Transform Analysis of LTI Systems

✧ Introduction

- Frequency response of LTI systems
- Linear constant-coefficient difference equations
- Magnitude and phase
- Minimum phase
- Linear phase
- An LTI system can completely be characterized in the time domain by its *impulse response*.

  (*Note:* assuming no initial conditions)

  Time: $y[n] = x[n] * h[n]$

  z-transform: $Y(z) = X(z) \cdot H(z)$; *H*(z) – *System function*

  Frequency response: $Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$; $H(e^{j\omega})$ -- *Frequency response*;

  Magnitude or *gain*: $|H(e^{j\omega})|$,

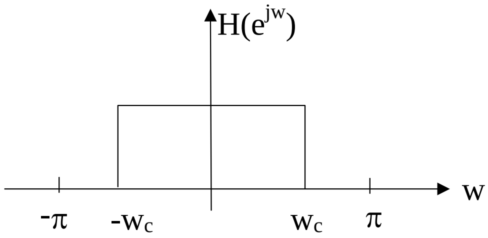
  Phase response or *phase shift*: $\angle H(e^{j\omega})$

✧ Frequency Response of LTI Systems

- **Frequency-selective filters**

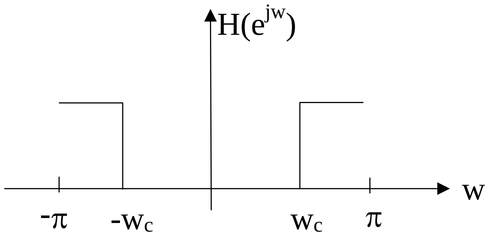
Ideal lowpass: $h_{lp}[n] = \dfrac{\sin(\omega_c n)}{\pi n}$

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$



Ideal highpass: $h_{hp}[n] = \delta[n] - \dfrac{\sin(\omega_c n)}{\pi n}$

$$H_{HP}(e^{j\omega}) = \begin{cases} 1, & \omega_c \le |\omega| \le \pi \\ 0, & \text{otherwise} \end{cases}$$



Remark: This definition includes the phase specification. That is, zero phase for all frequencies. Practical physical systems cannot achieve this specificaiton. In addition, it is non-causal and needs infinite input samples to compute the current output. They are not *computationally realizable*. Typically, the frequency-selective filters are referred to their *magnitude* specifications.

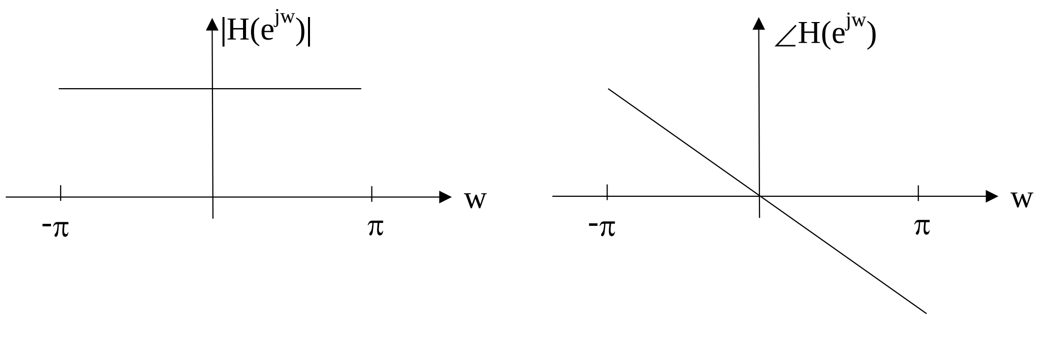
Terms: lowpass, highpass, bandpass,bandstop, all-pass filters

- **Phase delay**

$$h_{id}[n] = \delta[n - n_d] \;\leftrightarrow\; H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

The output is a delayed version of the input. (The shape of the waveform is not changed.)

$$\begin{cases} |H_{id}(e^{j\omega})| = 1 \\ \angle H_{id}(e^{j\omega}) = -\omega n_d, \ |\omega| < \pi \end{cases}$$



Linear phase: The phase repsonse is a linear function of ω (passing through the origin).

A frequency-selective filter with a linear phase is often acceptable and can be approximated by a practical system. That is,

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_1 \le |\omega| \le \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

Group delay:

$$\tau(\omega) \equiv \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\left\{\arg[H(e^{j\omega})]\right\}$$

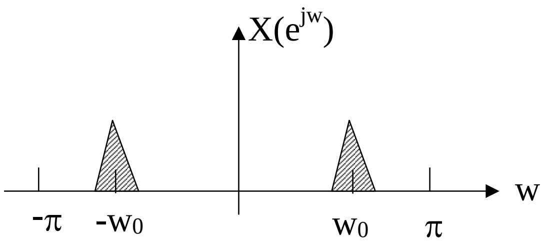
A convenient measure of the linearity of the phase. It is clear that for the ideal delay, $\tau(\omega) = n_d$ is a constant (independent of ω).

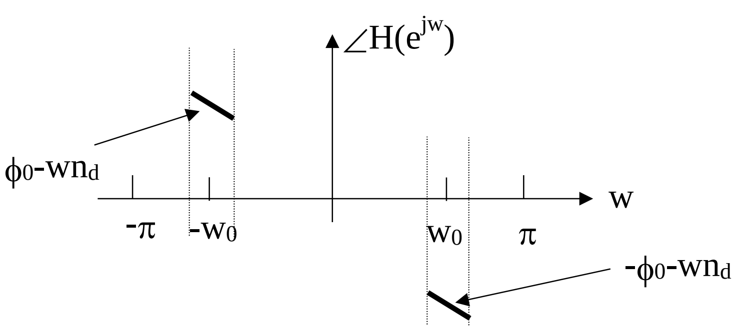
Motivation: Narrow-band signals in communication systems

$$\begin{aligned} &\text{Input}: && x[n] = s[n]\cos(\omega_0 n) \\ &\text{System}: && \angle H(e^{j\omega}) \approx -\phi_0 - \omega n_d \\ &\text{Output}: && y[n] = s[n-n_d]\cos(\omega_0 n - \phi_0 - \omega_0 n_d) \end{aligned}$$

$$X(e^{jw}) = \frac{1}{2}S(e^{j(w+w_0)}) + \frac{1}{2}S(e^{j(w-w_0)})$$

$$Y(e^{jw}) \approx \frac{1}{2}S(e^{j(w+w_0)})e^{j(\phi_0 - wn_d)} + \frac{1}{2}S(e^{j(w-w_0)})e^{j(-\phi_0 - wn_d)}$$



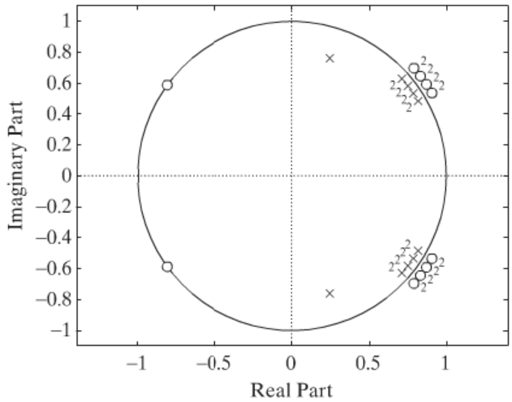


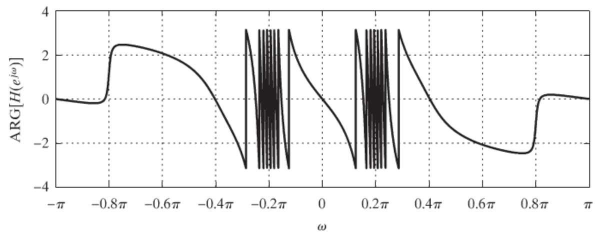
Remark: arg[H(.)]: *continuous phase* (|value| < π or > π)

ARG[H(.)]: *principal value* (|value| < π)

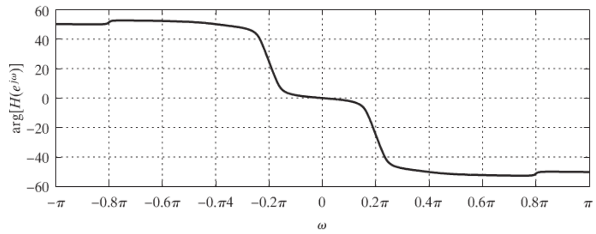
Example:

$$H(z)=\underbrace{\left(\frac{(1-0.98e^{j0.8\pi}z^{-1})(1-0.98e^{-j0.8\pi}z^{-1})}{(1-0.8e^{j0.4\pi}z^{-1})(1-0.8e^{-j0.4\pi}z^{-1})}\right)}_{H_1(z)}\underbrace{\prod_{k=1}^{4}\left(\frac{(c_k^*-z^{-1})(c_k-z^{-1})}{(1-c_kz^{-1})(1-c_k^*z^{-1})}\right)^2}_{H_2(z)}$$

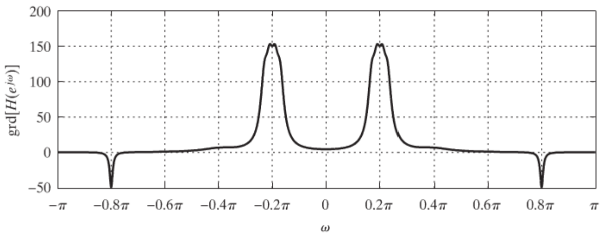




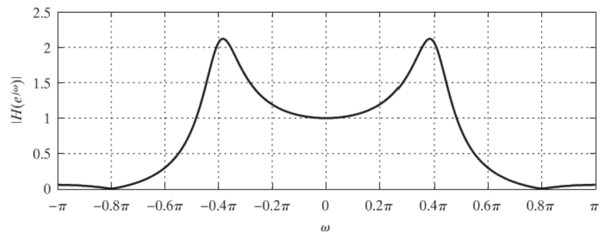
(a) Principle Value of Phase Response



(b) Unwrapped Phase Response



(a) Group delay of $H(z)$

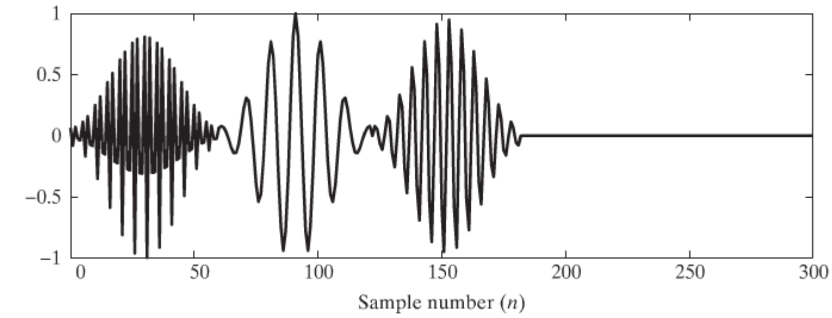


(b) Magnitude of Frequency Response

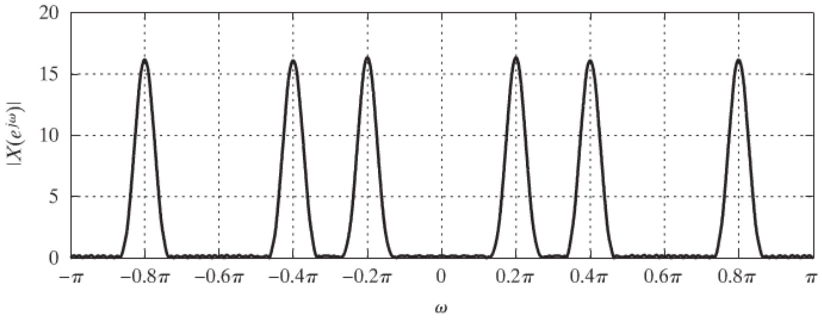
$$x_1[n] = w[n]\cos(0.2\pi n)$$
$$x_2[n] = w[n]\cos(0.4\pi n - \pi/2)$$
$$x_3[n] = w[n]\cos(0.8\pi n + \pi/5)$$

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le \text{n} \le \text{M} \\ 0, & \text{otherwise} \end{cases}$$

Input: $x[n] = x_3[n] + x_1[n - M - 1] + x_2[n - 2M - 2]$

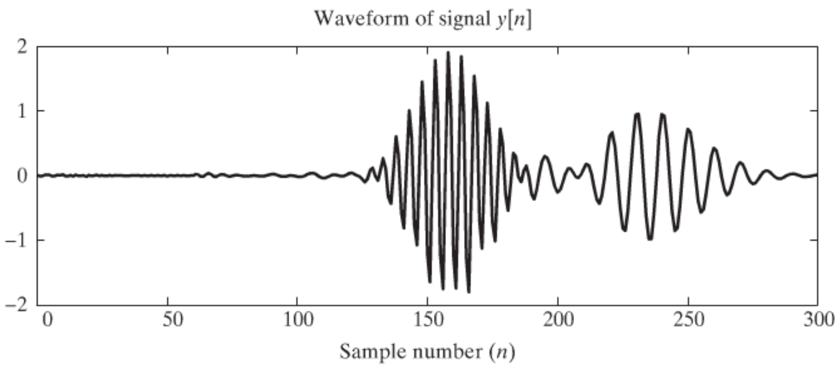


(a) Waveform of signal $x[n]$



(b) Magnitude of DTFT of $x[n]$

Output



✧ Systems Characterized by Linear Constant-coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \\ &= \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \end{aligned}$$

Note: X(z) and Y(z) have overlapping regions of convergence.

- **Stability** (BIBO)

  BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (absolutely summable)

  ⇔ ROC contains the unit circle.

- **Causality**

  Causal (right sequence) iff ROC: $|z| > r_{\max}$

- **Causal and Stable:** All poles are inside the unit circle.

  Example:

$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$$
$$\Rightarrow H(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - 2z^{-1})}$$

Remark: ***Schur-Cohn Stability Test***

To test if all the roots of A(z) lie inside the unit circle.

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

For $A_m(z) = 1 + a_m(1)z^{-1} + a_m(2)z^{-2} + \cdots + a_m(m)z^{-m}$

Define $B_m(z) = z^{-m} A_m(z^{-1}) = z^{-m} + a_m(1)z^{-(m-1)} + a_m(2)z^{-(m-2)} + \cdots + a_m(m)$

Set $A_N(z) = A(z)$

$$K_N = a_N(N)$$

$$\Rightarrow A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2} \qquad m = N, N-1, \cdots, 1$$

where $K_m = a_m(m)$

If $|K_m| < 1 \quad for \ m = 1, 2, \cdots, N$

$\Leftrightarrow$ all the roots of A(z) lie inside the unit circle.

Example:

$$H(z) = \frac{1}{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}$$

$$A_2(z) = 1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2} \qquad K_2 = -\frac{1}{2}$$

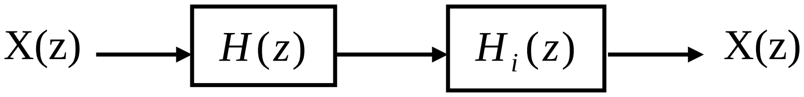
$$\Rightarrow \quad B_2(z) = -\frac{1}{2} - \frac{7}{4}z^{-1} + z^{-2}$$

$$\Rightarrow \quad A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2} = 1 - \frac{7}{2}z^{-1}$$

$$\Rightarrow \quad K_1 = -\frac{7}{2}$$

$\Rightarrow$ unstable

- **Inverse System:**



$$H(z)H_i(z) = 1$$

Or, $$H(z) = \frac{1}{H_i(z)}$$

Poles of $H(z)$ ➔ Zeros of $H_i(z)$; Zeros of $H(z)$ ➔ Poles of $H_i(z)$

$H_i(z)$ is *causal and stable* if all the zeros of $H(z)$ are inside the unit circle.
Minimum-phase: both poles and zeros are inside the unit circle.

- **Impulse Response for Rational System Functions**

IIR (Infinite Impulse Response):

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

At least one nonero pole $\{d_k\}$ is not cancelled by zero.

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k (d_k)^n u[n].$$

Example: $y[n] - ay[n-1] = x[n]$

$$H(z) = \frac{1}{1-az^{-1}}$$

FIR (Finite Impulse Response):

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

No nonzero poles (All poles are at "0").

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

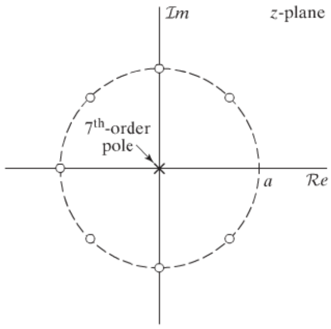
Ex.

$$h[n] = \begin{cases} a^n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1-a^{M+1} z^{-M-1}}{1-az^{-1}}$$

$$z_k = ae^{j2\pi k/(M+1)}, \qquad k = 0,1,\ldots,M$$

e.g. M = 7



✧ Frequency Response for Rational System Functions

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

<u>Log magnitude</u>: unit in *decibels* (dB)

$$20\log_{10}|H(e^{j\omega})| = 20\log_{10}\left|\left(\frac{b_0}{a_0}\right)\right| + \sum_{k=1}^{M} 20\log_{10}|1 - c_k e^{-j\omega}|$$
$$- \sum_{k=1}^{N} 20\log_{10}|1 - d_k e^{-j\omega}|$$

Clearly, $20\log_{10}|Y(e^{j\omega})| = 20\log_{10}|H(e^{j\omega})| + 20\log_{10}|X(e^{j\omega})|$

<u>Phase</u>:

$$\angle H(e^{j\omega}) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M}\angle(1 - c_k e^{-j\omega}) - \sum_{k=1}^{N}\angle(1 - d_k e^{-j\omega})$$

<u>Group delay</u>:

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N} \frac{d}{d\omega}(\arg[1 - d_k e^{-j\omega}]) - \sum_{k=1}^{M} \frac{d}{d\omega}(\arg[1 - c_k e^{-j\omega}])$$

- *Problem?* Phase ambiguity. The real filter phase is continuous and can be greater than π or smaller than $-\pi$. However, the calculated phase is the principal value,

$$-\pi < \mathrm{ARG}[H(e^{j\omega})] < \pi$$

Therefore, $\angle H(e^{j\omega}) = \mathrm{ARG}[H(e^{j\omega})] + 2\pi \cdot r(\omega)$ where $r(\omega)$ is an integer and can be a funciton of ω. How to find $r(\omega)$? It can be rather complicated and we skip the details here.

However, the phase *group delay* is identical in both cases (except for the discontinuties of $\mathrm{ARG}[H(e^{j\omega})]$).

$$\frac{d}{d\omega}\left\{\mathrm{ARG}[H(e^{j\omega})]\right\} = \frac{d}{d\omega}\left\{\arg[H(e^{j\omega})]\right\}$$

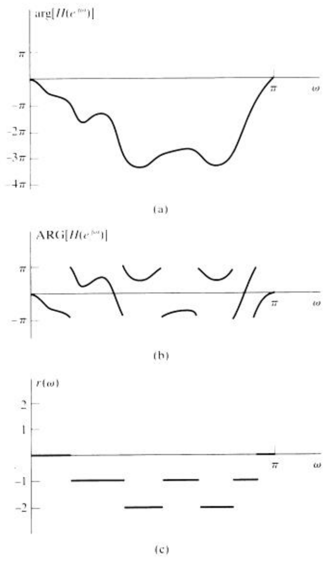


Figure 5.7 (a) Continuous-phase curve for a system function evaluated on the unit circle. (b) Principal value of the phase curve in part (a). (c) Integer multiples of 2π to be added to $\mathrm{ARG}[H(e^{j\omega})]$ to obtain $\arg[H(e^{j\omega})]$.

- **Single zero or pole**

$$H(z) = (\frac{b_0}{a_0})\frac{1}{1-dz^{-1}} \quad \text{or} \quad (\frac{b_0}{a_0})\frac{1-cz^{-1}}{1-dz^{-1}} \quad \text{or} \quad c_0(1-cz^{-1})$$

$$H(e^{jw}) = (\frac{b_0}{a_0})\frac{1}{1-de^{-jw}} \quad \text{or} \quad (\frac{b_0}{a_0})\frac{1-ce^{-jw}}{1-de^{-jw}} \quad \text{or} \quad c_0(1-ce^{-jw})$$

For the factor $(1-re^{j\theta}e^{-jw})$

$$20\log_{10}\left|1-re^{j\theta}e^{-jw}\right| = 10\log_{10}[1+r^2-2r\cos(w-\theta)]$$

$$ARG[1-re^{j\theta}e^{-jw}] = \tan^{-1}[\frac{r\sin(w-\theta)}{1-r\cos(w-\theta)}]$$

$$grd[1-re^{j\theta}e^{-jw}] = \frac{r^2-r\cos(w-\theta)}{1+r^2-2r\cos(w-\theta)}$$

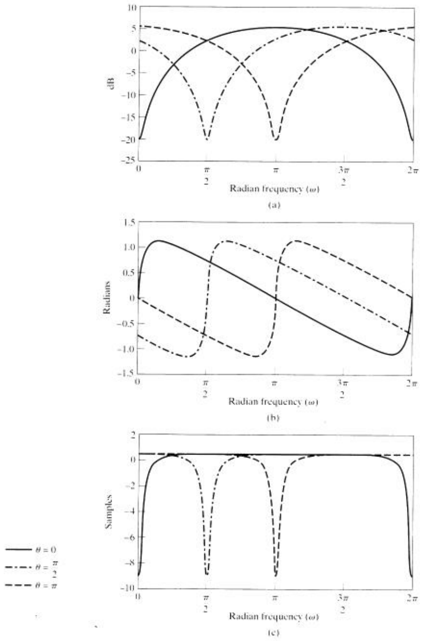


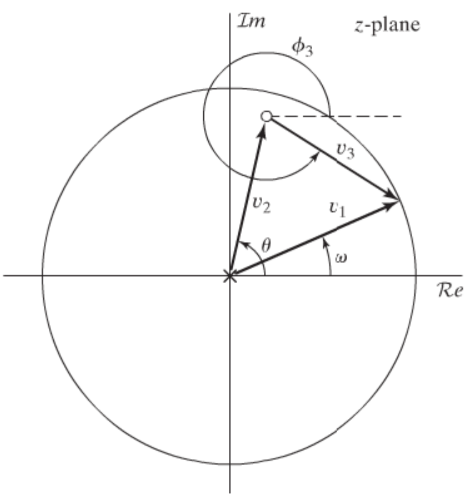
Figure 5.8 Frequency response for a single zero, with $r = 0.9$ and the three values of θ shown. (a) Log magnitude. (b) Phase. (c) Group delay.

$$H(z) = (1 - re^{j\theta} z^{-1}) = \frac{z - re^{j\theta}}{z}$$

$$H(e^{jw}) = \frac{e^{jw} - re^{j\theta}}{e^{jw}}$$

$$\left|1 - re^{j\theta} e^{-jw}\right| = \left|\frac{e^{jw} - re^{j\theta}}{e^{jw}}\right| = \frac{|\vec{v}_3|}{|\vec{v}_1|}$$

$$\angle(1 - re^{j\theta} e^{-jw}) = \angle(e^{jw} - re^{j\theta}) - \angle e^{jw} = \angle\vec{v}_3 - \angle\vec{v}_1 = \phi_3 - \omega$$

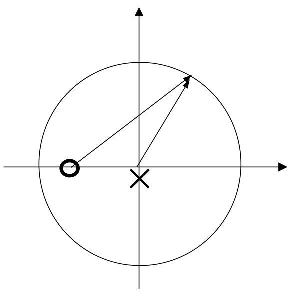


In general,

$$H(z)=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}=\left(\frac{b_0}{a_0}\right)z^{N-M}\,\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-d_k)}$$

$$\begin{cases}\left|H(e^{j\omega})\right|=\left|\dfrac{b_0}{a_0}\right|\dfrac{\prod_{k=1}^{M}|e^{j\omega}-c_k|}{\prod_{k=1}^{N}|e^{j\omega}-d_k|}\\ \angle H(e^{j\omega})=\angle\left(\dfrac{b_0}{a_0}\right)+\sum_{k=1}^{M}\angle(e^{j\omega}-c_k)-\sum_{k=1}^{N}\angle(e^{j\omega}-d_k)+(N-M)\omega\end{cases}$$

Example:



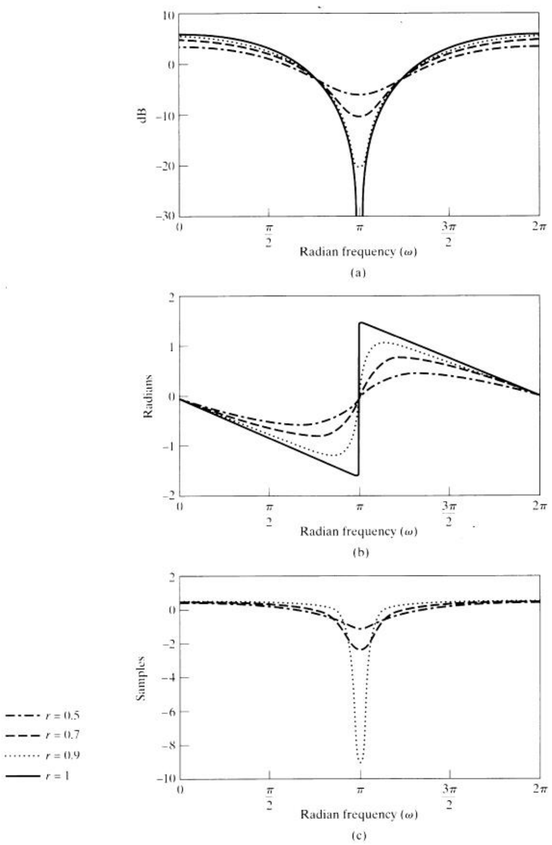


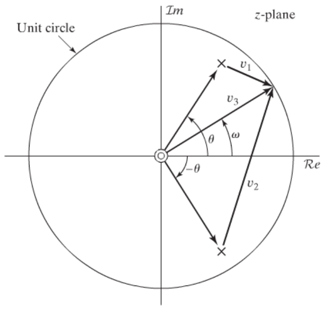
Figure 5.11 Frequency response for a single zero, with $\theta=\pi$, $r=1$, 0.9, 0.7, and 0.5. (a) Log magnitude. (b) Phase. (c) Group delay for $r=0.9$, 0.7, and 0.5.

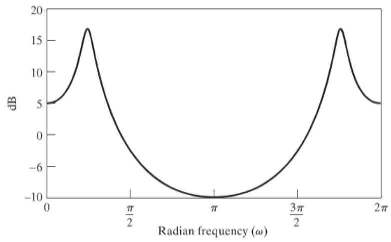
- **A second-order IIR system** (conjugate poles)

$$H(z)=\frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

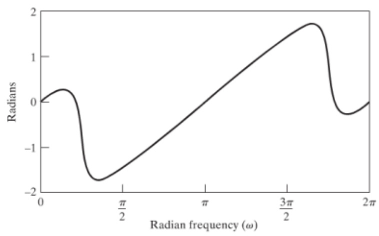
$$\begin{cases}\left|H(e^{j\omega})\right| == \dfrac{1}{|\vec{v_1}||\vec{v_2}|} \\ \angle H(e^{j\omega}) = -\angle(e^{j\omega}-re^{j\theta})-\angle(e^{j\omega}-re^{-j\theta})\end{cases}$$

Example:

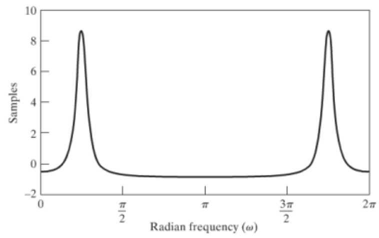




(a)



(b)



(c)

- **A second-order FIR system** (conjugate zeros)

$$H(z)=(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})=1-2r\cos\theta z^{-1}+r^2z^{-2}$$

✧ Relationship Between Magnitude and Phase

- For rational system functions, there is some constraint between magnitude and phase.
- Given the *number* of poles and zeros and the *magnitude (phase) response*, there are only a *finite* number of possible *phase (magnitude) responses*.
- Example: Given magnitude (square), try to decide its phase.

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega})$$
$$= H(z)H^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j\omega}}$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k^* z)}$$

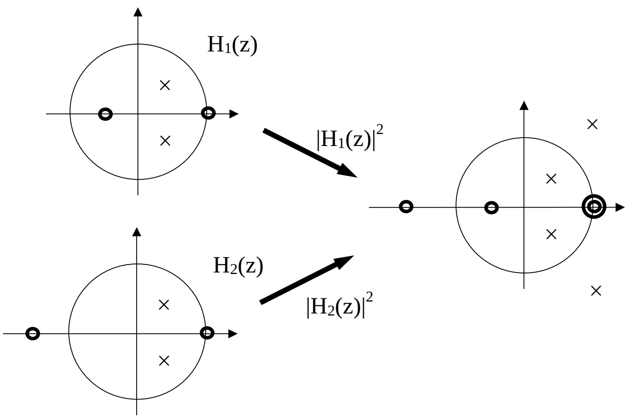
(because $(1-c_k z^{-1})^* = (1-c_k{}^*(z^{-1})^*) \Rightarrow (1-c_k{}^* z)$ for $z \to (z^*)^{-1}$)

Let

$$C(z) = H(z)H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}\frac{\prod_{k=1}^{M}(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k^* z)}$$

For each pole d_k of $H(z)$ ==> poles d_k and $(d_k^*)^{-1}$ of $C(z)$

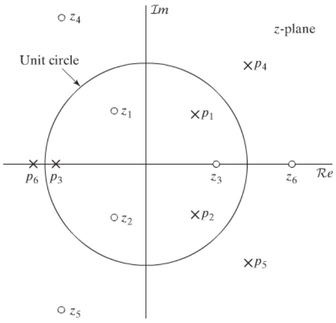
For each pole c_k of $H(z)$ ==> poles c_k and $(c_k^*)^{-1}$ of $C(z)$



If $H(z)$ is causal and stable, all its poles are inside the unit circle.

Given $C(z)$, we may want to find a causal, stable and real $H(z)$.

Example:



Possible zero patterns: z_3 or z_6 and (z_1,z_2) or (z_4,z_5)

If H(z) corresponds to a stable causal system ⇒ poles are p_1, p_2, and p_3.

Given $H_1(z)$,

let $H(z) = H_1(z)\dfrac{z^{-1}-a^*}{1-az^{-1}}$

then $|H(z)|^2 = |H_1(z)|^2$

If the number of poles and zeros of H(z) is unspecified ⇒ infinite choices of H(z).

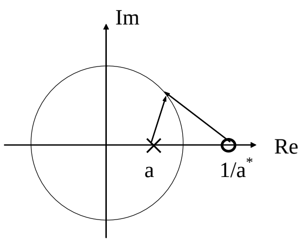
✧ All-pass Systems

- Magnitude = constant; all frequency components can pass through (but the phase is not linear).

$$H_{\text{ap}}(z) = \frac{z^{-1}-a^*}{1-az^{-1}} = -a^* \frac{z-\frac{1}{a^*}}{z-a}$$

pole : a
zero : $\dfrac{1}{a^*}$

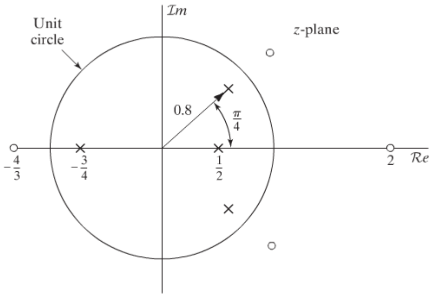
$$H_{\text{ap}}(e^{j\omega}) = \frac{e^{-j\omega}-a^*}{1-ae^{-j\omega}} = e^{-j\omega}\cdot\frac{1-a^*e^{j\omega}}{1-ae^{-j\omega}}$$



For $a = re^{j\theta}$,
$$\begin{cases} |H_{\text{ap}}(e^{j\omega})| = \text{constant} \\ \angle\left[\dfrac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right] = -\omega - 2\arctan\left[\dfrac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] \end{cases}$$

- A general all-pass filter is a product of the first-order and second-order factors.

  General form: $H_{ap}(z) = A \cdot \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$



- The phase of a stable and causal all-pass filter is non-positive.
  The group delay of a stable and causal all-pass system is non-negative.

(Pf) For $a = re^{j\theta}$, $grd\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{1 - r^2}{|1 - re^{j\theta}e^{-j\omega}|^2} \geq 0$

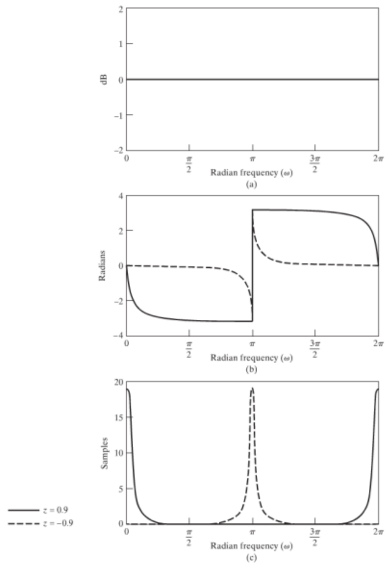
$$\arg[H_{\text{ap}}(e^{j\omega})] = -\int_0^{\omega} grd[H_{\text{ap}}(e^{j\phi})]d\phi + \arg[H_{\text{ap}}(e^{j0})] \leq 0, \text{ for } 0 \leq \omega < \pi$$

$$H_{\text{ap}}(e^{j0}) = A\prod_{k=1}^{M_r} \frac{1 - d_k}{1 - d_k} \times \prod_{k=1}^{M_c} \frac{|1 - e_k|^2}{|1 - e_k|^2} = A$$

- All-pass filters can be used as phase compensators. They are useful in transforming frequency-selective low-pass filters into other frequency-selective form and in obtaining variable-cutoff filters.

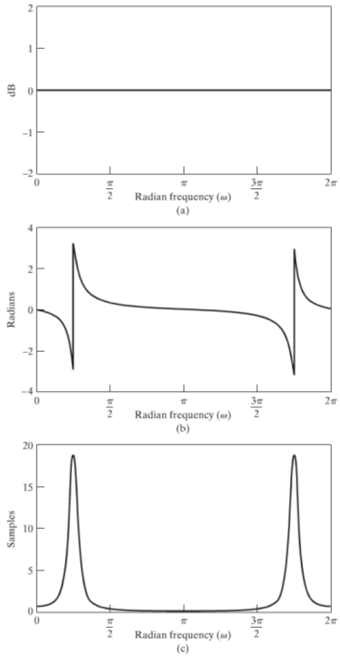
Example: $H_{ap}(z) = \frac{z^{-1} - 0.9}{1 - 0.9z^{-1}}$ Solid Line

$H_{ap}(z) = \frac{z^{-1} + 0.9}{1 + 0.9z^{-1}}$ Dashed Line



Example:

$$H_{ap}(z)=\frac{(z^{-1}-0.9e^{-j\frac{\pi}{4}})(z^{-1}-0.9e^{j\frac{\pi}{4}})}{(1-0.9e^{j\frac{\pi}{4}}z^{-1})(1-0.9e^{-j\frac{\pi}{4}}z^{-1})}$$



✧ Minimum-phase Systems

- *Minimum-phase system*:

  H(z) and its inverse 1/H(z) are both causal and stable.

  ➔ All poles and zeros are inside the unit circle.

  ➔ Given a magnitude-squared function, H(z) is uniquely determined.

- For any (stable, causal) rational system function $H(z)$, it can be expressed by

$$H(z)=H_{\min}(z)\cdot H_{ap}(z)$$

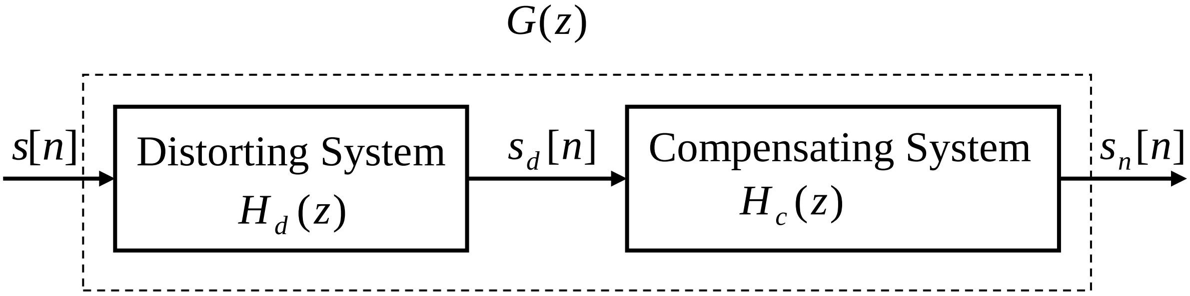
(Pf) Suppose $H(z)$ has one zero (z =1/c*, where |c|<1) outside the unite circle (and the remaining poles and zeros are inside the unit circle). Then,

$$H(z)=H_1(z)(z^{-1}-c^*)$$
$$=H_1(z)(1-cz^{-1})\frac{z^{-1}-c^*}{1-cz^{-1}}$$

Note: $H_1(z)(1-cz^{-1})$ is minimum phase.

The above procedure can be extended to general cases.

- A use of minimum-phase system – frequency-response compensation



Design $H_c(z)$ such that $G(z) = H_d(z) \cdot H_c(z)$ is desired.

For example, if we wish $G(z) = H_d(z) \cdot H_c(z) = const$

Let $H_d(z) = H_{d\min}(z) \cdot H_{ap}(z)$,

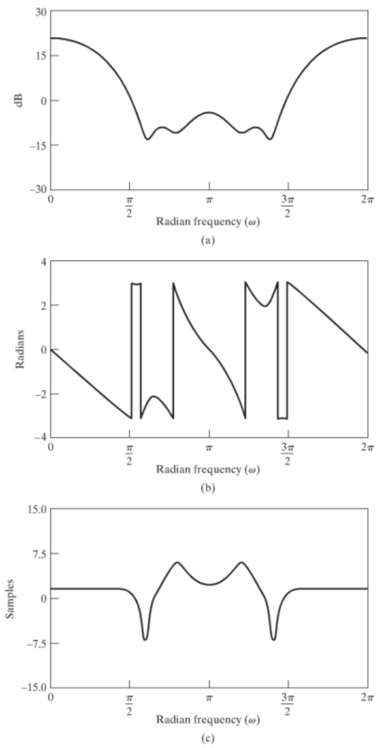
then choose $H_c(z) = \dfrac{1}{H_{d\min}(z)}$ instead.

$$G(z) = H_d(z)H_c(z) = H_{ap}(z)$$

$$\Rightarrow \quad |G(z)| = 1$$

$$\angle G(e^{jw}) = \angle H_{ap}(e^{jw})$$

Example: $H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})$

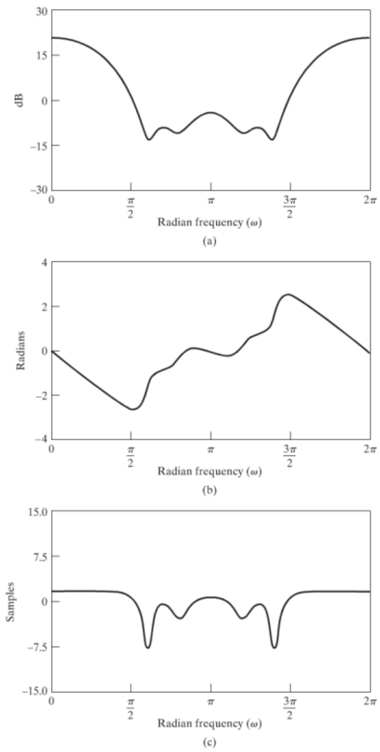


$$H(z) = H_{\min}(z)H_{ap}(z)$$

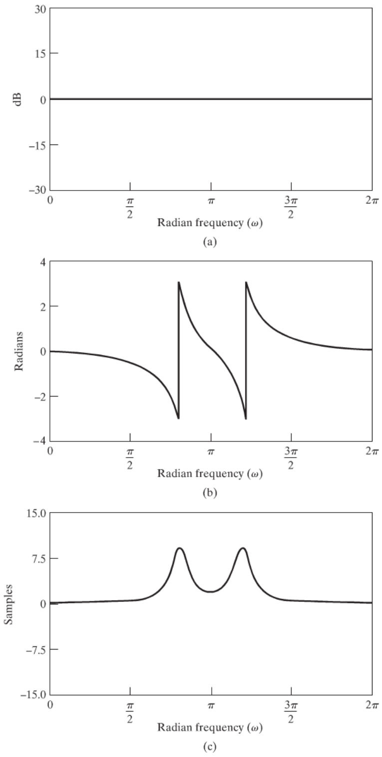
$$H_{\min}(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})$$

$$H_{ap}(z) = \frac{(z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi})}{(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})}$$

$H_{\min}(z)$



$H_{ap}(z)$



- **Properties**

  - ***Minimum Phase-lag Property***

    Why this type of systems is called "minimum phase"?

    Given the magnitude specification $|H(e^{j\omega})|$ of a system, find the one that has the least phase-lag. ➔ Minimum-phase system.

(Pf) $H(z) = H_{d\min}(z)\cdot H_{ap}(z)$

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

It was shown that for allpass filters, $\arg[H_{ap}(e^{j\omega})] < 0,\ 0 \le \omega < \pi.$

Thus, $\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \text{negative value}$

Hence, $H_{\min}(z)$ has the minimum phase-lag (less negative).

Remark: To ensure the minimum phase-lag property, (in addition to the pole and zero locations),

$$H(e^{j0}) = \sum_{n=-\infty}^{\infty} h[n] > 0$$

- ***Minimum Group-delay Property***

$$\text{grd}\left[H(e^{j\omega})\right] = \text{grd}\left[H_{\min}(e^{j\omega})\right] + \underbrace{\text{grd}\left[H_{\text{ap}}(e^{j\omega})\right]}_{>0} \qquad 0 \le \omega \le \pi$$

$$\Rightarrow \ \text{grd}\left[H_{\min}(e^{j\omega})\right] < \text{grd}\left[H(e^{j\omega})\right]$$

- ***Minimum Energy-delay Property***

Same $|H(e^{j\omega})|$ → Different $H(z)$ (differ in phase)

$(H(z) \leftrightarrow h[n])$

Define **partial energy** (of impulse response) to be

$$E[n] = \sum_{m=0}^{n} \left|h[m]\right|^2 .$$

Then the minimum-phase system $H_{\min}(z)$ has the largest $E[n]$ among all possible $H(z)$. That is, it accumulates more energy up to *n*.

Remarks: (1) Special case: for *n*=0, $|h[0]| < |h_{\min}[0]|$.

(2) Total energy is the same for all systems with the same magnitude response (Paseval's theorem).

$$\sum_{n=0}^{\infty} \left|h[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{jw})\right|^2 dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{\min}(e^{jw})\right|^2 dw = \sum_{n=0}^{\infty} \left|h_{\min}[n]\right|^2$$

(3) **Maximum-phase** system: All its zeros are outside the unit circle.

→ Maximum energy-delay.

Example:

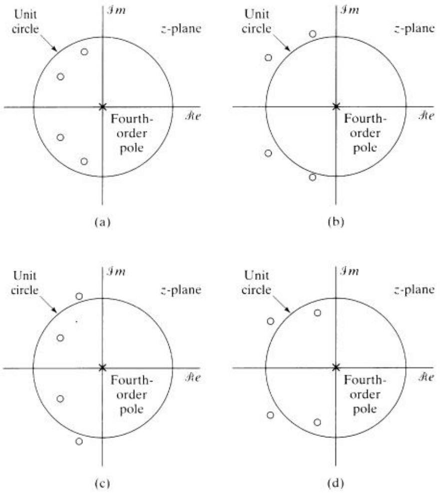


Figure 5.30 Four systems, all having the same frequency-response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals.

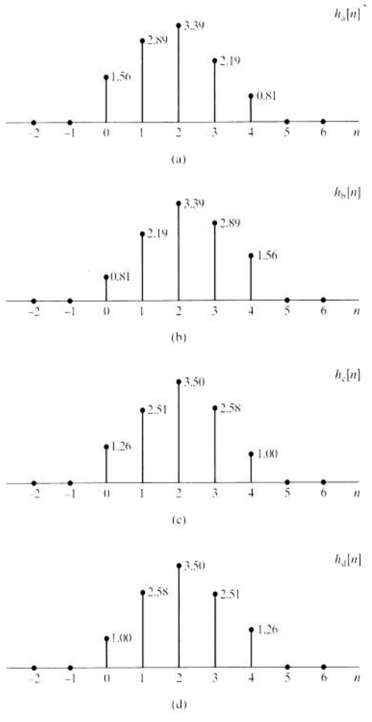
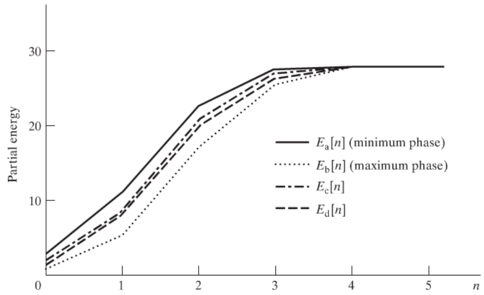


Figure 5.31 Sequences corresponding to the pole–zero plots of Figure 5.30.



✧ Generalized Linear Phase Systems

- For causal systems, zero phase is not attainable.

  ⇒ In many situations, it's particularly desirable to design systems to have exactly or approximately linear phase.

- Ideal Delay: $H_{\text{id}}(e^{j\omega}) = e^{-j\omega\alpha} \quad \leftrightarrow \quad h_{\text{id}}[n] = \delta[n-\alpha]$

  $$\begin{cases} \left|H_{\text{id}}(e^{j\omega})\right| = 1 \\ \angle H_{\text{id}}(e^{j\omega}) = -\omega\alpha \end{cases} \quad \leftrightarrow \quad \text{grd}\left[H_{\text{id}}(e^{j\omega})\right] = \alpha$$

- Linear phase $H(e^{j\omega}) = \left|H(e^{j\omega})\right| \cdot e^{-j\omega\alpha}, \quad |\omega| < \pi$

  $$\begin{cases} \left|H(e^{j\omega})\right| = \text{any} \\ \angle H(e^{j\omega}) = -\omega\alpha, \quad \alpha : \text{can be non-integer} \end{cases}$$

Ex., Symmetry of $h[n]$ in the ideal delay system:

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

$$h_{\text{lp}}[n] = \frac{\sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}$$

1. $\alpha = \text{integer}$

   $h_{\text{lp}}[n]: \text{even} \qquad \left(h[2\alpha - n] = h[n]\right)$
2. $\alpha = \text{integer} + \frac{1}{2}$

   $h_{\text{lp}}[n]: \text{even} \qquad \left(h[2\alpha - n] = h[n]\right)$
3. Othewise,

   no symmetry

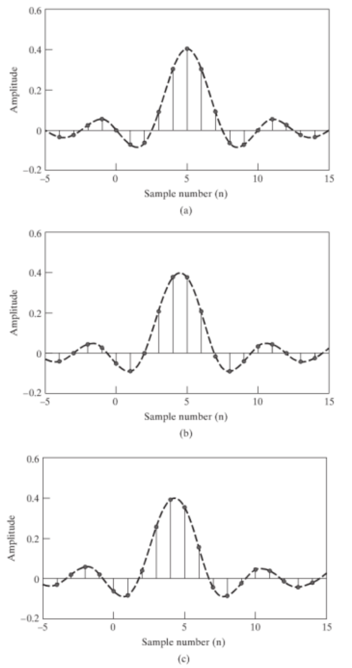


Figure 5.32 Ideal lowpass filter impulse responses, with $\omega_c = 0.4\pi$. (a) Delay $= \alpha = 5$. (b) Delay $= \alpha = 4.5$. (c) Delay $= \alpha = 4.3$.

We observe a symmetric property, $h[2\alpha - n] = h[n]$, for 2α = interger in the above linear-phase systems.

■ ***Generalized Linear Phase***

$$H(e^{j\omega}) = A(e^{j\omega}) \cdot e^{-j\alpha\omega + j\beta},\ \ \alpha, \beta : \text{real constants}$$

$$A(e^{j\omega}) : \text{real but can be negative}$$

$$\text{Special case}: \quad \beta = 0 \ \rightarrow \ \text{linear phase}$$

$$\begin{cases} \text{Group delay}: & \tau(\omega) = \text{grd}\left[H(e^{j\omega})\right] = \alpha \\ \text{Phase}: & \arg\left[H(e^{j\omega})\right] = \beta - \alpha\omega, \quad 0 < \omega < \pi \end{cases}$$

Essentially, this system has a *constant* group delay.

■ Symmetry property of a generalized system

$$H(e^{j\omega}) = A(e^{j\omega}) e^{j(\beta - \alpha\omega)}$$
$$= A(e^{j\omega}) \cos(\beta - \alpha\omega) - jA(e^{j\omega}) \sin(\beta - \alpha\omega)$$

On the other hand,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n] \cos \omega n - j \sum_{n=-\infty}^{\infty} h[n] \sin \omega n$$

$$\tan(\beta - \alpha\omega) = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \frac{-\sum_{n=-\infty}^{\infty} h[n] \sin \omega n}{\sum_{n=-\infty}^{\infty} h[n] \cos \omega n}$$

Note: $A(e^{j\omega})$ disappear!

Cross-multiplying:

$$\sin(\beta - \alpha\omega) \cdot \sum_{n=-\infty}^{\infty} h[n] \cos \omega n = -\cos(\beta - \alpha\omega) \cdot \sum_{n=-\infty}^{\infty} h[n] \sin \omega n$$

$$\sum_{n=-\infty}^{\infty} h[n] \left\{ \cos \omega n \sin(\beta - \alpha\omega) + \sin \omega n \cos(\beta - \alpha\omega) \right\} = 0$$

$$\sum_{n=-\infty}^{\infty} h[n] \sin(\omega n - \omega\alpha + \beta) = 0, \quad \forall \omega, \quad \text{eqn.}(*)$$

Remark: This is the necessary conditions of constant group delay. That is, there is a constraint on the key parameters: $h[n], \alpha, \beta$.

There are many solutions that satisfy eqn.(*).

<Solution 1>

$$\begin{cases} \beta = 0 \text{ or } \pi \\ 2\alpha = M = \text{integer} \\ h[2\alpha - n] = h[n] \end{cases}$$

To see this is a solution, we plug in.

If α is integer,

$$\sum_{n=-\infty}^{\infty} h[n]\sin\left(\omega\left(n-\frac{M}{2}\right)\right)$$
$$= h\left[\frac{M}{2}\right]\sin\left(\omega\left(\frac{M}{2}-\frac{M}{2}\right)\right) + \sum_{n=\frac{M}{2}+1}^{\infty} h[n]\left(\sin\left(\omega\left(n-\frac{M}{2}\right)\right)+\sin\left(\omega\left(M-n-\frac{M}{2}\right)\right)\right)$$
$$= \sum_{n=\frac{M}{2}+1}^{\infty} h[n]\left(\sin\left(\omega\left(n-\frac{M}{2}\right)\right)+\sin\left(\omega\left(\frac{M}{2}-n\right)\right)\right)$$
$$= 0$$

Note: We only require that h[n] satisfies a symmetric property. There is no constraint on the numerical values of h[n].

Remark: It can be shown that $A(e^{j\omega})$ is even (and real).

<Solution 2>

$$\begin{cases} \beta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\ 2\alpha = M = \text{integer} \\ h[2\alpha - n] = -h[n] \end{cases} \Rightarrow \sin\left(\omega\left(n-\frac{M}{2}\right)+\frac{\pi}{2}\right) = \cos\left(\omega\left(n-\frac{M}{2}\right)\right)$$

Check if this is a valid solution.

$$\sum_{n=-\infty}^{\infty} h[n]\cos\left(\omega\left(n-\frac{M}{2}\right)\right)$$
$$= \sum_{n=\frac{M}{2}+1}^{\infty} h[n]\left(\cos\left(\omega\left(n-\frac{M}{2}\right)\right)-\cos\left(\omega\left(\frac{M}{2}-n\right)\right)\right)$$
$$= 0$$

Remark: It can be shown that $A(e^{j\omega})$ is odd (and real).

There are *other possible* solutions, for example, fractional delay.

■ Causal Generalized Linear-phase Systems

If a generalized linear-phase system is also causal, then eqn.(*) becomes

$$\sum_{n=0}^{\infty} h[n]\sin(\omega n - \omega\alpha + \beta) = 0, \quad \forall \omega$$

Under this causal condition and $h[2\alpha - n] = h[n]$,

$$h[n] = 0, \quad n < 0 \text{ and } n > M .$$

Note: This is an FIR filter.
More precisely, if the filter length is M+1 and

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & otherwise \end{cases} \qquad \text{(symmetric w.r.t. } M/2)$$

Then, $H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$, where $A_e(e^{j\omega})$ is real and even.
If

$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M, \\ 0, & otherwise \end{cases} \qquad \text{(anti-symmetric w.r.t. } M/2)$$

then, $H(e^{j\omega}) = A_o(e^{j\omega})e^{-j\omega M/2 + j\pi/2}$, where $A_o(e^{j\omega})$ is real and odd.

Remark: Nearly all linear-phase filters are FIR filters. (There are special types of causal IIR filters that have generalized linear phase, but they cannot be implemented by difference equations) The above two cases are the most common ones.

■ Four Types of Linear-phase FIR Filters

Type I. $h[n] = h[M-n]$
M even (or $\frac{M}{2}$, an integer)

Type II. $h[n] = h[M-n]$
M odd (or $\frac{M}{2}$, a half integer)

Type III. $h[n] = -h[M-n]$
M even

Type IV. $h[n] = -h[M-n]$
M odd

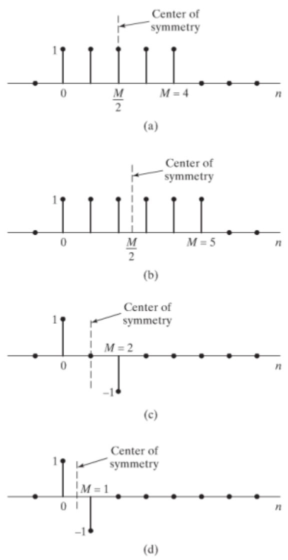


Figure 5.33 Examples of FIR linear-phase systems. (a) Type I, M even, $h[n] = h[M-n]$. (b) Type II, M odd, $h[n] = h[M-n]$. (c) Type III, M even, $h[n] = -h[M-n]$. (d) Type IV, M odd, $h[n] = -h[M-n]$.

Type I

$$\begin{cases} h[n] = h[M-n] & 0 \le n \le M \\ M \text{ is even} \end{cases}$$

$$H(e^{jw}) = \sum_{n=0}^{M} h[n]e^{-jwn} = e^{-jw\frac{M}{2}} \left(\sum_{k=0}^{\frac{M}{2}} a[k]\cos wk\right)$$

where $a[0] = h[\frac{M}{2}]$, and $a[k] = 2h[\frac{M}{2} - k], \quad k = 1,2,\ldots\frac{M}{2}$

Type II

$$\begin{cases} h[n] = h[M-n] & 0 \le n \le M \\ M \text{ is odd} \end{cases}$$

$$H(e^{jw}) = e^{-jw\frac{M}{2}} \left(\sum_{k=1}^{\frac{M+1}{2}} b[k]\cos w(k-\frac{1}{2})\right)$$

where $b[k] = 2h[\frac{M+1}{2} - k], \quad k = 1,2,\ldots\frac{M+1}{2}$

Type III

$$\begin{cases} h[n] = -h[M-n] & 0 \le n \le M \\ M \text{ is even} \end{cases}$$

$$H(e^{jw}) = je^{-jw\frac{M}{2}} \left(\sum_{k=1}^{\frac{M}{2}} c[k]\sin wk\right)$$

where $c[k] = 2h[\frac{M}{2} - k], \quad k = 1,2,\ldots\frac{M}{2}$

Type IV

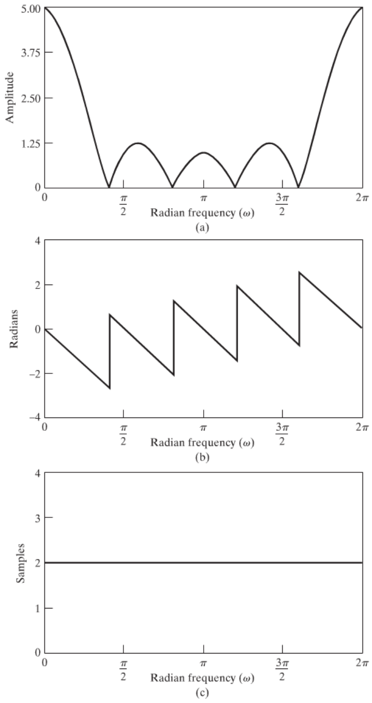
$$\begin{cases} h[n] = -h[M-n] & 0 \le n \le M \\ M \text{ is odd} \end{cases}$$

$$H(e^{jw}) = je^{-jw\frac{M}{2}} \left(\sum_{k=1}^{\frac{M+1}{2}} d[k]\sin(w(k-\frac{1}{2}))\right)$$

where $d[k] = 2h[\frac{M+1}{2} - k], \quad k = 1,2,\ldots\frac{M+1}{2}$

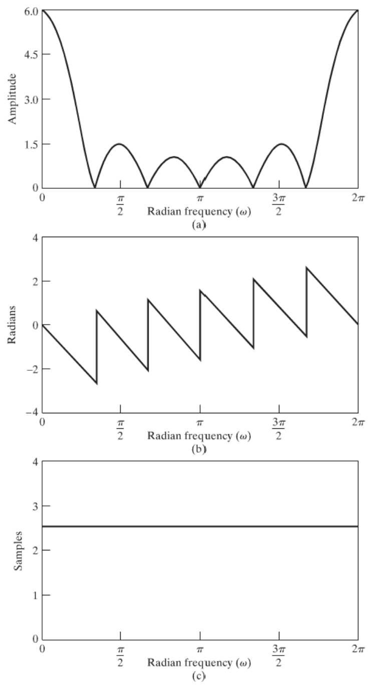
Example: (type I)

$$h[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad H(e^{jw}) = e^{-j2w} \frac{\sin(\frac{5w}{2})}{\sin(\frac{w}{2})}$$



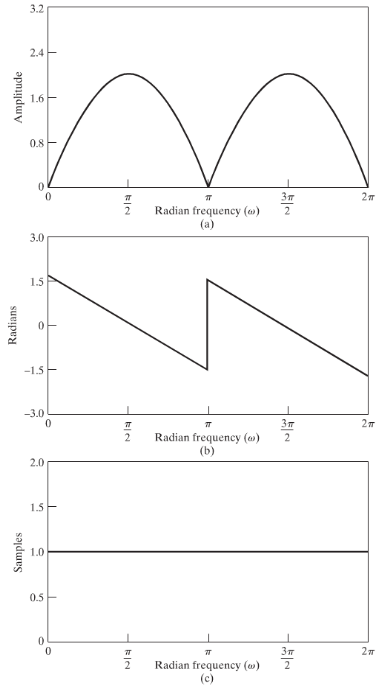
Example: (type II)

$$h[n] = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad H(e^{jw}) = e^{-j\frac{5}{2}w} \frac{\sin(3w)}{\sin(\frac{w}{2})}$$



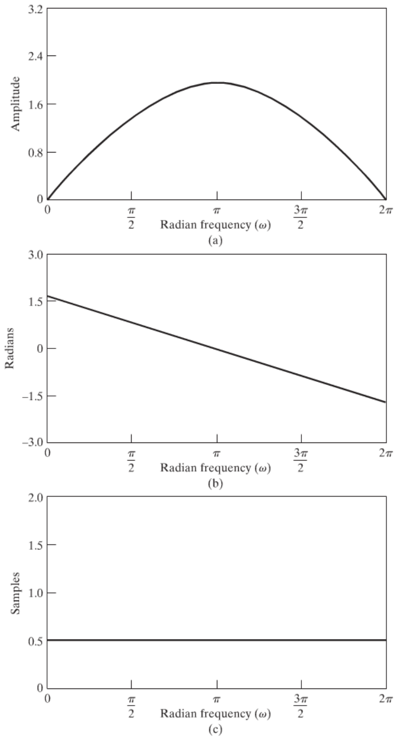
Example: (type III)

$$h[n] = \delta[n] - \delta[n-2] \quad \Rightarrow \quad H(e^{jw}) = 1 - e^{-j2w} = j[2\sin w]e^{-jw}$$



Example: (type IV)

$$h[n] = \delta[n] - \delta[n-1] \quad \Rightarrow \quad H(e^{jw}) = 1 - e^{-jw} = j[2\sin \frac{w}{2}]e^{-j\frac{w}{2}}$$



- **Zeros of FIR Linear Phase Systsems**
  - Causal, stable, FIR systems ➔ no non-zero poles
  - Zeros are conjugate pairs (because $h[n]$ is real).
  - Zeros are conjugate reciprocals pairs (z_0, z_0^{-1}).

(Pf) (A) Type I and II systems

$$H(z) = \sum_{n=0}^{M} h[n] \cdot z^{-n}$$

(k=M-n)

$$= \sum_{n=0}^{M} h[M-n] \cdot z^{-n}$$

$$= \sum_{k=M}^{0} h[k] \cdot z^{k} z^{-M}$$

$$= z^{-M} H\left(z^{-1}\right)$$

If z_0 is a zero, i.e., $H(z_0) = 0$ then $z_0^{-M} H(z_0^{-1}) = 0$.

That is, its reciprocal is a zero too.

(B) Type III and IV systems

$$H(z) = -z^{-M} H\left(z^{-1}\right)$$

If z_0 is a zero, so does its reciprocal.

- Type II (symmetry, odd) ➔ zero at "-1"

Type III (anti-symmetry, even) ➔ zero at "1" and "-1"

Type IV (anti-symmetry, odd) ➔ zero at "1"

(Pf) (A) Type I and II:

$$H(z) = z^{-M} H\left(z^{-1}\right)$$

Let z=-1, $H(-1) = (-1)^{-M} H(-1)$

$$H(-1) = (-1) H(-1)$$

When M is odd, H(-1) must be 0.

(B) Type III and IV:

$$H(z) = -z^{-M} H\left(z^{-1}\right)$$

Let z=1, $H(1) = -H(1)$ ➔ $H(1)$ must be 0.

Let z=-1, $H(-1) = -(-1)^{-M} H(-1)$.

When M is even, $H(-1) = -H(-1)$ must be 0.

Remark: These constraints on the zeros are useful in designing FIR linear-phase systems.

For example, for highpass systems, zero should not be at "-1".

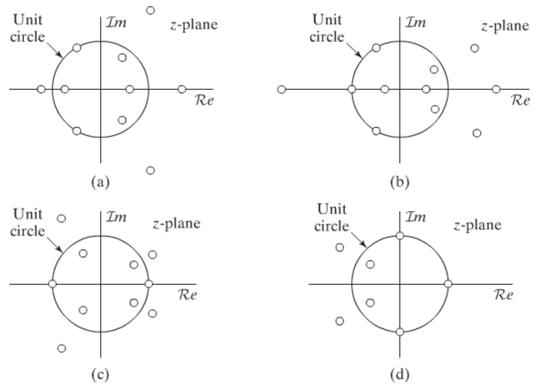


Figure 5.38 Typical plots of zeros for linear-phase systems. (a) Type I. (b) Type II. (c) Type III. (d) Type IV.

- Any FIR linear-phase system can be expressed as follows:

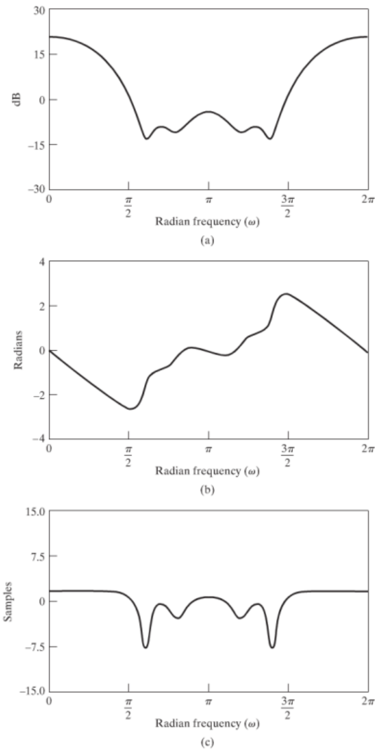
$$\underbrace{H(z)}_{\text{linear phase}} = \underbrace{H_{\min}(z)}_{\text{min. phase}} \cdot \underbrace{H_{\text{uc}}(z)}_{\text{unit circle}} \cdot \underbrace{H_{\max}(z)}_{\text{max. phase}}$$

Because zeros are reciprocals, $H_{\max}(z) = z^{-Mi} H_{\min}\left(z^{-1}\right)$, where Mi is the number of zeros of $H_{\min}(z)$.

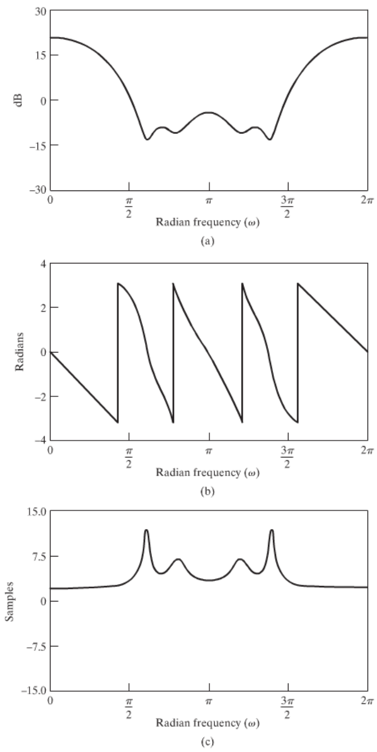
Example:

$$H_{\min}(z) = (1.25)^2 (1-0.9e^{j0.6\pi} z^{-1})(1-0.9e^{-j0.6\pi} z^{-1})(1-0.8e^{j0.8\pi} z^{-1})(1-0.8e^{-j0.8\pi} z^{-1})$$

$$H_{\max}(z) = (0.9)^2 (1-1.1111e^{j0.6\pi} z^{-1})(1-1.1111e^{-j0.6\pi} z^{-1})(1-1.25e^{j0.8\pi} z^{-1})(1-1.25e^{-j0.8\pi} z^{-1})$$



$H_{\min}(z)$



$H_{\max}(z)$

$$H(z) = H_{\min}(z) H_{\max}(z)$$

$$\Rightarrow \quad 20\log_{10}\left|H(e^{jw})\right| = 20\log_{10}\left|H_{\min}(e^{jw})\right| + 20\log_{10}\left|H_{\max}(e^{jw})\right| = 40\log_{10}\left|H_{\min}(e^{jw})\right|$$

$$\angle H(e^{jw}) = \angle H_{\min}(e^{jw}) + \angle H_{\max}(e^{jw}) = \angle H_{\min}(e^{jw}) + (-wM_i - \angle H_{\min}(e^{jw})) = -wM_i$$