Computation of DFT

- Efficient *algorithms* for computing DFT – Fast Fourier Transform.
  (a) Compute only a few points out of all N points
  (b) Compute all N points
- What are the efficiency criteria?
  - Number of multiplications
  - Number of additions
  - Chip area in VLSI implementation

✧ DFT as a Linear Transformation

- Matrix representation of DFT
  Definition of DFT:

$$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{kn}, \qquad k=0,1,\ldots,N-1$$

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad n=0,1,\ldots,N-1$$

where

Let $\mathbf{x}_N=\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1)\end{bmatrix}, \quad \mathbf{X}_N=\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1)\end{bmatrix},$

and

$$\mathbf{W}_N=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Thus,

$$\begin{aligned} \mathbf{X}_N &= \mathbf{W}_N \mathbf{x}_N && N\text{ - point DFT} \\ \mathbf{x}_N &= \mathbf{W}_N^{-1}\mathbf{X}_N && N\text{ - point IDFT} \\ &= \frac{1}{N}\mathbf{W}_N^{*}\mathbf{X}_N \end{aligned}$$

Because the matrix (transformation) $\mathbf{W}_N$ has a specific structure and because W_N^k has particular values (for some k and n), we can reduce the number of arithmetic operations for computing this transform.

Example $x[n] = [0\ 1\ 2\ 3]$

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Only additions are needed to compute this specific transform.
(This is a well-known *radix-4 FFT*)

Thus, the DFT of $x[n]$ is $\mathbf{X}_4 = \mathbf{W}_4 \mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$

✧ Fast Fourier Transform

-- Highly efficient algorithms for computing DFT

- General principle: *Divide-and-conquer*
- Specific properties of W_N^k
  - Complex conjugate symmetry: $W_N^{-kn} = (W_N^{kn})^*$
  - Symmetry: $W_N^{k+N/2} = -W_N^k$
  - Periodicity: $W_N^{k+N} = W_N^k$
  - Particular values of *k* and *n*: e.g., radix-4 FFT (no multiplications)
- Direct computation of DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{kn}, \quad k = 0,1,\ldots,N-1$$
$$= \sum_{n=0}^{N-1} \left\{ \begin{array}{l} \left[\mathrm{Re}(x[n]) \cdot \mathrm{Re}\left(W_N^{kn}\right) - \mathrm{Im}(x[n]) \cdot \mathrm{Im}\left(W_N^{kn}\right)\right] + \\ j\left[\mathrm{Re}(x[n]) \cdot \mathrm{Im}\left(W_N^{kn}\right) + \mathrm{Im}(x[n])\mathrm{Re}\left(W_N^{kn}\right)\right] \end{array} \right\}$$

For each *k*, we need *N* complex multiplications and *N*-1 complex additions. → 4*N* real multiplications and 4*N*-2 real additions.

We will show how to use the properties of W_N^k to reduce computations.

- Radix-2 algorithms: Decimation-in-time; Decimation-in-frequency
- Composite *N* algorithms: Cooley-Tukey; Prime factor
- Winograd algorithm
- Chirp transform algorithm

✧ Radix-2 Decimation-in-time Algorithms

-- Assume *N*-point DFT and $N = 2^{\nu}$

- Idea: *N*-point DFT → $N/2$-point DFT → $N/4$-point DFT

  $N/4$-point DFT

  $N/2$-point DFT → $N/4$-point DFT

  $N/4$-point DFT

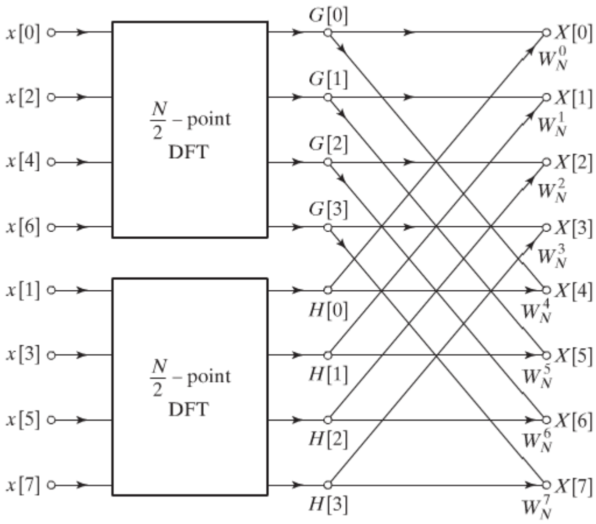
- Sequence: $x[0]\ \ x[1]\ \ x[2]\ \ x[3]\ \ \cdots\ \ x[n/2]\ \ \cdots\ \ x[N-1]$

  Even index: $x[0]\ \ x[2]\ \ \cdots\ \ x[N-2]$
  Odd index: $x[1]\ \ x[3]\ \ \cdots\ \ x[N-1]$

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad k = 0,1,\ldots,N-1 \\
&= \underbrace{\sum_{n\text{ even}} x[n]W_N^{kn}}_{n=2r} + \underbrace{\sum_{n\text{ odd}} x[n]W_N^{kn}}_{n=2r+1} \\
&= \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{(2r+1)k}
\end{aligned}
$$

$$\because W_N^2 = e^{-2j\left(\frac{2\pi}{N}\right)} = e^{-2j\left(\frac{\pi}{N/2}\right)} = W_{N/2}$$

$$
\begin{aligned}
X[k] &= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}} + \underbrace{W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}} \\
&= G[k] + W_N^k H[k]
\end{aligned}
$$



- Comparison:

(a) Direct computation of N-point DFT (N frequency samples):

~ N^2 complex multiplications and N^2 complex adds

(b) Direct computation of $N/2$-point DFT:

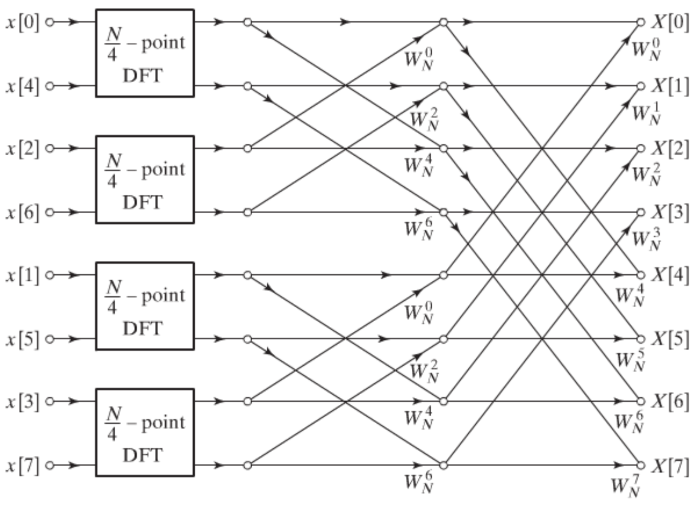
~ $\left(\frac{N}{2}\right)^2$ complex multiplications and $\left(\frac{N}{2}\right)^2$ complex adds

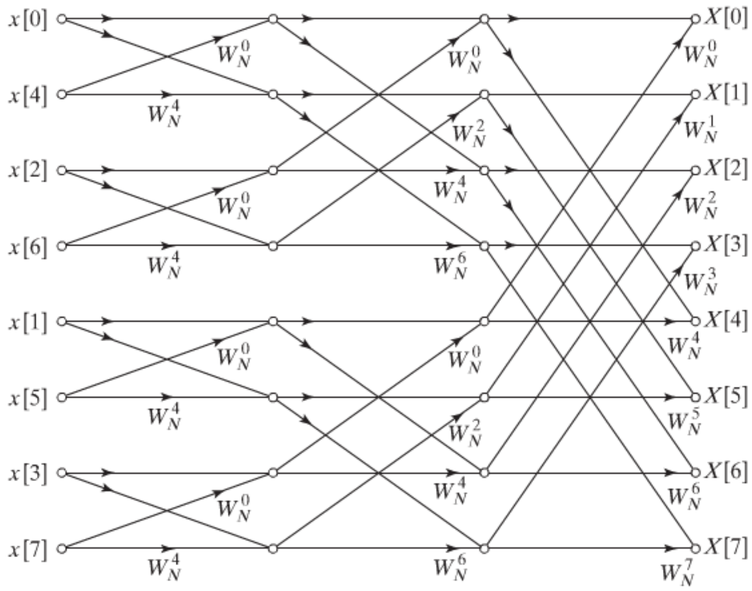
+ additional N complex multis and N complex adds

~ (Total:) $N+2\left(\frac{N}{2}\right)^2 = N+\frac{N^2}{2}$ complex multis and adds

(c) $\log_2 N$ -stage FFT

Since $N = 2^\nu$, we can further break $N/2$-point DFT into two $N/4$-point DFT and so on.



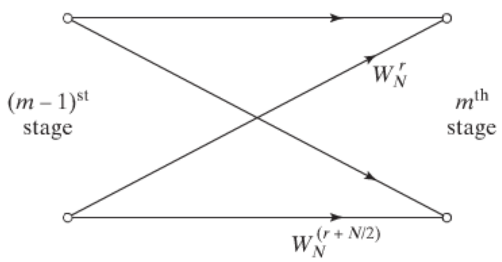


At each stage: ~ N complex multis and adds

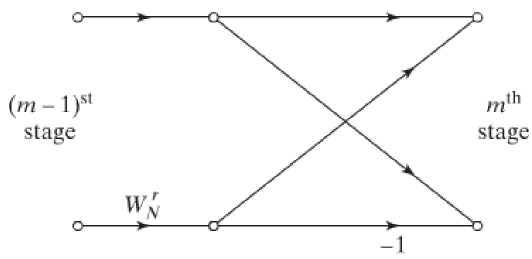
Total: ~ $N\log_2 N$ complex multis and adds (--> $\frac{N}{2}\log_2 N$)

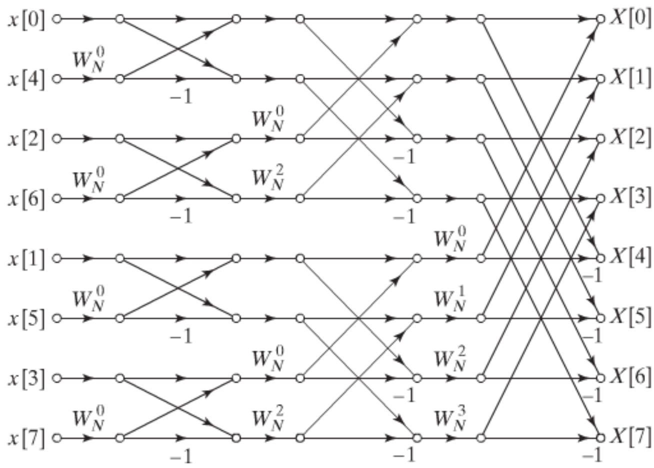
Number of points, N	Direct Computation: Complex Multis	FFT: Complex Multis	Speed Im-provement Factor
4	16	4	4.0
8	64	12	5.3
16	256	32	8
64	4,096	192	21.3
256	65,536	1,024	64.0
1024	1,048,576	5,120	204.8

- **Butterfly**: Basic unit in FFT
  Two multiplications:



One multiplication:

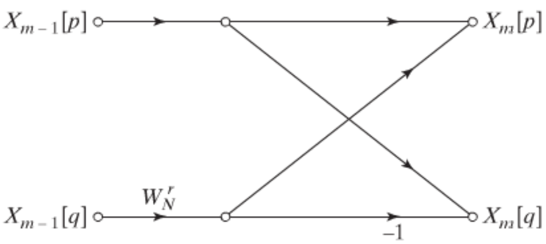




- **In-place** computations

  Only two registers are needed for computing a butterfly unit.

$$X_m[p] = X_{m-1}[p] + W_N^r X_{m-1}[q]$$
$$X_m[q] = X_{m-1}[p] - W_N^r X_{m-1}[q]$$



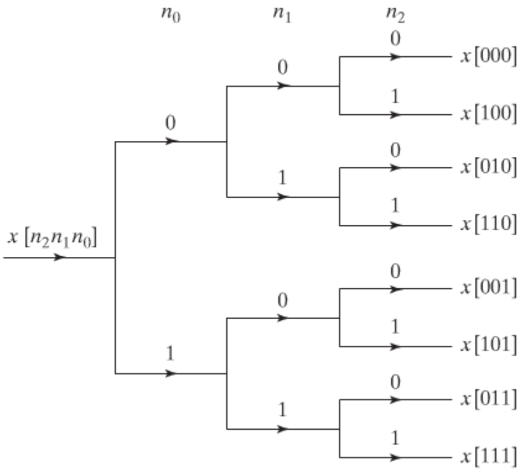
Advantage: less storage!

- In order to retain the in-place computation property, the input data are accessed in the **bit-reversed** order.

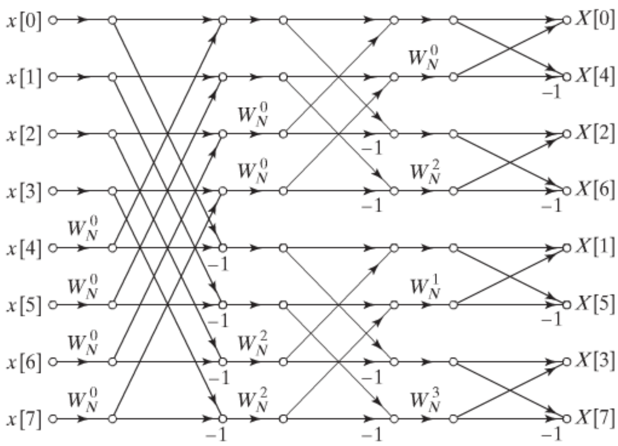
Note: The outputs are in the normal order (same as the "position")

Position	Binary equivalent	Bit reversed	Sequence index
6	110	011	3
2	010	010	2

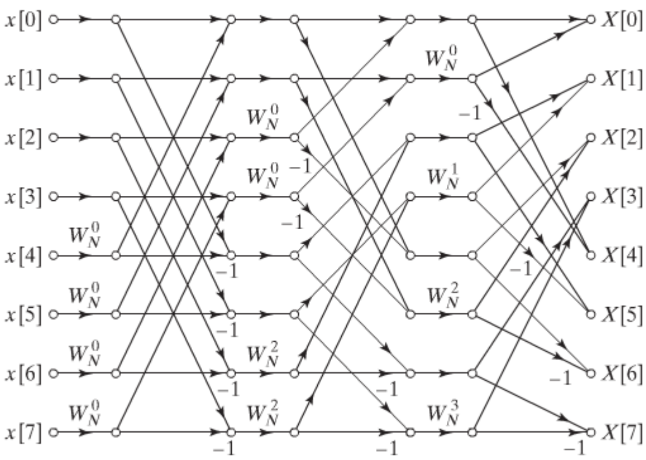
Remark: Index 3 input data is placed at position 6.

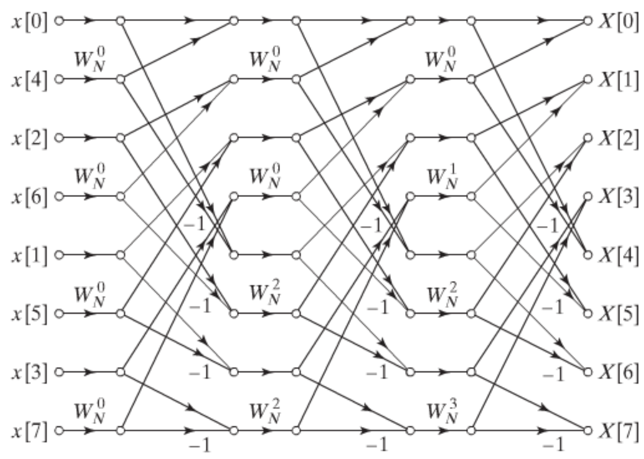


We may also place the inputs in the normal order; then the outputs are in the bit-reversed order.



- If we try to maintain the normal order of both inputs and outputs, then in-place computation structure is destroyed.





✧ Radix-2 Decimation-in-frequency Algorithms

- Dividing the output sequence $X[k]$ into smaller pieces.

$$X(k)=\sum_{n=0}^{N-1}x(n)W_N^{kn},\quad k=0,1,\ldots,N-1$$

If k is even, $k=2r$.

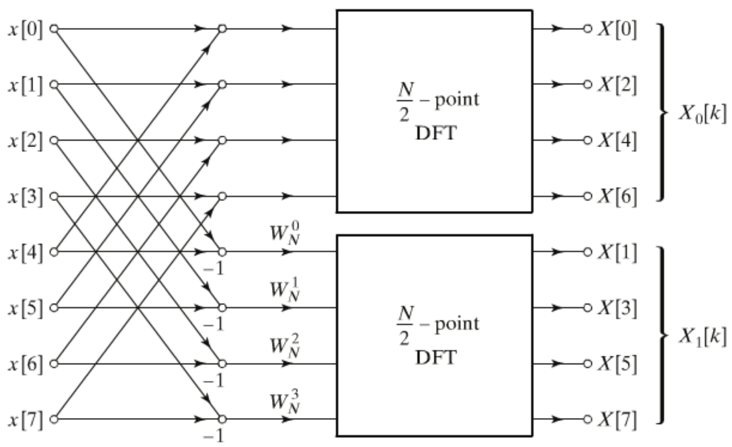
$$\begin{aligned}
X[2r]&=\sum_{n=0}^{N-1}x[n]W_N^{kn},\qquad r=0,1,\cdots,\frac{N}{2}-1\\
&=\sum_{n=0}^{\frac{N}{2}-1}x[n]W_N^{2nr}+\sum_{n=\frac{N}{2}}^{N-1}x[n]W_N^{2nr}\qquad n\leftarrow(n+N/2)\\
&=\sum_{n=0}^{\frac{N}{2}-1}x[n]W_N^{2nr}+\sum_{n=0}^{\frac{N}{2}-1}x\left[n+\frac{N}{2}\right]\cdot W_N^{2r\left(n+\frac{N}{2}\right)}\\
&\because W_N^{2r[n+N/2]}=W_N^{2rn}W_N^{rN}=W_N^{2rn}\\
&=\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]+x\left[n+\frac{N}{2}\right]\right)\cdot W_N^{2nr}\\
&=\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]+x\left[n+\frac{N}{2}\right]\right)\cdot W_{N/2}^{nr}
\end{aligned}$$

Similarly, if k is odd, $k=2r+1$.

$$X[2r+1]=\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]-x\left[n+\frac{N}{2}\right]\right)\cdot W_N^{n}\cdot W_{N/2}^{nr}$$

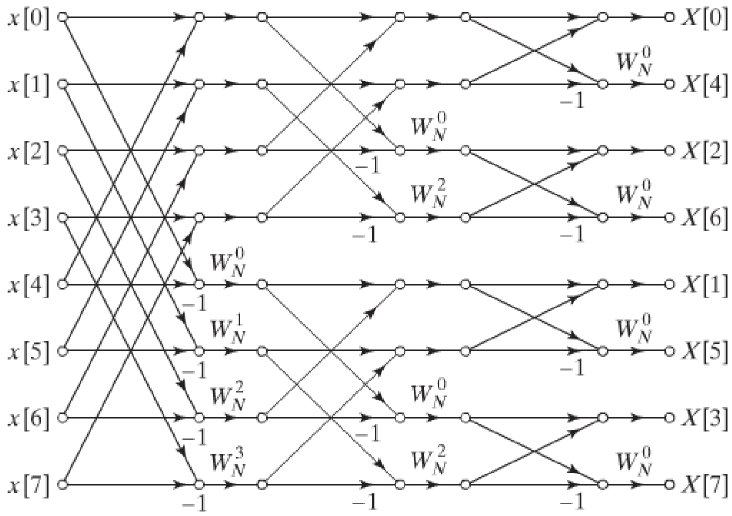
$$\begin{cases} X[2r] = \sum_{n=0}^{\frac{N}{2}-1}\left(x[n] + x\left[n+\frac{N}{2}\right]\right)\cdot W_{N/2}^{nr} \\ X[2r+1] = \sum_{n=0}^{\frac{N}{2}-1}\left(x[n] - x\left[n+\frac{N}{2}\right]\right)\cdot W_N^{n}\cdot W_{N/2}^{nr} \end{cases}$$

Let $\begin{cases} g[n] = x[n] + x\left[n+\frac{N}{2}\right] \\ h[n] = x[n] - x\left[n+\frac{N}{2}\right] \end{cases}$



We can further break $X[2r]$ into even and odd groups …

Again, we can reduce the two-multiplication butterfly into one multiplication. Hence, the computational complexity is bout $\frac{N}{2}\log_2 N$. The in-place computation property holds if the outputs are in bit-reversed order (when inputs are in the normal order).



✧ FFT for Composite N

-- Cooley-Tukey Algorithm: $N = N_1 N_2$

$$\begin{cases} \text{Time index}: & n = N_2 n_1 + n_2 & \begin{cases} 0 \le n_1 \le N_1 - 1 \\ 0 \le n_2 \le N_2 - 1 \end{cases} \\ \text{Freq. index}: & k = k_1 + N_1 k_2 & \begin{cases} 0 \le k_1 \le N_1 - 1 \\ 0 \le k_2 \le N_2 - 1 \end{cases} \end{cases}$$

Remark: $n \leftrightarrow (n_1, n_2)$ and $k \leftrightarrow (k_1, k_2)$

- **Goal:** Decompose *N*-point DFT into two stages:

N_1-point DFT $\otimes$ N_2-point DFT

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \le k \le N-1 \\ &= X[k_1 + N_1 k_2] \\ &= \sum_{n_2=0}^{N_2-1} \sum_{n_1=0}^{N_1-1} x[N_2 n_1 + n_2] \cdot W_N^{(k_1+N_1k_2)(N_2n_1+n_2)} \\ &= \sum_{n_2=0}^{N_2-1} \sum_{n_1=0}^{N_1-1} x[N_2 n_1 + n_2] \cdot \underbrace{W_N^{N_2k_1n_1}}_{W_{N_1}^{k_1n_1}} \cdot W_N^{k_1n_2} \cdot \underbrace{W_N^{k_1N_1n_2}}_{W_{N_2}^{k_2n_2}} \cdot \underbrace{W_N^{N_1N_2k_2n_1}}_{1} \\ &= \underbrace{\sum_{n_2=0}^{N_2-1} \left\{ \underbrace{\left[\sum_{n_1=0}^{N_1-1} x[N_2 n_1 + n_2] \cdot W_{N_1}^{k_1n_1} \right]}_{N_1\text{-point}} \cdot \underbrace{W_N^{k_1n_2}}_{\substack{\text{twiddle} \\ \text{factor}}} \right\} \cdot W_{N_2}^{k_2n_2}}_{N_2\text{-point}} \end{aligned}$$

- **Procedure**

(1) Compute N_1-point DFT: (row transform)

$$G[n_2, k_1] = \sum_{n_1=0}^{N_1-1} x[N_2 n_1 + n_2] \cdot W_{N_1}^{k_1n_1}$$

(2) Multiply twiddle factors:

$$\tilde{G}[n_2, k_1] = W_N^{k_1n_2} \cdot G[n_2, k_1]$$

(3) Compute N_2-point DFT: (column transform)

$$X[k_1 + N_1 k_2] = \sum_{n_2=0}^{N_2-1} \tilde{G}[n_2, k_1] \cdot W_{N_2}^{k_2n_2}$$

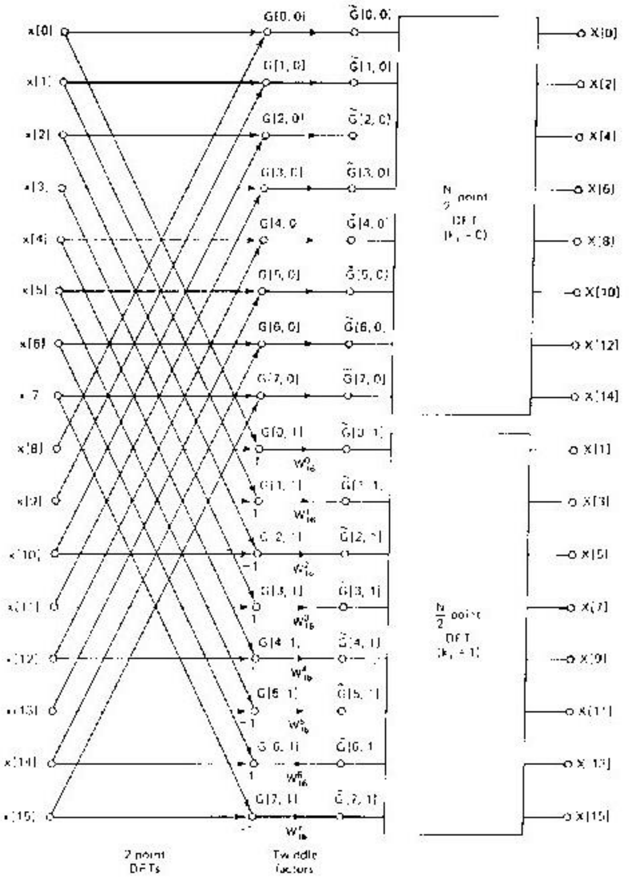
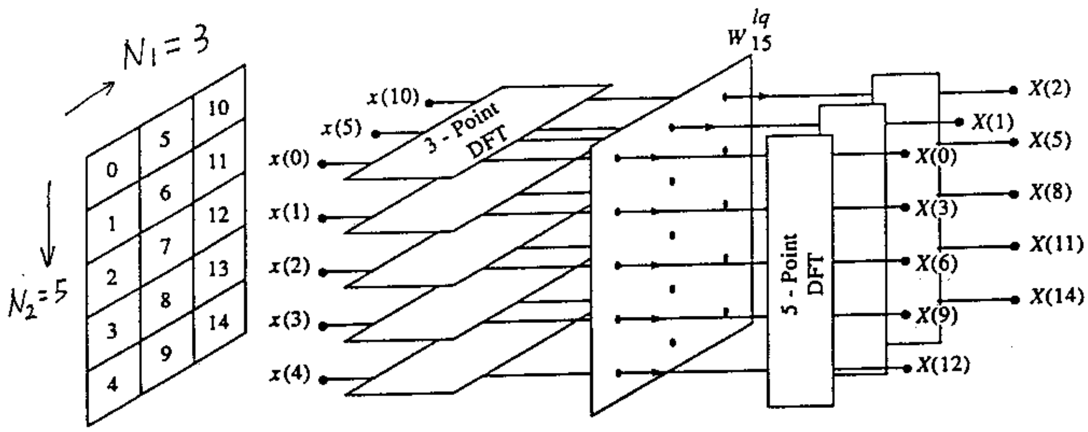


Figure 9.27 Flow graph of decomposition of an N-point DFT into two $(N/2)$-point DFTs ($N = 16$)

(Computation of N=15-point DFT by means of 3-point and 5-point DFTs.)



■ **Extension:** $N = N_1 N_2 \cdots N_\nu$

Let $\mu(N) \equiv$ number of multiplications for N - point DFT

If $N = N_1 N_2$

$$\begin{cases} \text{1. row transform}: & N_2 \cdot \mu(N_1) \\ \text{2. twiddle factors}: & N_1 N_2 = N \\ \text{3. column transfrm}: & N_1 \cdot \mu(N_2) \end{cases}$$

$$\mu(N) = N_2 \cdot \mu(N_1) + N_1 \cdot \mu(N_2) + N$$
$$= N\left(\frac{\mu(N_1)}{N_1} + \frac{\mu(N_2)}{N_2} + 1\right)$$

In general, $\mu(N) = N\left(\sum_{i=1}^{\nu} \frac{\mu(N_i)}{N_i} + (\nu - 1)\right)$

In fact, the term $(\nu - 1)$ should be $(\nu-1)/2$ because rearranging the butterfly structure would make half of the branches becoming “1”.

■ Special Case: $N_1 = N_2 = \cdots = N_\nu = 2$

Radix-2: $N_1 = N_2 = \cdots = N_\nu = 2$ and $\nu = \log_2 N$

$\mu(N) = N(\nu-1)/2$ multiplications because $\mu(2)$ requires no multiplications.

Radix-4: $N_1 = N_2 = \cdots = N_\nu = 4$ and $\nu = \log_4 N$

$\mu(N) = N(\nu-1)/2$ multiplications because $\mu(4)$ requires no multiplications. This FFT has fewer stages than Radix-2 ==> fewer multiplications.

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

✧ Inverse FFT

- IDFT: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\cdot W_N^{-kn}$ (*)

  DFT: $X[k] = \sum_{n=0}^{N-1} x[n]\cdot W_N^{nk}$

  Hence, take the conjugate of (*) :

$$
\begin{aligned}
x^*[n] &= \frac{1}{N}\left(\sum_{k=0}^{N-1} X[k]\cdot W_N^{-kn}\right)^* \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(X[k]\cdot W_N^{-kn}\right)^* \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(X^*[k]\cdot W_N^{kn}\right) \\
&= \frac{1}{N}\,\mathrm{DFT}\left[X^*(k)\right]
\end{aligned}
$$

  Take the conjugate of the above equation:

$$
\begin{aligned}
x[n] &= \frac{1}{N}\left(\mathrm{DFT}\left[X^*(k)\right]\right)^* \\
&= \frac{1}{N}\left(\mathrm{FFT}\left[X^*(k)\right]\right)^*
\end{aligned}
$$

Thus, we can use the FFT algorithm to compute the inverse DFT.

✧ The Goertzel Algorithm

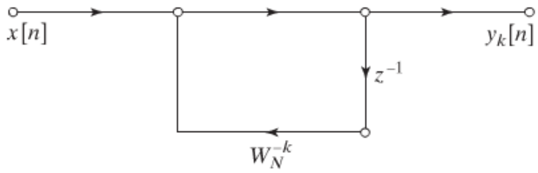
$$W_N^{-kN} = e^{j(\frac{2\pi}{N})Nk} = e^{j2\pi k} = 1$$

$$X[k] = W_N^{-kN}\sum_{r=0}^{N-1} x[r]W_N^{kr} = \sum_{r=0}^{N-1} x[r]W_N^{-k(N-r)}$$

If we define x[n] = 0 for n < 0 and n ≥ N

and $y_k[n] = \sum_{r=-\infty}^{\infty} x[r]W_N^{-k(N-r)}u[n-r] = x[n]*(W_N^{-kn}u[n])$,

$$\Rightarrow \quad X[k] = y_k[n]\Big|_{n=N}$$



$$H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}}$$

If x[n] is complex, we need 4 real multiplications and 4 real additions to compute each $y_k[n]$.

To compute $y_k[N]$, we need to compute $y_k[1]$, $y_k[2]$, …, $y_k[N-1]$.

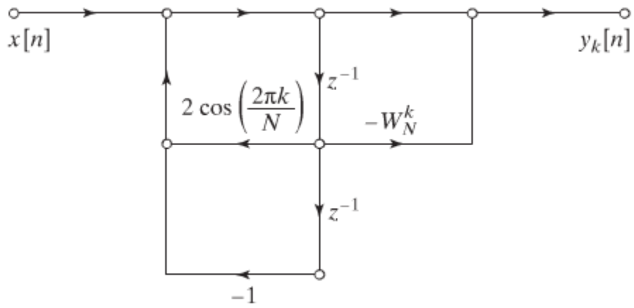
⇒ We need 4N real multiplications and 4N real additions to compute X[k].

Remarks:

- less efficient than the direct method.
- Avoid the computation or storage of the coefficients W_N^{kn}.

To reduce the number of multiplications,

$$H_k(z) = \frac{1 - W_N^k z^{-1}}{(1 - W_N^{-k} z^{-1})(1 - W_N^k z^{-1})} = \frac{1 - W_N^k z^{-1}}{1 - 2\cos(2\pi k / N) z^{-1} + z^{-2}}$$



If x[n] is complex, we only need 2 real multiplications and 4 real additions to implement the poles of the system.

(The complex multiplication by $-W_N^k$ needs not be performed at every iteration.)

⇒ To compute X[k], we need 2N real multiplications and 4N real additions for the poles and 4 real multiplications and 4 real additions for the zero.

Remarks:

- Avoid the computation or storage of the coefficients W_N^{kn}.
- Only need to compute and save W_N^k and $\cos(2\pi k / N)$.