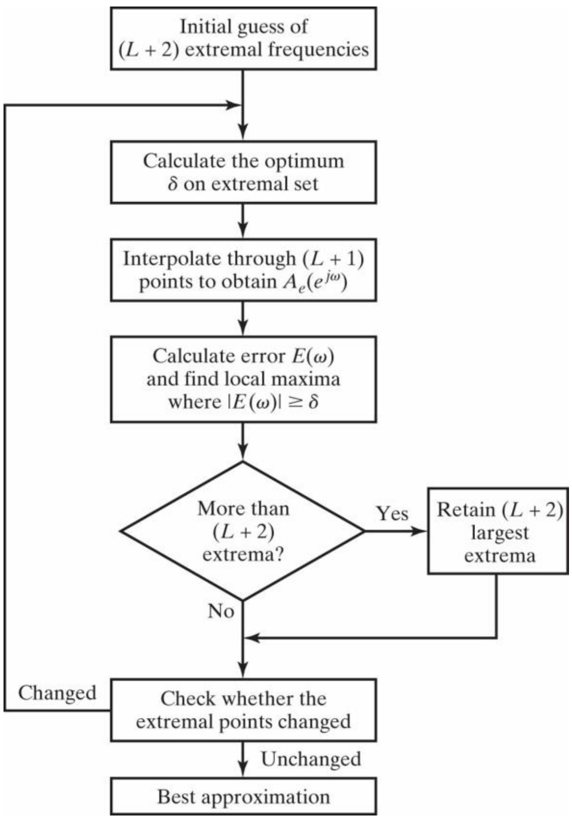
However, there is short cut. We can calculate $A_e\left(e^{j\omega}\right)$ for all ω directly based on $W(\omega_k), H_d\left(e^{j\omega_k}\right)$ and ω_k without solving for $\{a_i\}$.

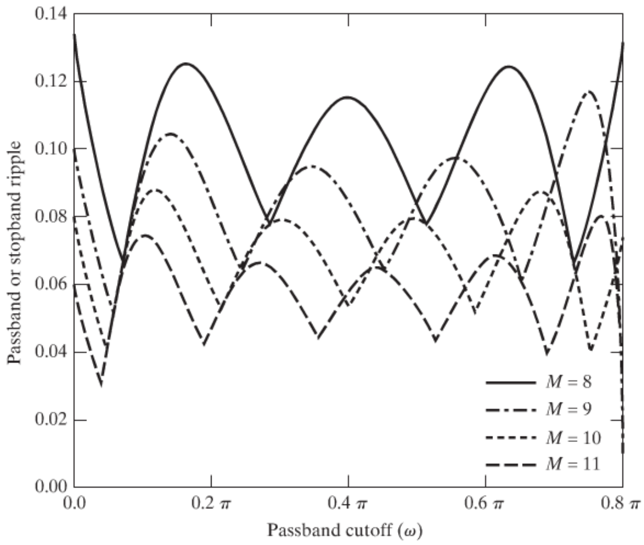
$$A_e\left(e^{j\omega}\right) = P(\cos\omega) = \frac{\sum_{k=1}^{L+1}\left[d_k \big/ (x - x_k) \right] c_k}{\sum_{k=1}^{L+1}\left[d_k \big/ (x - x_k) \right]},$$

where $c_k = H_d\left(e^{j\omega_k}\right) - \frac{(-1)^{k+1}\delta}{W(\omega_k)}$,

$$d_k = \prod_{i=1, i\neq k}^{L+1} \frac{1}{(x_k - x_i)}$$





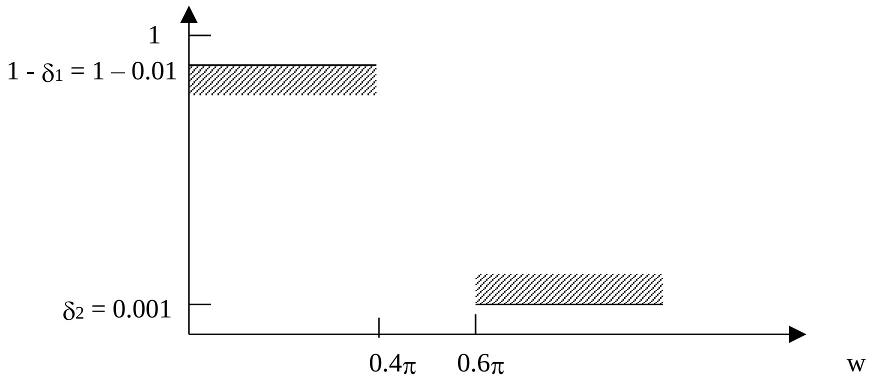


-- How to decide *M* (for lowpass)? (Experimental formula)

$$M = \frac{-10\log_{10}(\delta_1\delta_2)-13}{2.324\cdot\Delta\omega}$$

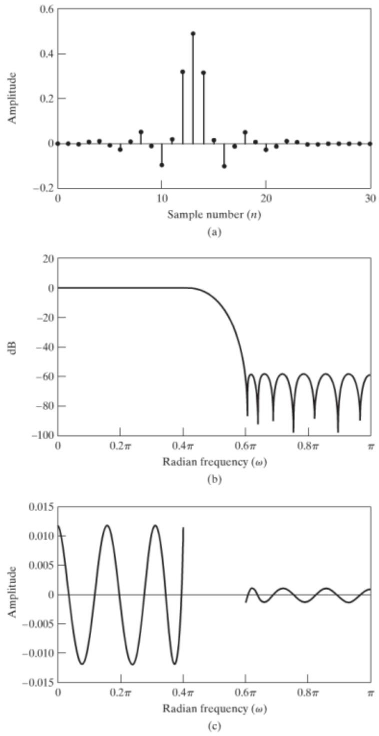
$$\Delta\omega = \omega_s - \omega_p$$

Example: Lowpass Filter



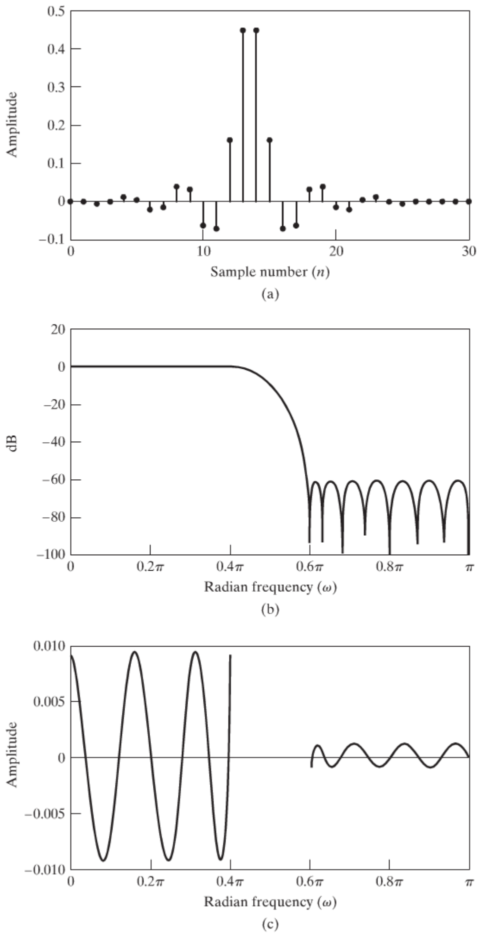
$$K = \frac{\delta_1}{\delta_2} = 10$$

$$M = \frac{-10\log_{10}(\delta_1\delta_2)-13}{2.324\cdot\Delta\omega} \Rightarrow \text{M} = 26$$



But the maximum errors in the passband and stopband are 0.0116 and 0.00116, respectively.

$\Rightarrow$ M = 27



Remark: The Kaiser window method requires a value M = 38 to meet or exceed the same specifications.

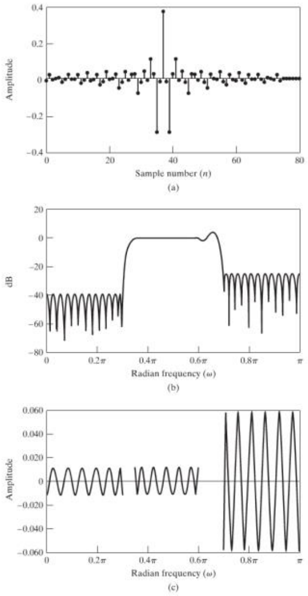
Example: Bandpass filter

Note: (1) From the alternation theorem

$\Rightarrow$ the minimum number of alternations for the optimum approximation is L + 2.

(2) Multiband filters can have more than L+3 alternations.

(3) Local extrema can occur in the transition regions.



IIR vs. FIR Filters

Property	FIR	IIR
Stability	Always stable	Incorporate stability constraint in design
Analog design	No meaningful analog equiv-alent	Simple transformation from an-alog filters
Phase linearity	Can be exact linear	Nonlinear typically
Computation	More multiplications and ad-ditions	Fewer
Storage	More coefficients	Fewer
Sensitivity to coefficient inaccuracy	Low sensitivity	Higher
Adaptivity	Easy	Difficult