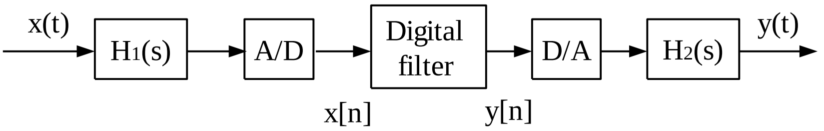
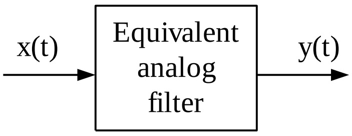
Discrete-Time Signals and Systems

✧ Introduction

- **Signal processing (system analysis and design)**
  - Analog
  - Digital
- **History**

  Before 1950s: analog signals/systems
  - 1950s: Digital computer
  - 1960s: Fast Fourier Transform (FFT)
  - 1980s: Real-time VLSI digital signal processors
- **Discrete-time signals are represented as sequences of numbers**
- **A typical digital signal processing system**





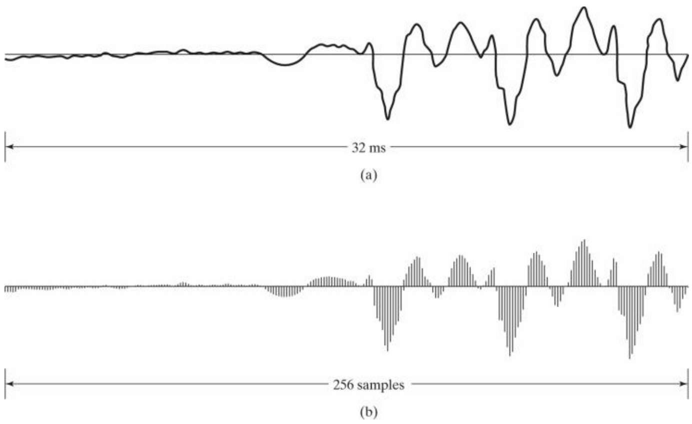
✧ 2.1 Discrete-time Signals: Sequences

- **Continuous-time signal** – Defined along a continuum of times: $x(t)$

  Continuous-time system – Operates on and produces continuous-time signals.

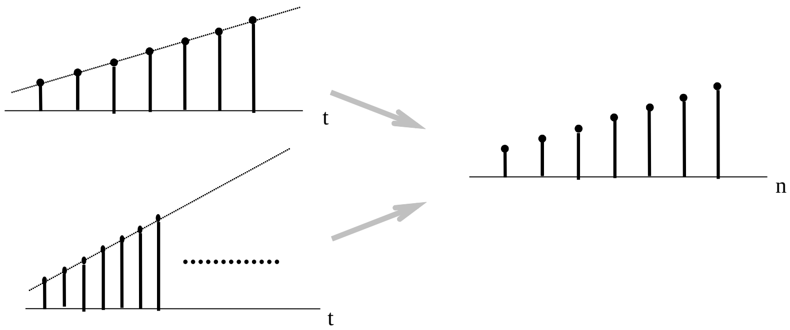
  Discrete-time signal – Defined at discrete times: $x[n]$

  Discrete-time system – Operates on and produces discrete-time signals.



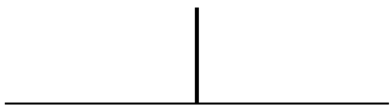
Remarks: **Digital signals** usually refer to the ***quantized*** discrete-time signals.

- **Sampling:** Very often, $x[n]$ is obtained by sampling $x(t)$. "the nth sample of the sequence" That is, $x[n] = x(nT)$, T: is the sampling period. But T is often not important in the discrete-time signal analysis.



- **Basic Sequences**:
  - **Unit Sample Sequence**

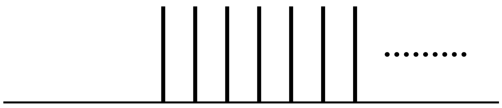
$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$



Remark: It is often called the *discrete-time impulse* or simply *impulse*. (Some books call it *unit pulse sequence*.)

  - **Unit Step Sequence**

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$



Note 1: $u[0]$=1, well-defined.

Note 2: $u[n] = \sum_{m=-\infty}^{n} \delta[m]$; accumulated sum of all previous impulses

$$\delta[n] = u[n] - u[n-1]$$

- **Exponential Sequences**

  $x[n] = A\alpha^n$ A and α are real numbers

  -- Combining basic sequences:

  $$x[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}, \quad ➔ \quad x[n] = A\alpha^n u[n]$$

- **Sinusoidal Sequences**

  $$x[n] = A\cos(\omega_0 n + \phi) \qquad \text{for all } n$$

  A: amplitude, $\omega_0 = 2\pi f_0$: frequency, ϕ: phase

  - It can be viewed as a sampled continuous-time sinusoidal. *However,* it is *not always* periodic!
  - Condition for being periodic with period N: $x[n] = x[n+N]$

    That is, $A\cos(\omega_0 n + \phi) = A\cos(\omega_0 (n+N) + \phi)$

    Or, $\omega_0 (n+N) = \omega_0 n + 2\pi k$, where k, n are integers (k, a fixed number; n, a running index, $-\infty < n < \infty$).

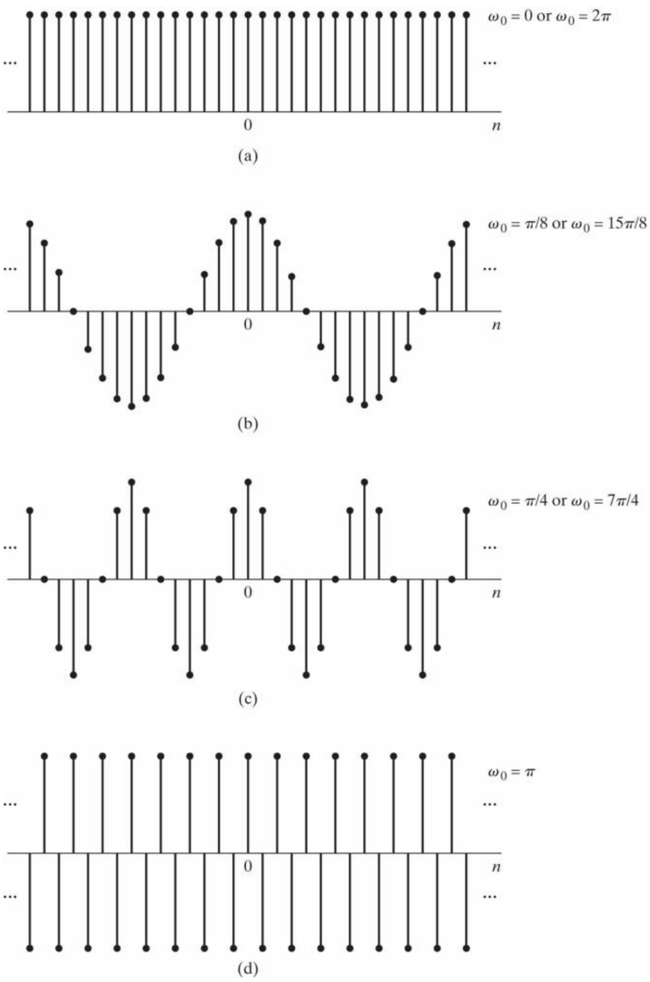
    ➔ $\omega_0 N = 2\pi k$ ➔ $\omega_0 = 2\pi k / N$.

    Hence, f_0 must be a rational number.
  - One discrete-time sinusoid corresponds to multiple continuous-time sinusoids of different frequencies.

    $$\begin{aligned} x[n] &= A\cos(\omega_0 n + \phi) \\ &= A\cos((\omega_0 + 2\pi r)n + \phi) \qquad \text{for all } n \end{aligned}$$

    where r is any integer

    Typically, we pick up the lowest frequency (r=0) under the assumption that the original continuous-time sinusoidal has a limited frequency value, $0 \leq \omega_0 < 2\pi$ or $-\pi \leq \omega_0 < \pi$. This is the *unambiguous* frequency interval.



- **Complex Exponential Sequences**

$x[n] = A\alpha^n, \ \ A = |A|e^{j\phi},$ and $\alpha = |\alpha|e^{j\omega_0}$

Hence,

$$x[n] = |A||\alpha|^n e^{j(\omega_0 n + \phi)} = |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi)$$

✧ 2.2 Discrete-Time Systems

- A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$.

$$y[n] = T\{x[n]\}$$

- **Ideal Delay**

$$y[n] = x[n - n_d], \quad -\infty < n < \infty,$$

where n_d is a fixed positive integer called the delay of the system.

- **Moving Average**

$$y[n]=\frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}x[n-k]$$

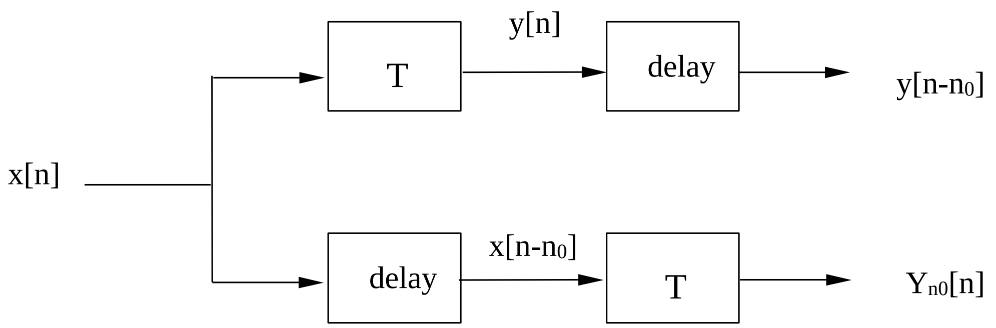
- **Memoryless:** If the output $y[n]$ at every value of *n* depends only on the input $x[n]$ at the same value of *n*.

- **Linear:** If it satisfies the principle of *superposition*.

(a) Additivity: $T\{x_1[n]+x_2[n]\}=T\{x_1[n]\}+T\{x_2[n]\}$

(b) Homogeneity or scaling: $T\{ax[n]\}=aT\{x[n]\}$

- **Time-invariant** (shift-invariant): A time shift or delay of the input sequence causes a corresponding shift in the output sequence.



e.g. y[n] = x[αn] is not time-invariant.

- **Causality:** For *any* n_0, the output sequence value at the index $n=n_0$ depends only on the input sequence values for $n\le n_0$

- **Stability** in the bounded-input, bounded-output sense (BIBO): If and only if every bounded input sequence produces a bounded output sequence.

✧ Linear Time-invariant (LTI) Systems

- A linear system is completely characterized by its impulse response.

(1) Sequence as a sum of delayed impulses: $x[n]=\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$

(2) An LTI system due to $\delta[n]$ input

$x[n]=\delta[n]$ yields $y[n]=h[n]$ (**impulse response**)

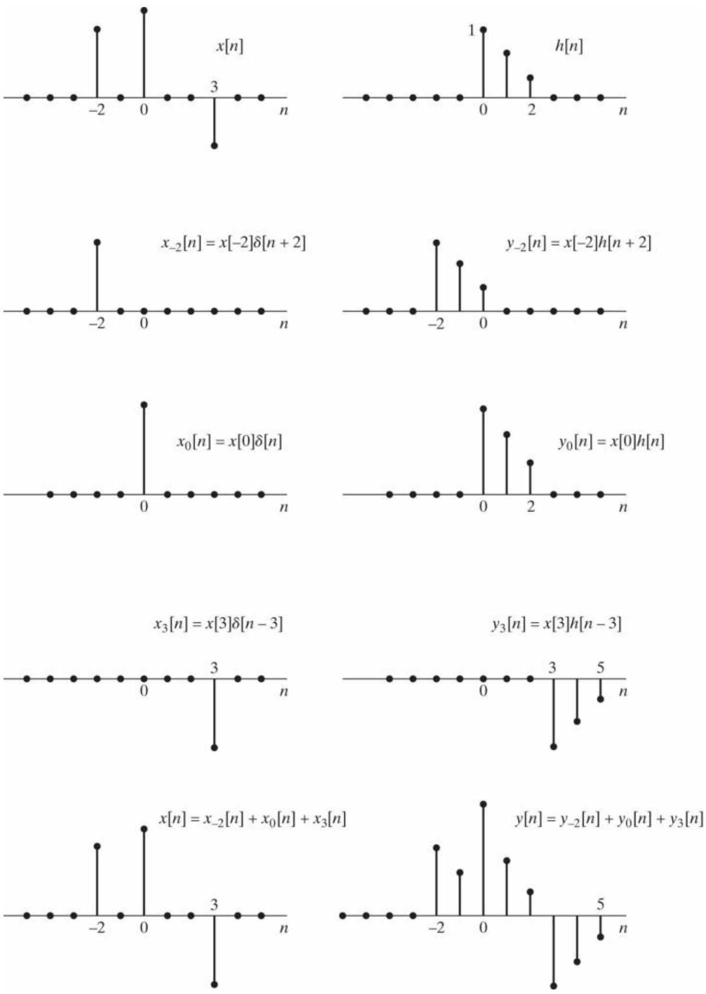
(3) $x[n]=\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$ yields $y[n]=\sum_{m=-\infty}^{\infty} x[m]h[n-m]$

- **Convolution sum**: $f_3[n]=\sum_{m=-\infty}^{\infty} f_1[m]f_2[n-m]=f_1[n]*f_2[n]$

  ■ Procedure of convolution

  1. Time-reverse: $h[m]$ → $h[-m]$
  2. Choose an *n* value
  3. Shift $h[-m]$ by *n*: $h[n-m]$
  4. Multiplication: $x[n]\cdot h[n-m]$
  5. Summation over *m*: $y[n]=\sum_{m=-\infty}^{\infty} x[m]h[n-m]$

  Choose another *n* value, go to Step 3.



✧ Properties of LTI Systems

- The properties of an LTI system can be observed from its impulse response.
- **Commutative**: $x[n] * h[n] = h[n] * x[n]$
- **Distributive**: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
- **Cascade** connection: $h[n] = h_1[n] * h_2[n]$
- **Parallel** connection: $h[n] = h_1[n] + h_2[n]$
- **BIBO stability:** If $h[n]$ is *absolutely summable* , i.e.,

$$\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$$

- **Casual** sequence → **Causal** system: $h[n] = 0, \quad n < 0$
- **Memoryless LTI:** $h[n] = k\delta[n]$

- Some frequently used systems:

  -- **Ideal Delay**

$$y[n] = x[n - n_d] \qquad\qquad h[n] = \delta[n - n_d]$$

  -- **Moving Average**

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \qquad h[n] = \begin{cases} \frac{1}{M_1 + M_2 + 1}, & -M_1 \le n \le M_2 \\ 0, & \text{otherwise} \end{cases}$$

  -- **Accumulator**

$$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad\qquad h[n] = u[n], \text{ unit step}$$

- **Finite-duration Impulse Response (FIR):**

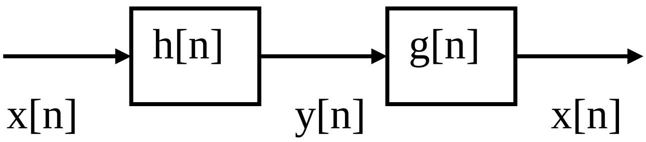
  Its impulse response has only a finite number of nonzero samples.

  -- FIR systems are always stable.

- **Infinite-duration Impulse Response (IIR):**

  Its impulse response is infinite in duration.

- **Inverse System**:



System $g[n]$ is the inverse of $h[n]$

$$h[n] * g[n] = \delta[n]$$

✧ Linear Constant-Coefficient Difference Equations

- An important class of LTI system is described by linear constant-coefficient equation.
  - **Difference Equation:** (general form)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

First-order system: $y[n] = ay[n-1] + bx[n]$

Solution:

$$y[n] = y_p[n] + y_h[n] = \text{particular solution} + \text{homogeneous solution}$$

Homogeneous solution: $\sum_{k=0}^{N} a_k y[n-k] = 0$ ($x[n]=0$)

Particular solution: (experience!)

✧ Frequency-Domain Representation

- Eigenfunction and eigenvalue

  What is eigenfunction of a system $T\{.\}$?

  $Cf[n] = T\{f[n]\}$, where C is a complex constant, *eigenvalue*.

  The output waveform has the same shape of the input waveform.

  The complex exponential sequence is the eigenfunction of any LTI system.

$$x[n] = e^{j\omega n} \longrightarrow \boxed{\text{LTI } h[n]} \longrightarrow y[n] = H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Magnitude: $\left|H(e^{j\omega})\right|$ Phase: $\angle H(e^{j\omega})$

- $H(e^{j\omega})$ is periodic.

- The above eigenfunction analysis is valid when the input is applied to the system at $n = -\infty$.

✧ Fourier Transform of Sequences

- Interpretation: Decompose an "arbitrary" sequence into "sinusoidal components" of different frequencies.

  - **DTFT: Discrete-time Fourier Transform**

Analysis: $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \equiv F\{x[n]\} \quad -\pi < \omega \le \pi$$

Synthesis: $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \equiv F^{-1}\{X(e^{j\omega})\}$$

$x[n] \leftrightarrow X(e^{j\omega})$ Discrete-Time Fourier Transform pair

Remarks: Fourier transform is also called *Fourier spectrum.*

Magnitude spectrum: $|X(e^{j\omega})|$

Phase spectrum: $\angle X(e^{j\omega})$

$X(e^{j\omega})$ is continuous in frequency, ω.

$X(e^{j\omega})$ is "periodic" with period 2π.

- Does every $x[n]$ have DTFT?

Convergence conditions: "error"→0 as N (samples)→∞

(A) Absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{(uniform convergence)}$$

(B) Finite energy (square-summable) ⟹ mean-square error →0

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad \text{(mean-square convergence)}$$

Gibbs phenomenon

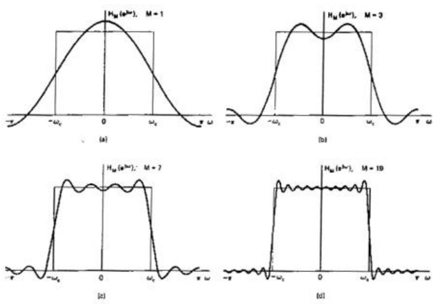


Figure 2.20 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

- DTFT of Special Functions

-- Impulse

$$\delta[n] \leftrightarrow 1$$

$$\delta[n-n_0] \leftrightarrow e^{-j\omega n_0}$$

-- Constant

$$1 \leftrightarrow \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$; An periodic impulse train.

Note: This is the analog impulse (delta) function.

-- Cosine sequence

$$\cos(\omega_0 n + \theta) \leftrightarrow \sum_{k=-\infty}^{\infty} \pi\left[e^{j\theta}\delta(\omega - \omega_0 + 2\pi k) + e^{-j\theta}\delta(\omega + \omega_0 + 2\pi k)\right]$$

-- Complex exponential

$$e^{j\omega_0 n} \leftrightarrow \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi r)$$

-- Unit step

$$u[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \pi \sum_{r=-\infty}^{\infty} \delta(\omega + 2\pi r)$$

✧ Symmetry Properties of Fourier Transform

Any (complex) $x[n]$ can be decomposed into $x[n] = x_e[n] + x_0[n]$

where *Conjugate-symmetric part*: $x_e[n] = (x[n] + x^*[-n])/2$

Conjugate-antisymmetric part: $x_0[n] = (x[n] - x^*[-n])/2$

Remark: $x[n]$ is *conjugate-symmetric* if $x[n] = x^*[-n]$

$x[n]$ is *conjugate-antisymmetric* if $x[n] = -x^*[-n]$

On the other hand, $X(e^{j\omega}) = \mathrm{Re}[X(e^{j\omega})] + j\,\mathrm{Im}[X(e^{j\omega})]$

Key 1: $x_e[n] \leftrightarrow \mathrm{Re}[X(e^{j\omega})]$, $x_o[n] \leftrightarrow j\,\mathrm{Im}[X(e^{j\omega})]$

Similarly, $X(e^{j\omega})$ can be decomposed into

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

where $X_e(e^{j\omega})$ is the *conjugate-symmetric part* and

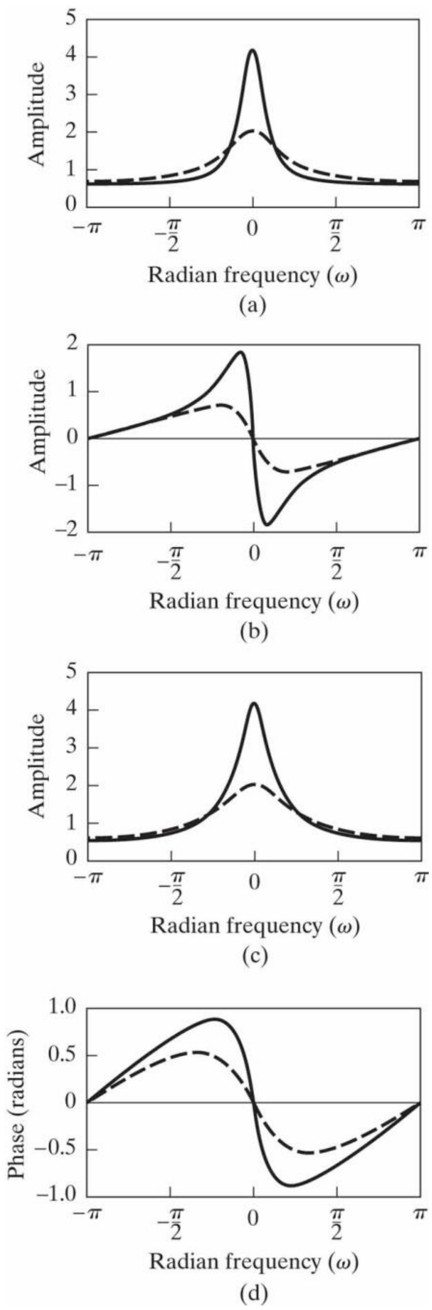
$X_o(e^{j\omega})$ is the *conjugate-antisymmetric part*

Key 2: $\mathrm{Re}[x[n]] \leftrightarrow X_e(e^{j\omega}), \quad j\,\mathrm{Im}[x[n]] \leftrightarrow X_o(e^{j\omega})$

Special case 1*:* If $x[n]$ is real, $X(e^{j\omega})$ is conjugate symmetric

(magnitude –even, phase – odd)

Special case 2: If $x[n]$ is conjugate-symmtric, $X(e^{j\omega})$ is real.



Real

Imaginary

Magnitude

Phase

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

✧ Fourier Transform Theorems

-- Linearity

If $x[n] \leftrightarrow X(e^{j\omega})$ and $y[n] \leftrightarrow Y(e^{j\omega})$

then $ax[n]+by[n] \leftrightarrow aX(e^{j\omega})+bY(e^{j\omega})$

-- Time Shift

If $x[n] \leftrightarrow X(e^{j\omega})$

then $x[n-n_d] \leftrightarrow X(e^{j\omega})e^{-j\omega n_d}$

-- Frequency Modulation

If $x[n] \leftrightarrow X(e^{j\omega})$

then $e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$

--Time Reversal

If $x[n] \leftrightarrow X(e^{j\omega})$

then $x[-n] \leftrightarrow X(e^{-j\omega})$

-- Differentiation in frequency

If $x[n] \leftrightarrow X(e^{j\omega})$

then $nx[n] \leftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega}$

-- Convolution

If $x[n] \leftrightarrow X(e^{j\omega})$ *and* $h[n] \leftrightarrow H(e^{j\omega})$

then $x[n]*h[n] \leftrightarrow X(e^{j\omega})H(e^{j\omega})$

-- Multiplication

If $x[n] \leftrightarrow X(e^{j\omega})$ *and* $w[n] \leftrightarrow W(e^{j\omega})$

then $x[n]w[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$

-- Parseval's Theorem

If $x[n] \leftrightarrow X(e^{j\omega})$

then $E=\sum_{n=-\infty}^{\infty}|x[n]|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2\, d\omega$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty}x[n]y^*[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$