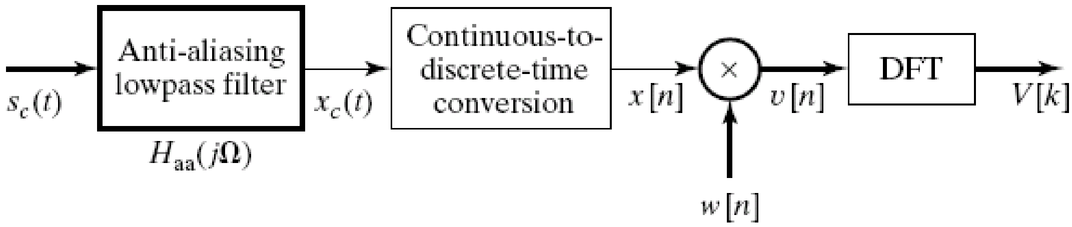
Fourier Analysis of Signals Using the DFT

✧ Fourier Analysis of Signals



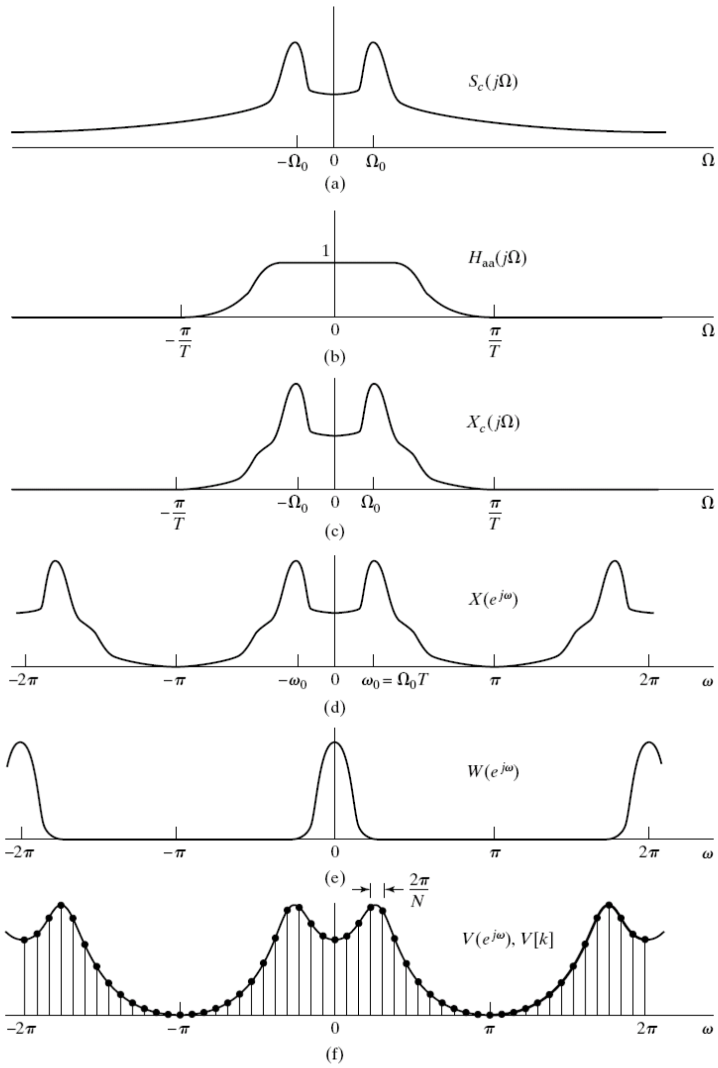


Figure 10.2 Illustration of the Fourier transforms of the system of Figure 10.1. (a) Fourier transform of continuous-time input signal. (b) Frequency response of antialiasing filter. (c) Fourier transform of output of antialiasing filter. (d) Fourier transform of sampled signal. (e) Fourier transform of window sequence. (f) Fourier transform of windowed signal segment and frequency samples obtained using DFT samples.

✧ DFT Analysis of Sinusoidal Signals

⊙ Effect of Windowing

Given a continuous-time signal $s_c(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 \cos(\Omega_1 t + \theta_1)$

Assume ideal sampling with no aliasing and no quantization error

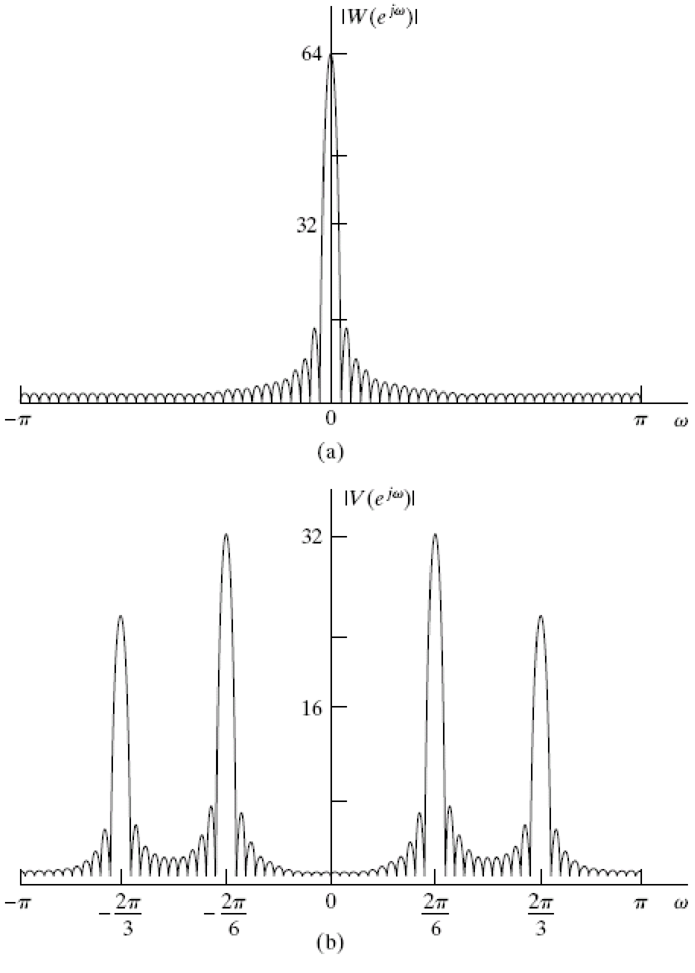
$$x[n] = A_0 \cos(w_0 n + \theta_0) + A_1 \cos(w_1 n + \theta_1) \quad \text{where } w_0 = \Omega_0 T\text{ , } w_1 = \Omega_1 T$$

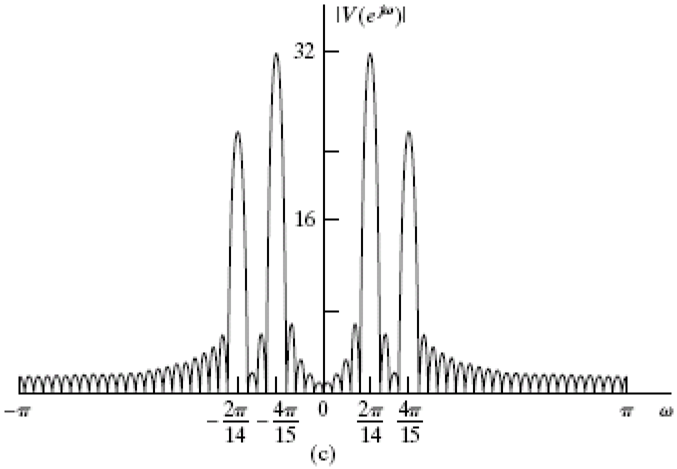
After windowing

$$v[n] = A_0 w[n] \cos(w_0 n + \theta_0) + A_1 w[n] \cos(w_1 n + \theta_1)$$

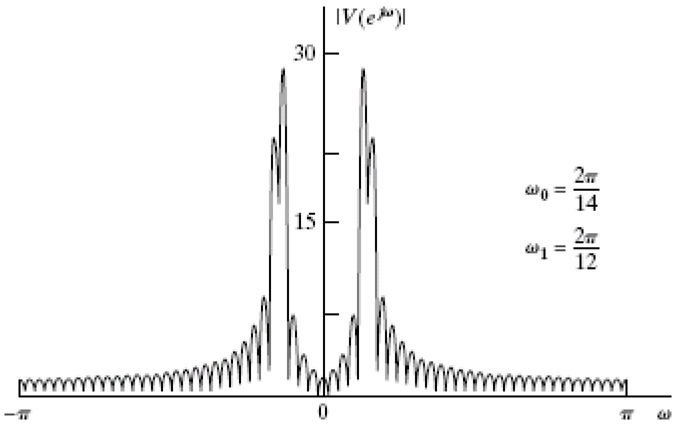
$$v[n] = \frac{A_0}{2} w[n] e^{j\theta_0} e^{jw_0 n} + \frac{A_0}{2} w[n] e^{-j\theta_0} e^{-jw_0 n} + \frac{A_1}{2} w[n] e^{j\theta_1} e^{jw_1 n} + \frac{A_1}{2} w[n] e^{-j\theta_1} e^{-jw_1 n}$$

$$V(e^{jw}) = \frac{A_0}{2} e^{j\theta_0} W(e^{j(w-w_0)}) + \frac{A_0}{2} e^{-j\theta_0} W(e^{j(w+w_0)}) + \frac{A_1}{2} e^{j\theta_1} W(e^{j(w-w_1)}) + \frac{A_1}{2} e^{-j\theta_1} W(e^{j(w+w_1)})$$

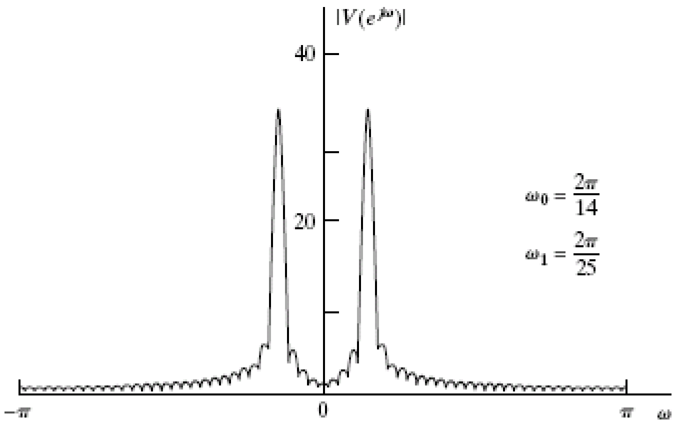




(c)



(d)



(e)

Reduced resolution: influenced primarily by the width of the main lobe of $W(e^{jw})$.

Leakage: depend on the relative amplitude of the main lobe and the side lobes of $W(e^{jw})$.

⊙ Effect of Spectral Sampling

After DFT, we obtain samples of $V(e^{jw})$ at the N equally spaced discrete-time frequencies

$$w_k = \frac{2\pi k}{N}, k = 0,1,\ldots,N-1.$$

These are equivalent to the continuous-time frequencies

$$\Omega_k = \frac{2\pi k}{NT}, k = 0,1,\ldots,\frac{N}{2}$$

and $\Omega_k = \dfrac{-2\pi(N-k)}{NT}, k = \dfrac{N}{2}+1,\ldots,N-1$ (Assume N is even)

Example: $v[n] = \begin{cases} \cos(\frac{2\pi}{14}n) + 0.75\cos(\frac{4\pi}{15}n) & 0 \le n \le 63 \\ 0 & \text{otherwise} \end{cases}$

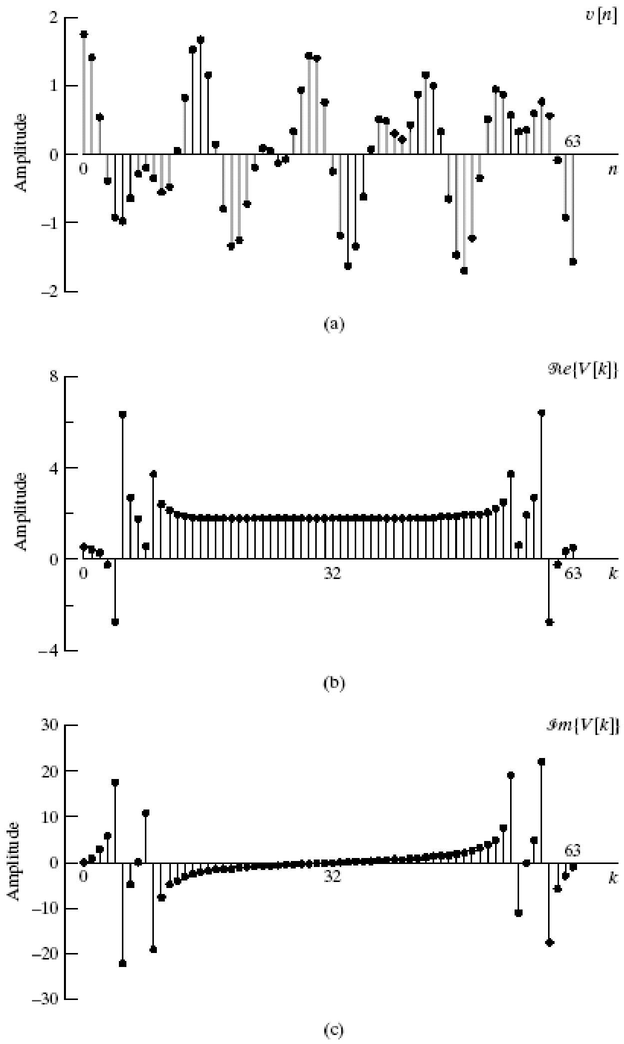


Figure 10.5 Cosine sequence and discrete Fourier transform with a rectangular window. (a) Windowed signal. (b) Real part of DFT. (c) Imaginary part of DFT.

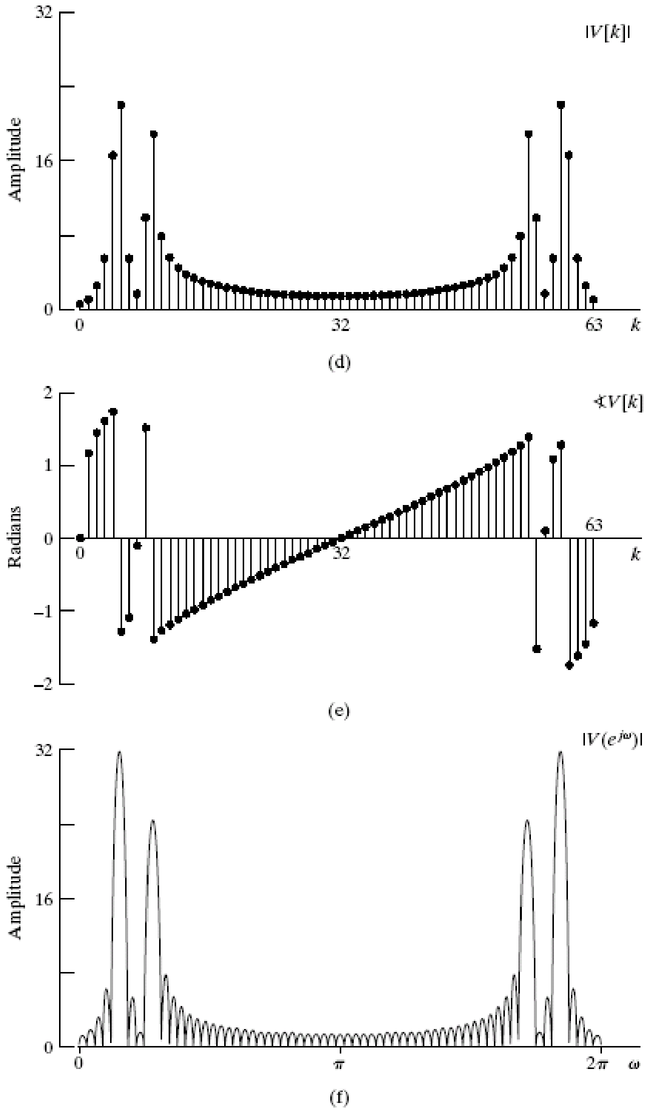


Figure 10.5 (*continued*) (d) Magnitude of DFT. (e) Phase of DFT. (f) Magnitude of discrete-time Fourier transform.

Example: Spectral Sampling with Frequencies Matching DFT Frequencies

$$v[n] = \begin{cases} \cos(\frac{2\pi}{16}n) + 0.75\cos(\frac{2\pi}{8}n) & 0 \le \text{n} \le 63 \\ 0 & \text{otherwise} \end{cases}$$

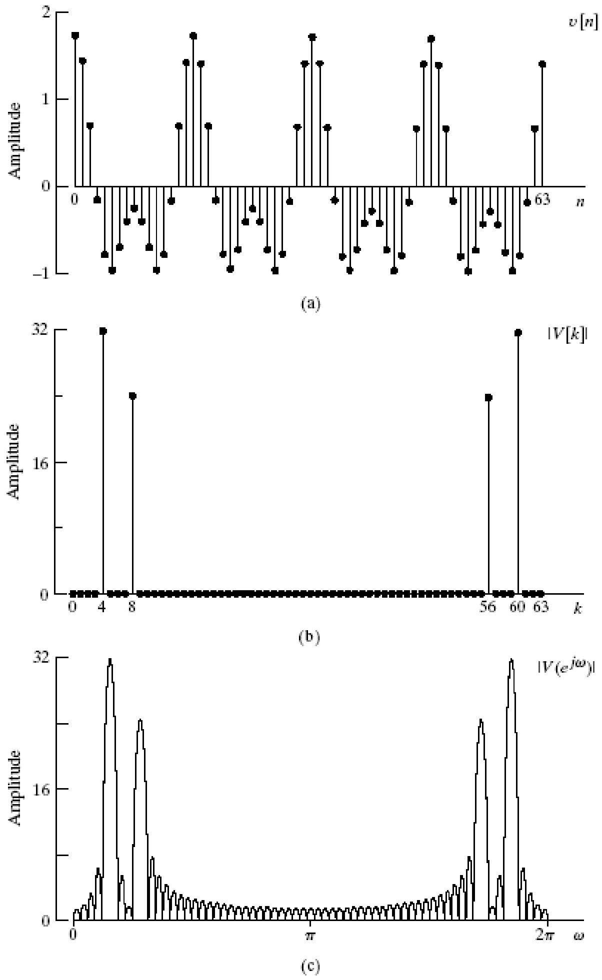


Figure 10.6 Discrete Fourier analysis of the sum of two sinusoids for a case in which the Fourier transform is zero at all DFT frequencies except those corresponding to the frequencies of the two sinusoidal components. (a) Windowed signal. (b) Magnitude of DFT. (c) Magnitude of discrete-time Fourier transform ($|V(e^{j\omega})|$).

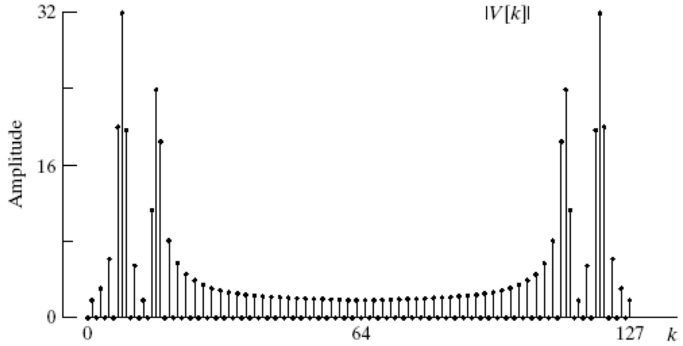


Figure 10.7 DFT of the signal as in Figure 10.6(a), but with twice the number of frequency samples used in Figure 10.6(b).

Example: DFT Analysis of Sinusoidal Signals Using a Kaiser Window

$$v[n] = w_K[n]\cos(\frac{2\pi}{14}n) + 0.75 w_K[n]\cos(\frac{4\pi}{15}n)$$

Kaiser window: $\beta = 5.48$

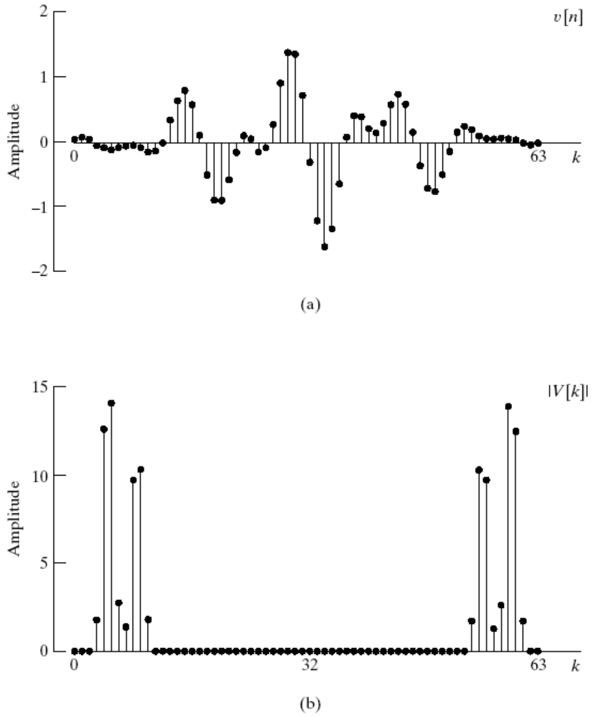


Figure 10.8 Discrete Fourier analysis with Kaiser window. (a) Windowed sequence for $L = 64$. (b) Magnitude of DFT for $L = 64$.

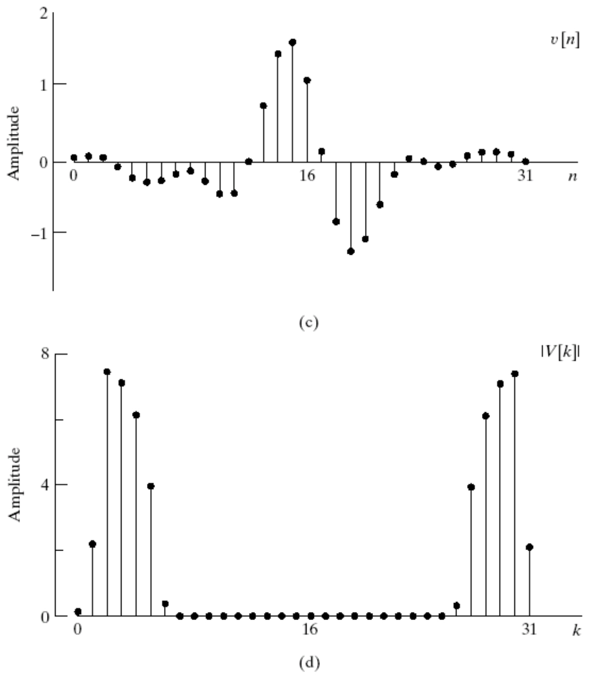


Figure 10.8 (*continued*) (c) Windowed sequence for $L = 32$. (d) Magnitude of DFT for $L = 32$.

Example: DFT Analysis with 32-point Kaiser Window and Zero-Padding

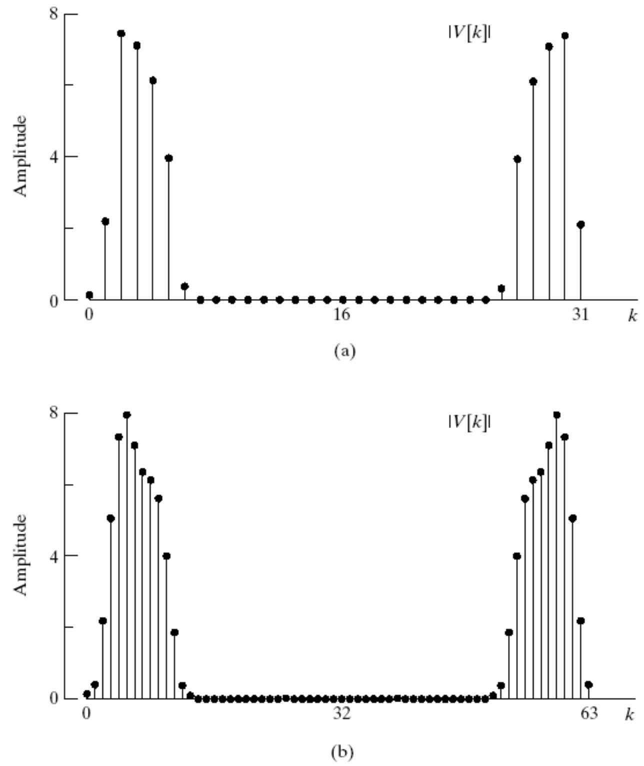


Figure 10.9 Illustration of effect of DFT length for Kaiser window of length $L = 32$. (a) Magnitude of DFT for $N = 32$. (b) Magnitude of DFT for $N = 64$.

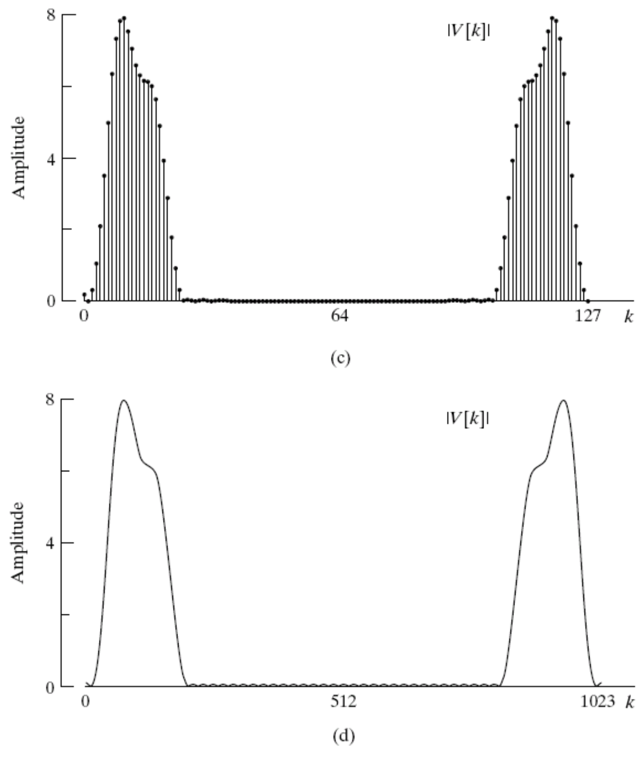


Figure 10.9 (*continued*) (c) Magnitude of DFT for $N = 128$. (d) Magnitude of DFT for $N = 1024$. (DFT values are linearly interpolated to obtain a smooth curve.)

Example: Oversampling and Linear Interpolation for Frequency Estimation

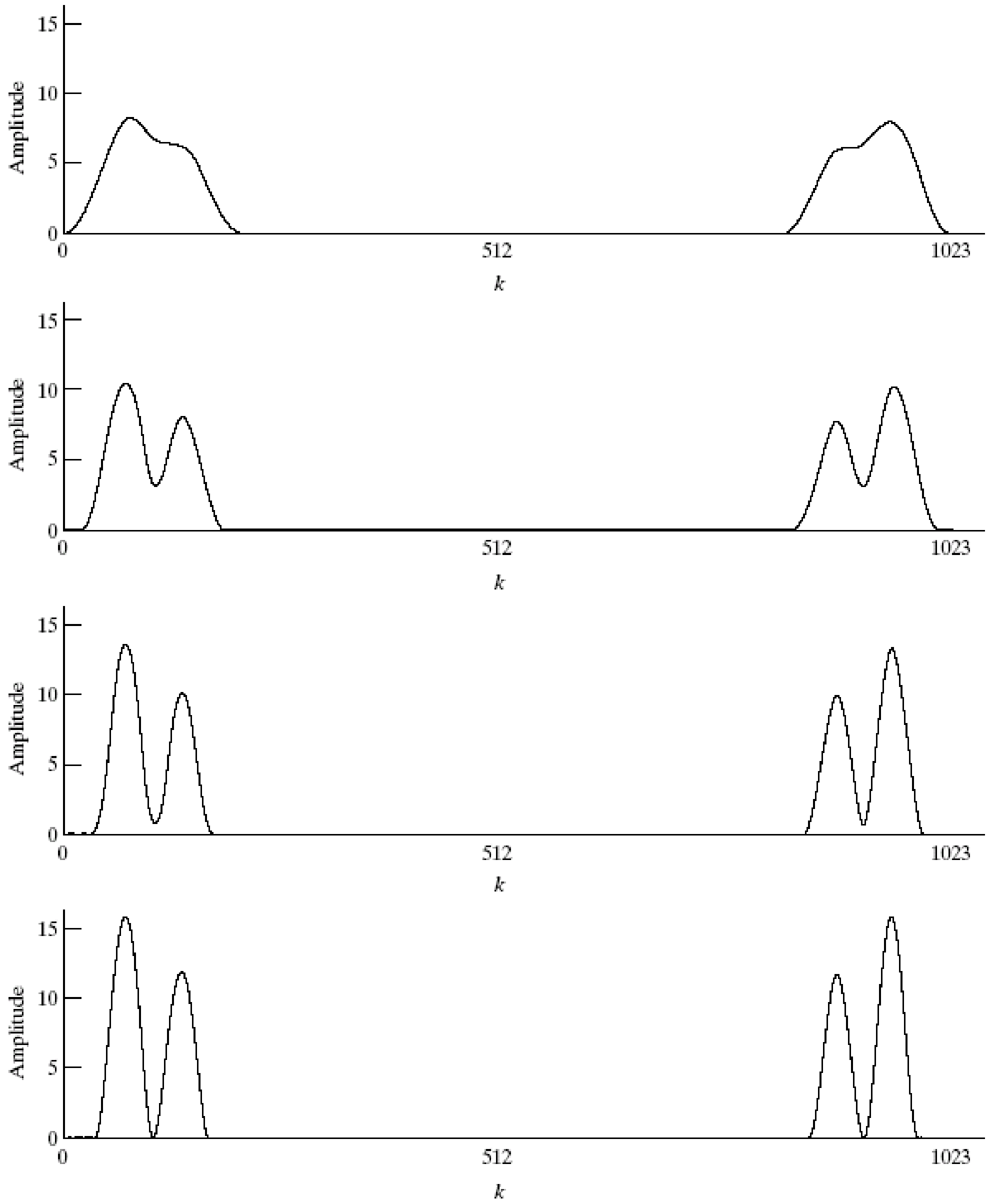


Figure 10.10 Illustration of the computation of the DFT for $N \gg L$ with linear interpolation to create a smooth curve (a) $N = 1024,\ L = 32$ (b) $N = 1024,\ L = 42$ (c) $N = 1024,\ L = 54$ (d) $N = 1024,\ L = 64$.

✧ Time-Dependent Fourier Transform

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

Example: $x[n] = \cos(w_0 n^2)$ $\quad w_0 = 2\pi \times 7.5 \times 10^{-6}$

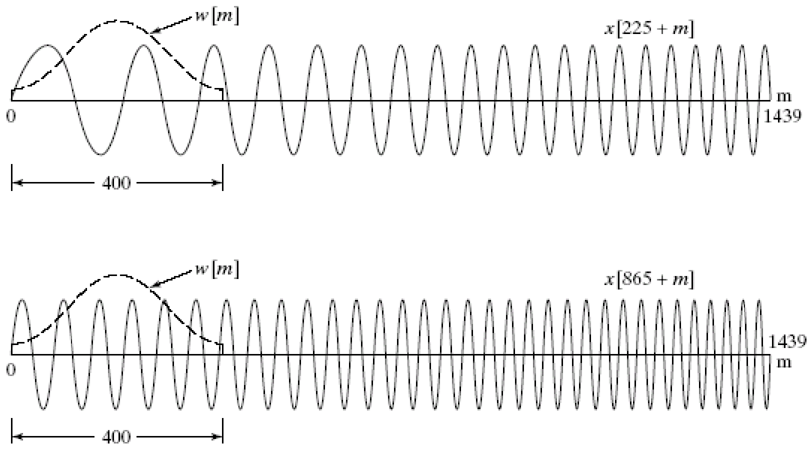


Figure 10.11 Two segments of the linear chirp signal $x[n] = \cos(\omega_0 n^2)$ with the window superimposed. $X[n, \lambda)$ at $n = 225$ is the discrete-time Fourier transform of the top trace multiplied by the window. $X[865, \lambda)$ is the discrete-time Fourier transform of the bottom trace multiplied by the window.

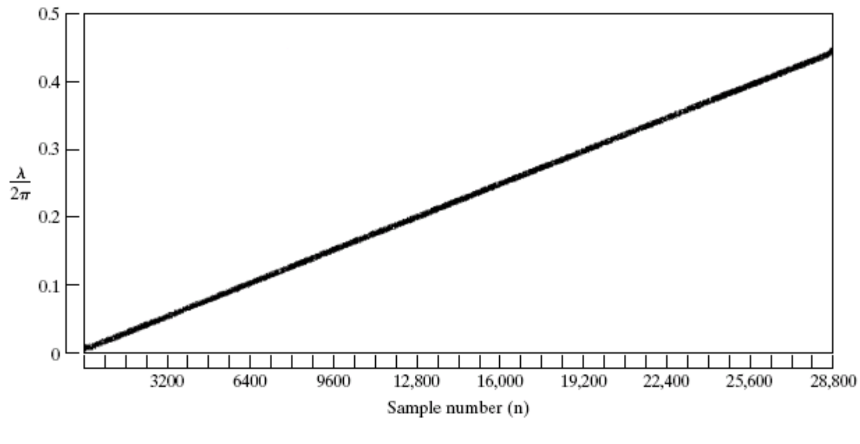


Figure 10.12 The magnitude of the time-dependent Fourier transform of $x[n] = \cos(\omega_0 n^2)$ using a Hamming window of length 400.

$$x[n+m]w[m] = \frac{1}{2\pi}\int_0^{2\pi} X[n,\lambda)e^{j\lambda m}d\lambda \qquad -\infty < m < \infty$$

$$x[n+m] = \frac{1}{2\pi w[m]}\int_0^{2\pi} X[n,\lambda)d\lambda \qquad \text{if } \ w[m]\neq 0$$

$$X[n,\lambda) = \sum_{m'=-\infty}^{\infty} x[m']w[-(n-m')]e^{j\lambda(n-m')}$$

$$\Rightarrow \quad X[n,\lambda) = x[n]*h_\lambda[n] \qquad \text{where } \ h_\lambda[n] = w[-n]e^{j\lambda n}$$

The time-dependent Fourier transform can be interpreted as the output of a linear time-invariant filter with impulse response $h_\lambda[n]$, or equivalently, with frequency response

$$H_\lambda(e^{jw}) = W(e^{j(\lambda-w)})$$

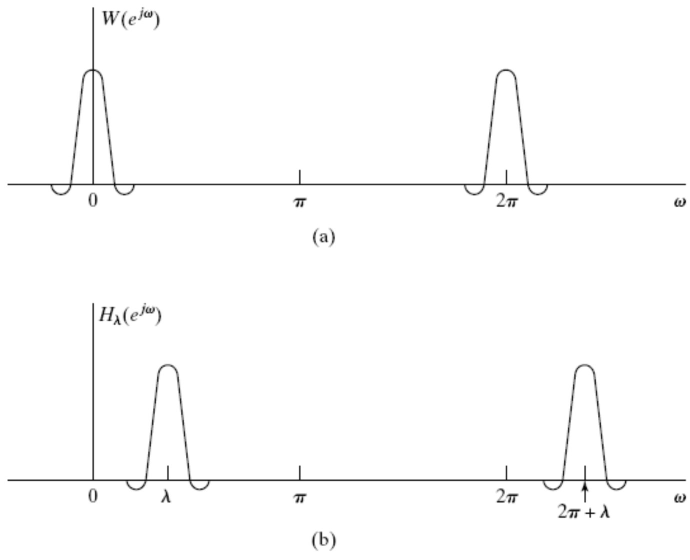


Figure 10.13 (a) Fourier transform of window in time-dependent Fourier analysis. (b) Equivalent bandpass filter for time-dependent Fourier analysis.

Remark: Another definition of time-dependent Fourier transform.

$$\breve{X}[n,\lambda) = \sum_{m=-\infty}^{\infty} x[m]w[m-n]e^{-j\lambda m} \qquad \Rightarrow \qquad \breve{X}[n,\lambda) = e^{-j\lambda n}X[n,\lambda)$$

⊙ Effect of the Window

Assume the window is unity for all m; i.e., assume there is no window at all.

$$X[n,\lambda) = X(e^{j\lambda})e^{j\lambda n}$$

$$X[n,\lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{j\theta n} X(e^{j\theta})W(e^{j(\lambda-\theta)})d\theta$$

The ability to resolve two narrowband signal components depends on the width of the main lobe of the Fourier transform of the window, while the degree of leakage of one component into the vicinity of the other depends on the relative side-lobe amplitude.

⊙ Sampling in Time and Frequency

Suppose the window has length L with samples beginning at m = 0.

$w[m] = 0$ outside the interval

If we sample X[n,λ) at N equally spaced frequencies $\lambda_k = 2\pi k/N$, with N ≥ L.

$$X[n,k] = X[n, 2\pi k/N) = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km} \qquad 0 \le k \le N-1$$

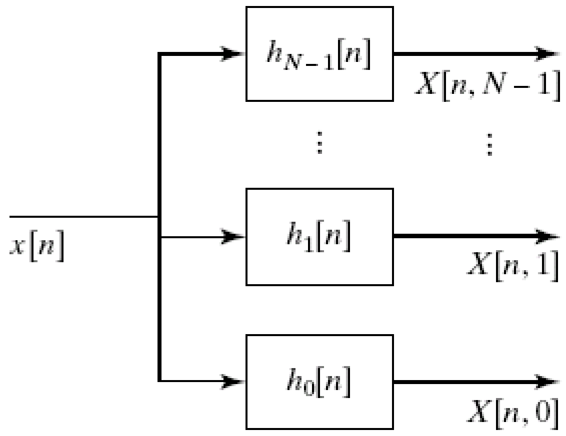
Using the inverse DFT, we have

$$x[n+m]w[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[n,k]e^{j(2\pi/N)km} \qquad 0 \le m \le L-1$$

$$\Rightarrow \quad x[n+m] = \frac{1}{N \cdot w[m]}\sum_{k=0}^{N-1} X[n,k]e^{j(2\pi/N)km} \quad \text{if } w[m] \ne 0 \text{ for } 0 \le m \le L-1$$

$$X[n,k] = x[n] * h_k[n] \qquad 0 \le k \le N-1 \quad \text{where } h_k[n] = w[-n]e^{j(2\pi/N)kn}$$

$$H_k(e^{jw}) = W(e^{j[(2\pi k/N)-w]})$$



$$X[rR,k] = X[rR, 2\pi k/N) = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}$$

$$X_r[k] = X[rR,k] = X[rR,\lambda_k) \qquad -\infty < r < \infty,\ 0 \le k \le N-1$$

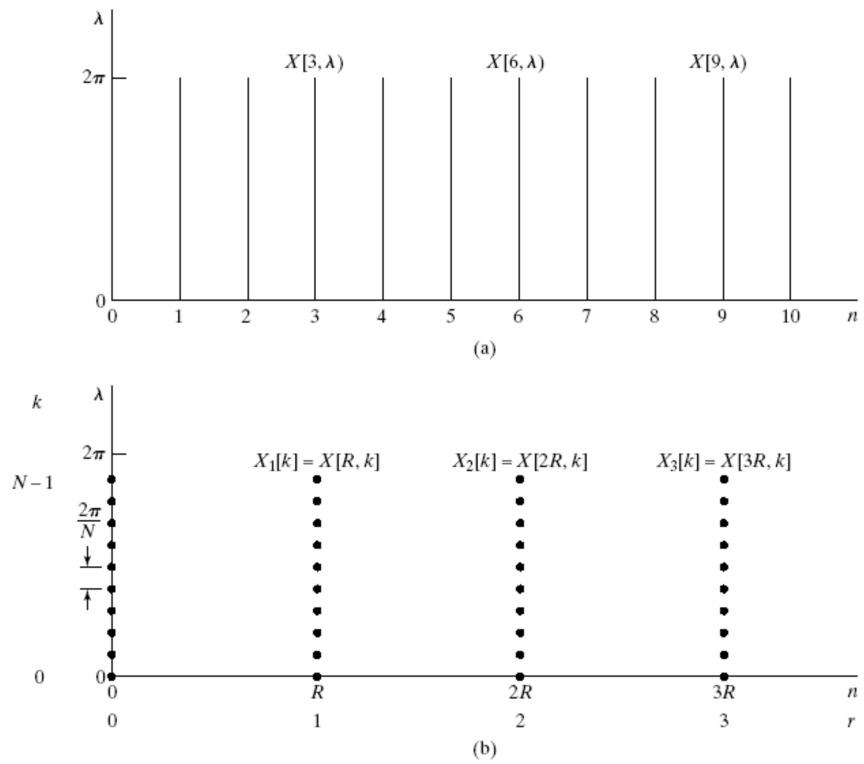


Figure 10.15 (a) Region of support for $X[n, \lambda)$. (b) Grid of sampling points in the $[n, \lambda)$-plane for the sampled time-dependent Fourier transform with $N = 10$ and R 3.

Remark: $N \ge L \ge R$

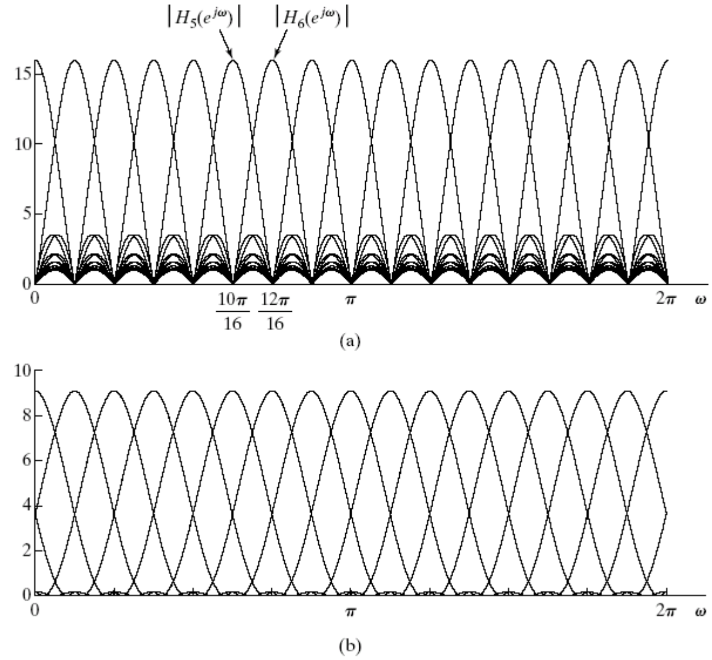
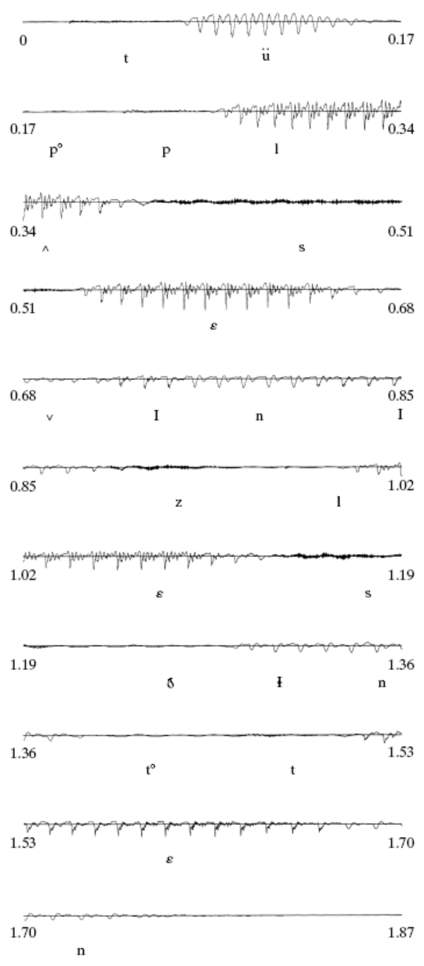


Figure 10.16 Filterbank frequency response. (a) Rectangular window. (b) Kaiser window.

Example: Speech signal "Two plus seven is less than ten".



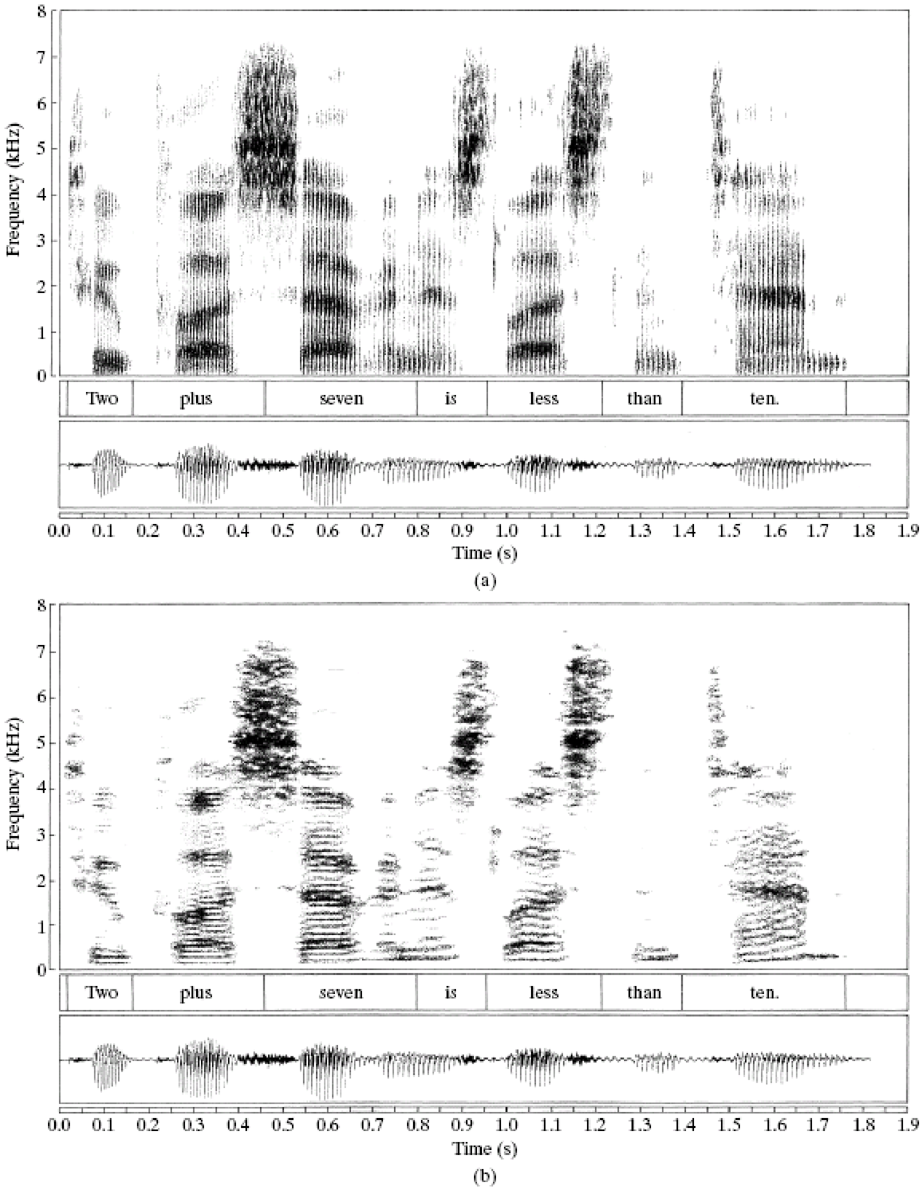


Figure 10.18 (a) Wideband spectrogram of waveform of Figure 10.17. (b) Narrowband spectrogram.