

Discrete Fourier Transform

- What is Discrete Fourier Transform (DFT)?
(Note: It's not DTFT – discrete-time Fourier transform)
 - A linear transformation (matrix)
 - Samples of the Fourier transform (DTFT) of an aperiodic (with finite duration) sequence
 - Extension of Discrete Fourier Series (DFS)
- Review: FT, DTFT, FS, DFS

<i>Time signal</i>	<i>Transform</i>	<i>Coeffs. (periodic/aperiodic)</i>	<i>Coeffs. (con-ti./discrete)</i>
Analog aperiodic	FT	Aperiodic	Continuous
Analog periodic	FT FS	Aperiodic Aperiodic	Continuous (impulse) Discrete
Discrete aperiodic	DTFT	Periodic	Continuous
Discrete periodic	DFS	Periodic	Discrete
Discrete finite-duration	DFT		

✧ The Discrete Fourier Series

- Properties of W_N

$$W_N = e^{-j2\pi/N}, \text{ thus } W_N^k = e^{-j\frac{2\pi}{N}k}$$

-- W_N is periodic with period N . (It is essentially cos and sin) : $W_N^k = W_N^{k\pm N} = W_N^{k\pm 2N} = \dots$

$$\text{-- } \sum_{k=0}^{N-1} W_N^{lk} = \begin{cases} N, & \text{if } l = mN \\ 0, & \text{if } l \neq mN \end{cases}$$

(Pf) (i) If $l = m \cdot N$, $W_N^{lk} = W_N^{mk \cdot N} = W_N^0 = 1$

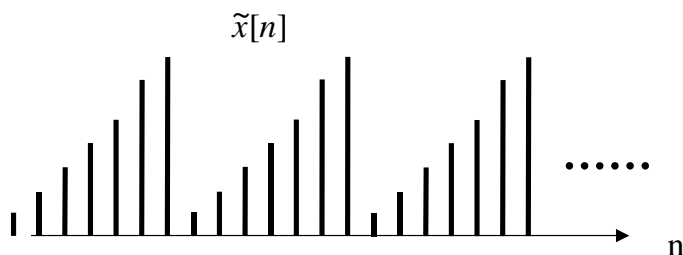
$$\sum_{k=0}^{N-1} W_N^{lk} = \sum_{k=0}^{N-1} 1 = N$$

(ii) If $l \neq m \cdot N$, $W_N^l \neq 1$

$$\sum_{k=0}^{N-1} W_N^{lk} = \frac{1 - W_N^{l \cdot N}}{1 - W_N^l} = \frac{1 - 1}{1 - W_N^l} = 0$$

$$\text{-- } Y[l] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{lk} = \sum_{m=-\infty}^{\infty} \delta[l - mN]$$

- **DFS** for periodic sequences



$$\tilde{x}[n] = \tilde{x}[n + rN], \quad \text{period } N$$

Its DFS representation is defined as follows:

Synthesis equation:
$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

Analysis equation:
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Note: The tilde in \tilde{x} indicates a periodic signal.

$\tilde{X}[k]$ is periodic of period N .

Pf)
$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

Pick an r ($0 \leq r < N$)

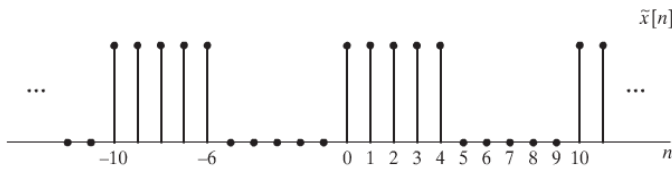
$$\times W_N^{rn} \Rightarrow \tilde{x}[n] W_N^{rn} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \cdot W_N^{rn}$$

$$\begin{aligned} \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{rn} &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \cdot W_N^{rn} \\ \sum_{n=0}^{N-1} \Rightarrow &= \frac{1}{N} \sum_{k=0}^{N-1} (\tilde{X}[k] \sum_{n=0}^{N-1} W_N^{(r-k)n}) \\ &= \tilde{X}[0] \cdot 0 + \tilde{X}[1] \cdot 0 + \dots + \tilde{X}[k=r] \cdot 1 + \dots \\ &= \tilde{X}[r] \end{aligned}$$

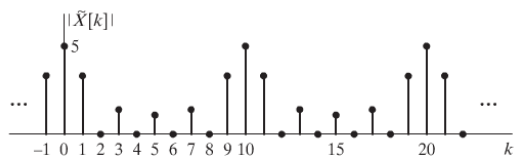
That is,
$$\tilde{X}[r] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{rn} \cdot$$

QED

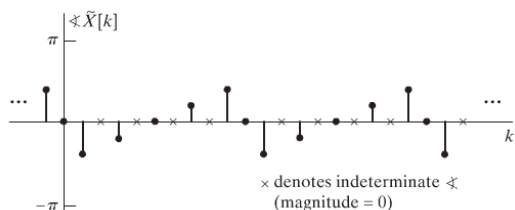
Example: Periodic Rectangular Pulse Train



$$\tilde{X}[k] = \sum_{n=0}^4 W_{10}^{kn} = \frac{1 - W_{10}^{5k}}{1 - W_{10}^k} = e^{-j\frac{4\pi k}{10}} \frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{10}\right)}$$



(a)



(b)

✧ Sampling the Fourier Transform

Compare two cases:

- (1) Periodic sequence $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$
- (2) Finite duration sequence $x[n] = \text{one period of } \tilde{x}[n]$

An aperiodic sequence:

$x[n] \xrightarrow{FT} X(e^{j\omega})$ $\updownarrow ?$ $\tilde{x}[n] \xleftarrow{IDFS} \tilde{X}[k] = X\left(e^{j\omega}\right)\Big _{\omega=\frac{2\pi}{N}k}$	<p style="text-align: center;">Compare:</p> $x(t) \xrightarrow{FT} X(j\Omega)$ $\downarrow \text{ samples } \quad \updownarrow ?$ $x[n] \xrightarrow{DTFT} X(e^{j\omega})$
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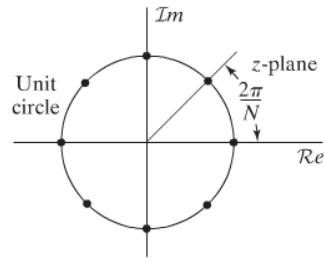
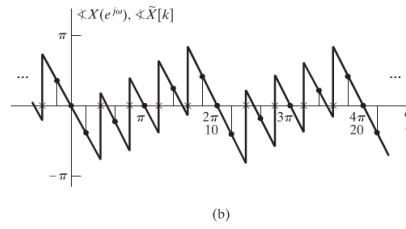
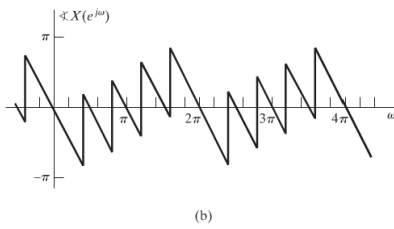
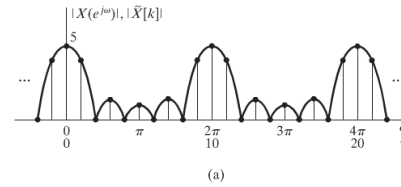
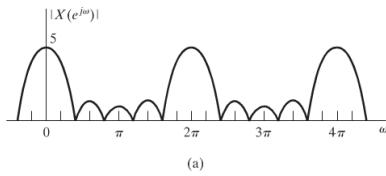


Figure 8.7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Example: $x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$

$$\tilde{x}[n] = \begin{cases} 1, & r10 \leq n \leq 4 + r10 \\ 0, & 5 + r10 \leq n \leq 9 + r10 \end{cases}$$

$r = \text{integer}$



$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \quad (\text{IDFS})$$

$$= \frac{1}{N} \sum_k \left(X(e^{j\omega}) \Big|_{\omega=\frac{2\pi}{N}k} \right) W_N^{-kn} \quad (\text{Sampling})$$

$$= \frac{1}{N} \sum_k \left(\sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} \right) \Big|_{\omega=\frac{2\pi}{N}k} W_N^{-kn} \quad (\text{FT})$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}km} \right) W_N^{-kn}$$

$$= \frac{1}{N} \sum_{m=-\infty}^{\infty} x[m] \left\{ \sum_{k=0}^{N-1} W_N^{km} W_N^{-kn} \right\} \quad (\text{Interchange } \sum)$$

$$= x[n] * \sum_{r=-\infty}^{\infty} \delta[n + rN] = \sum_{r=-\infty}^{\infty} x[n + rN]$$

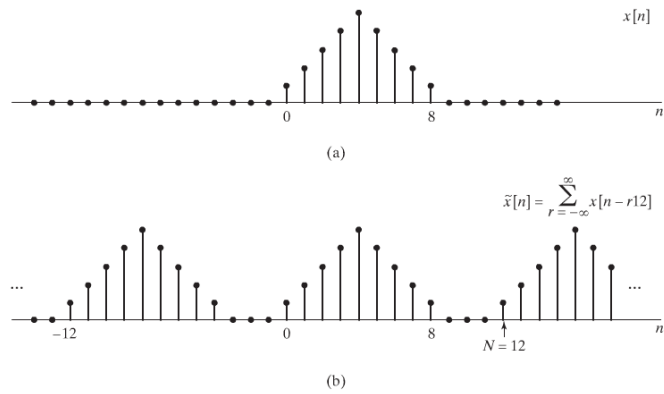


Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

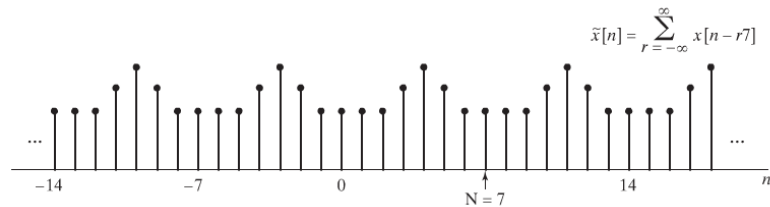


Figure 8.9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ in Figure 8.8(a) with $N = 7$.

If $x[n]$ has finite length and we take a sufficient number of equally spaced samples of its Fourier Transform (a number greater than or equal to the length of $x[n]$), then $x[n]$ is recoverable from $\tilde{x}[n]$.

- Two ways (equivalently) to define DFT:
 - (1) N samples of the DTFT of a finite duration sequence $x[n]$
 - (2) Make the periodic replica of $x[n] \rightarrow \tilde{x}[n]$

Take the DFS of $\tilde{x}[n]$

Pick up one segment of $\tilde{X}[k]$

$$\begin{array}{lcl}
 x[n] & \rightarrow DFT \rightarrow & X[k] \\
 \downarrow \text{periodic} & & \uparrow \text{one segment} \\
 \tilde{x}[n] & \rightarrow DFS \rightarrow & \tilde{X}[k]
 \end{array}$$

✧ Properties of the Discrete Fourier Series

-- Similar to those of FT and z-transform

- **Linearity**

$$\left. \begin{aligned} \tilde{x}_1[n] &\leftrightarrow \tilde{X}_1[k] \\ \tilde{x}_2[n] &\leftrightarrow \tilde{X}_2[k] \end{aligned} \right\} \Rightarrow a\tilde{x}_1[n] + b\tilde{x}_2[n] \leftrightarrow a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

- **Shift**

$$\begin{aligned} \tilde{x}[n] \leftrightarrow \tilde{X}[k] &\Rightarrow \tilde{x}[n-m] \leftrightarrow W_N^{km} \tilde{X}[k] \\ &W_N^{-nl} \tilde{x}[n] \leftrightarrow \tilde{X}[k-l] \end{aligned}$$

- **Duality**

$$\text{Def: } \begin{cases} \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} & (*) \\ \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} & (\#) \end{cases}$$

$$\begin{cases} \tilde{x}[n] \leftrightarrow \tilde{X}[k] \\ \tilde{X}[k] \leftrightarrow N\tilde{x}[-k] \end{cases}$$

- **Symmetry** $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$

$$\text{Re}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_e[k] \left(= \frac{1}{2} (\tilde{X}[k] + \tilde{X}^*[-k]) \right)$$

$$j \text{Im}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_o[k] \left(= \frac{1}{2} (\tilde{X}[k] - \tilde{X}^*[-k]) \right)$$

$$\tilde{x}_e[n] = \frac{1}{2} (\tilde{x}[n] + \tilde{x}^*[-n]) \leftrightarrow \text{Re}\{\tilde{X}[k]\}$$

$$\tilde{x}_o[n] = \frac{1}{2} (\tilde{x}[n] - \tilde{x}^*[-n]) \leftrightarrow j \text{Im}\{\tilde{X}[k]\}$$

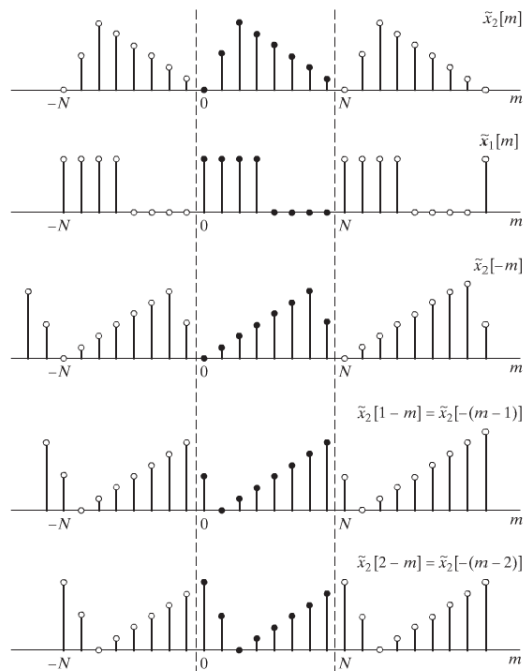
$$\begin{aligned} \text{If } \tilde{x}[n] \text{ is real, } \tilde{X}[k] &= \tilde{X}^*[-k]. & \Rightarrow \begin{cases} |\tilde{X}[k]| = |\tilde{X}[-k]| \\ \angle \tilde{X}[k] = -\angle \tilde{X}[-k] \end{cases} \\ & & \Rightarrow \begin{cases} \text{Re}\{\tilde{X}[k]\} = \text{Re}\{\tilde{X}[-k]\} \\ \text{Im}\{\tilde{X}[k]\} = -\text{Im}\{\tilde{X}[-k]\} \end{cases} \end{aligned}$$

• **Periodic Convolution**

$\tilde{x}_1[n], \tilde{x}_2[n]$ are periodic sequences with period N

$$\sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{x}_2[n-m] \leftrightarrow \tilde{X}_1[k] \tilde{X}_2[k]$$

$$\tilde{x}_3[n] = \tilde{x}_1[n] \tilde{x}_2[n] \leftrightarrow \frac{1}{N} \sum_{l=0}^{N-1} \tilde{X}_1[l] \tilde{X}_2[k-l]$$



✧ **Discrete Fourier Transform**

• *Definition*

$x[n]$: length $N, 0 \leq n \leq N - 1$

Making the periodic replica:

$$\begin{aligned} \tilde{x}[n] &= \sum_{r=-\infty}^{\infty} x[n + rN] \\ &\equiv x[(n \text{ modulo } N)] \\ &\equiv x[((n))_N] \end{aligned}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Keep one segment (finite duration)

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \leq k \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{That is, } \tilde{X}[k] = X[((k))_N]$$

This finite duration sequence $X[k]$ is the **discrete Fourier transform** (DFT) of $x[n]$

Analysis eqn:
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \leq k \leq N-1$$

Synthesis eqn:
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \leq n \leq N-1$$

Remark: DFT formula is the same as DFS formula. Indeed, many properties of DFT are derived from those of DFS.

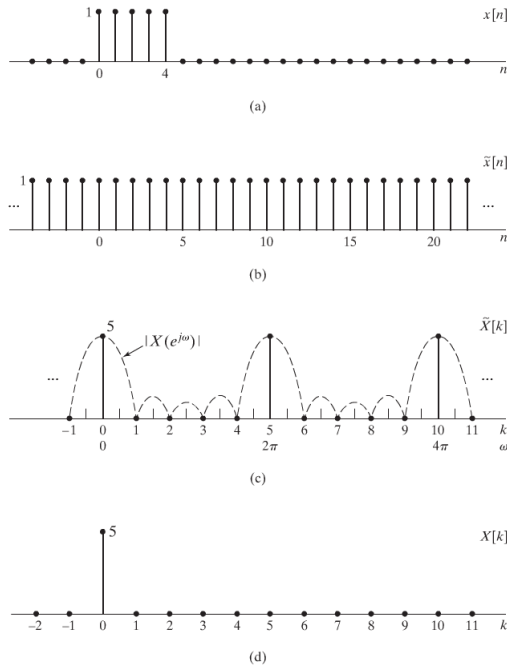


Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

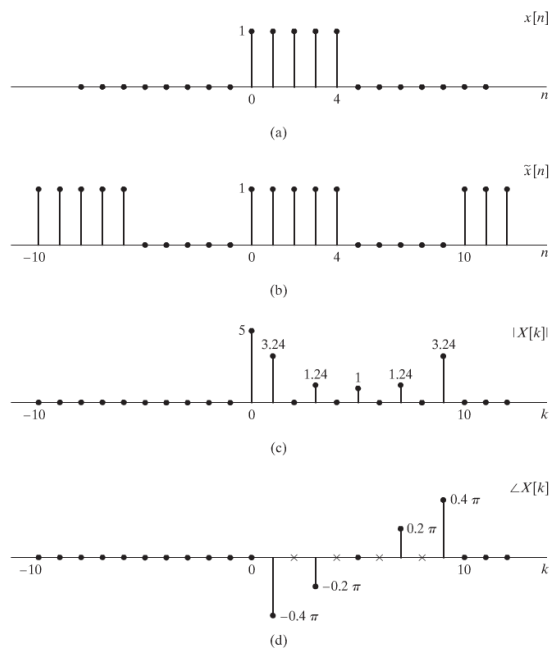


Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)

✧ Properties of Discrete Fourier Transform

- **Linearity**

$$\left. \begin{aligned} x_1[n] &\leftrightarrow X_1[k] \\ x_2[n] &\leftrightarrow X_2[k] \end{aligned} \right\} \Rightarrow ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k]$$

$$length = \max[N_1, N_2]$$

- **Circular Shift**

$$x[n] \leftrightarrow X[k] \Rightarrow \begin{aligned} x[((n-m))_N] &\leftrightarrow W_N^{km} X[k] \\ W_N^{-ln} x[n] &\leftrightarrow X[((k-l))_N] \end{aligned}$$

(Pf) From the right side of the 2nd eqn.

$$W_N^{km} X[k] = e^{j\frac{2\pi}{N}km} X[k] \rightarrow e^{j\frac{2\pi}{N}km} \tilde{X}[k] \quad \text{QED}$$

$$\begin{array}{ccc} \Downarrow \text{DFT} & & \Downarrow \text{IDFS} \\ x[((n-m))_N] & \leftarrow & x[((n-m))_N] = \tilde{x}[n-m] \end{array}$$

Remark: This is *circular* shift not *linear* shift. (Linear shift of a periodic sequence = circular shift of a finite sequence.)

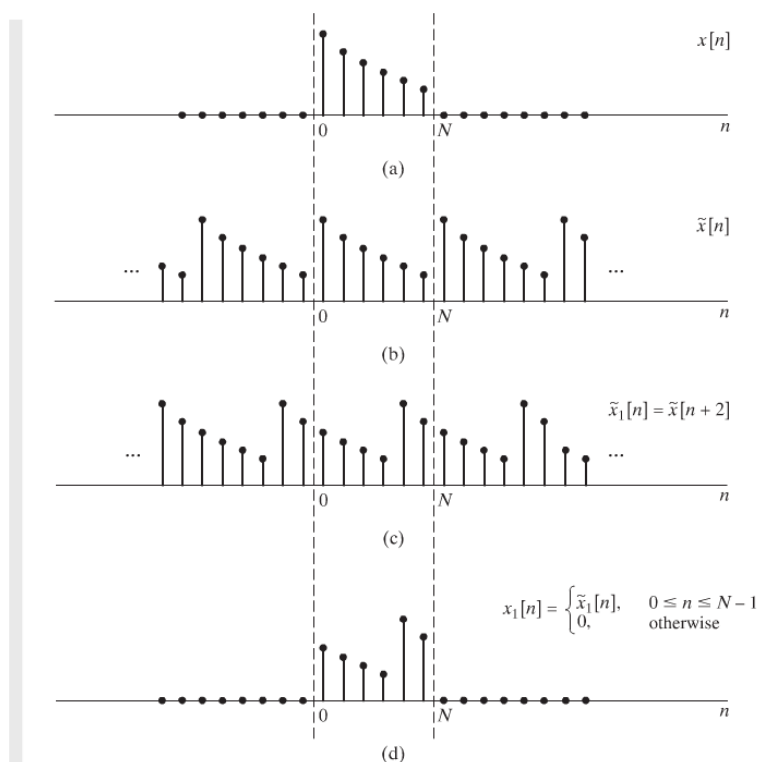


Figure 8.12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear-phase factor.

• **Duality**

$$x[n] \leftrightarrow X[k]$$

$$X[n] \leftrightarrow Nx[((-k))_N], \quad 0 \leq k \leq N-1$$

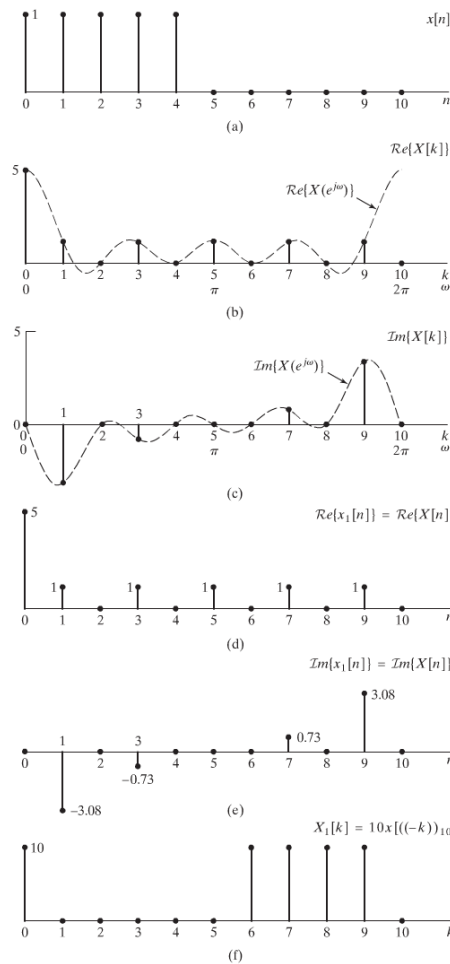


Figure 8.13 Illustration of duality. (a) Real finite-length sequence $x[n]$. (b) and (c) Real and imaginary parts of corresponding DFT $X[k]$. (d) and (e) The real and imaginary parts of the dual sequence $x_1[n] = X[n]$. (f) The DFT of $x_1[n]$.

• **Symmetry Properties**

$$x_{ep}[n] = \text{periodic conjugate-symmetric}$$

$$\equiv \tilde{x}_e[n]$$

$$= \frac{1}{2} \{x[((n))_N] + x^*[((n))_N]\}, \quad 0 \leq n \leq N-1$$

$$= \begin{cases} \frac{1}{2} \{x[n] + x^*[N-n]\}, & 1 \leq n \leq N-1 \\ \text{Re}\{x[0]\}, & n = 0 \end{cases}$$

$x_{op}[n]$ = periodic conjugate - antisymmetric

$$= \begin{cases} \frac{1}{2} \{x[n] - x^*[N-n]\}, & 1 \leq n \leq N-1 \\ \text{Im}\{x[0]\}, & n = 0 \end{cases}$$

$$x_{ep}[n] \leftrightarrow \text{Re}\{X[k]\} \qquad x_{op}[n] \leftrightarrow j \text{Im}\{X[k]\}$$

If $x[n]$ real, $X[k] = X^*[((-k))_N]$, $0 \leq k \leq N-1$

$$\Rightarrow \begin{cases} |X[k]| = |X[((-k))_N]| \\ \angle\{X[k]\} = -\angle\{X[((-k))_N]\} \end{cases} \Rightarrow \begin{cases} \text{Re}\{X[k]\} = \text{Re}\{X[((-k))_N]\} \\ \text{Im}\{X[k]\} = -\text{Im}\{X[((-k))_N]\} \end{cases}$$

$$\begin{cases} \text{Re}\{x[n]\} \leftrightarrow X_{ep}[k] = \frac{1}{2} \{X[((k))_N] + X^*[((-k))_N]\} \\ \text{Im}\{x[n]\} \leftrightarrow X_{op}[k] = \frac{1}{2} \{X[((k))_N] - X^*[((-k))_N]\} \end{cases}$$

• **Circular Convolution**

$$x_3[n] = x_1[n] \otimes x_2[n]$$

$$\equiv \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N] \quad N\text{-point circular convolution}$$

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1[k] X_2[k]$$

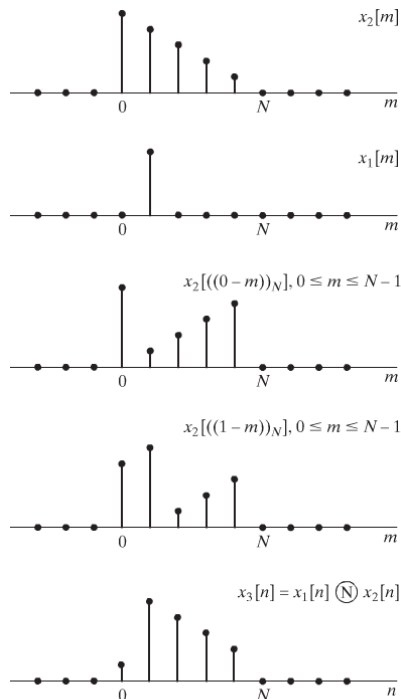


Figure 8.14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse, $x_1[n] = \delta[n-1]$.

Example: N-point circular convolution of two constant sequences of length N

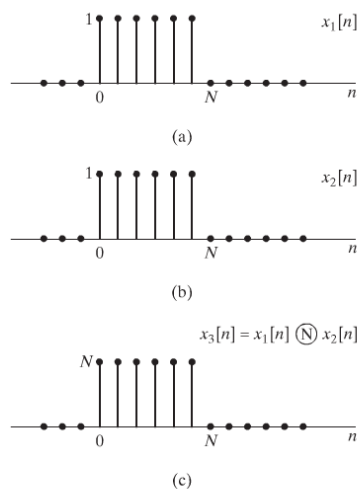
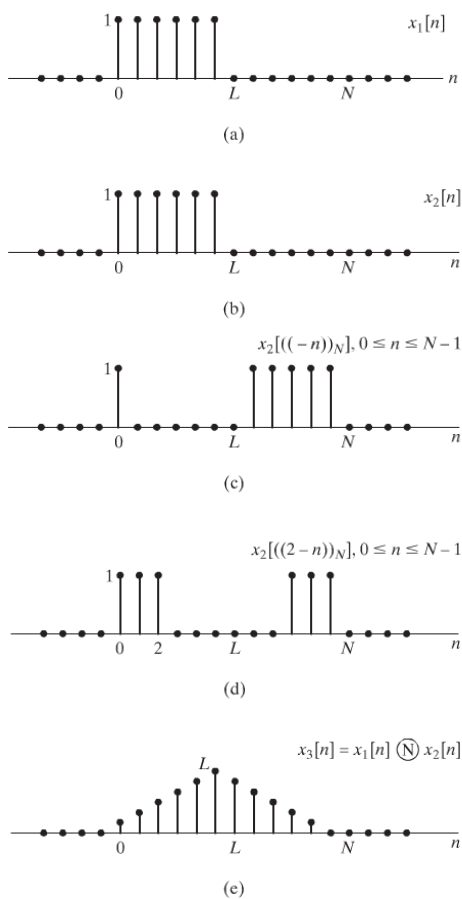


Figure 8.15 N-point circular convolution of two constant sequences of length N.

2L-point circular convolution of two constant sequences of length L



✧ Linear Convolution Using DFT

- Why using DFT? There are fast DFT algorithms (FFT)

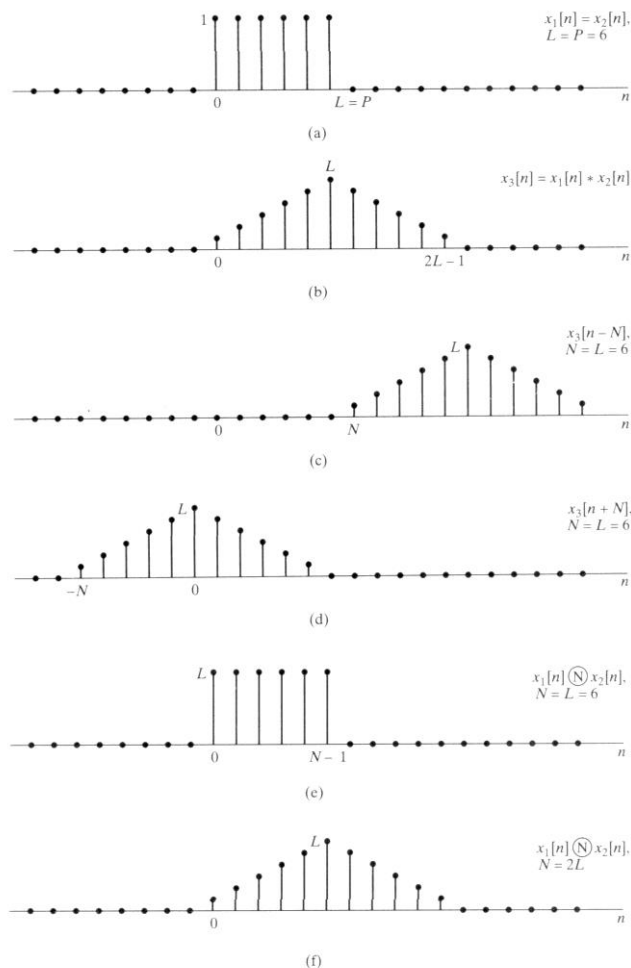


Figure 8.18 Illustration that circular convolution is equivalent to linear convolution followed by aliasing. (a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved. (b) The linear convolution of $x_1[n]$ and $x_2[n]$. (c) $x_3[n - N]$ for $N = 6$. (d) $x_3[n + N]$ for $N = 6$. (e) $x_1[n] \textcircled{\otimes} x_2[n]$, which is equal to the sum of (b), (c), and (d) in the interval $0 \leq n \leq 5$. (f) $x_1[n] \textcircled{\otimes} x_2[n]$.

- How to do it?
 - (1) Compute the N -point DFT of $x_1[n]$ and $x_2[n]$ separately
 $\rightarrow X_1[k]$ and $X_2[k]$
 - (2) Compute the product $X_3[k] = X_1[k]X_2[k]$
 - (3) Compute the N -point IDFT of $X_3[k] \rightarrow x_3[n]$
- Problems: (a) Aliasing
 (b) Very long sequence

• **Aliasing**

$x_1[n]$, length L (nonzero values)

$x_2[n]$, length P

In order to avoid aliasing, $N \geq L + P - 1$

(What do we mean avoid aliasing? The preceding procedure is *circular* convolution but we want *linear* convolution. That is, $x_3[n]$ equals to the linear convolution of $x_1[n]$ and $x_2[n]$)

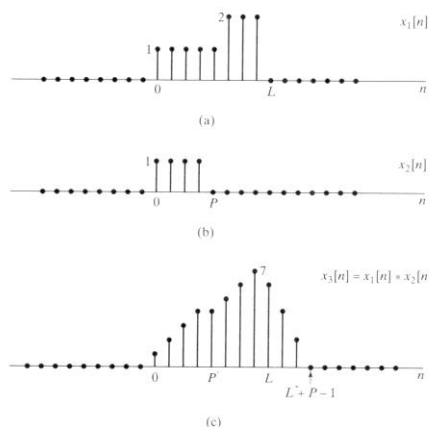


Figure 8.19 An example of linear convolution of two finite-length sequences.

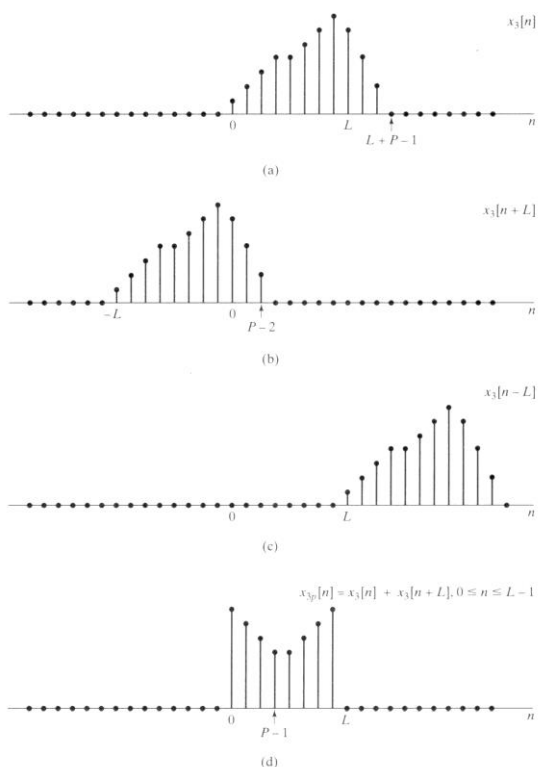


Figure 8.20 Interpretation of circular convolution as linear convolution followed by aliasing for the circular convolution of the two sequences $x_1[n]$ and $x_2[n]$ in Figure 8.19.

$x_1[n]$ pad with zeros \rightarrow length N
 $x_2[n]$ pad with zeros \rightarrow length N

Interpretation: (Why call it aliasing?)

$X_3[k]$ has a (time domain) bandwidth of size $L + P - 1$

(That is, the nonzero values of $x_3[n]$ can be at most $L + P - 1$)

Therefore, $X_3[k]$ should have at least $L + P - 1$ samples. If the sampling rate is insufficient, aliasing occurs on $x_3[n]$.

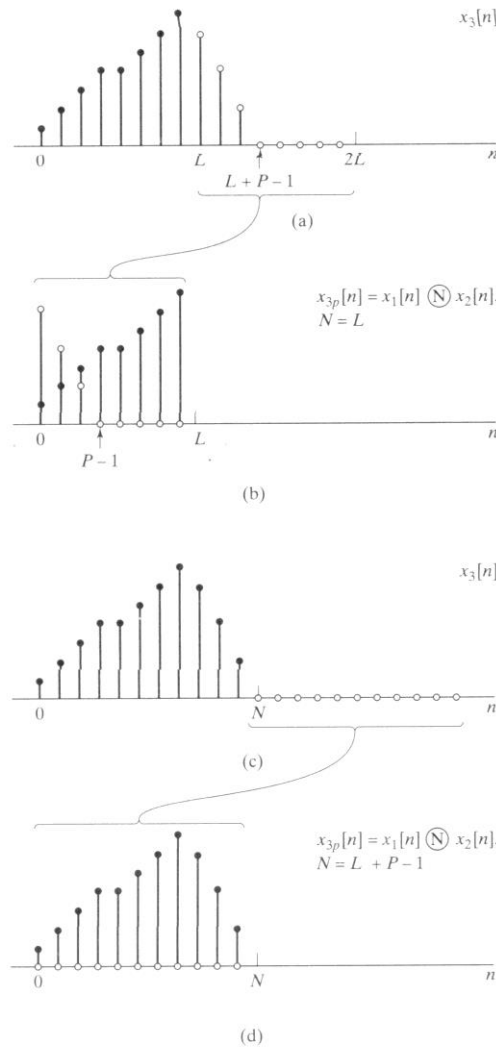


Figure 8.21 Illustration of how the result of a circular convolution “wraps around.” (a) and (b) $N = L$, so the aliased “tail” overlaps the first $(P - 1)$ points. (c) and (d) $N = (L + P - 1)$, so no overlap occurs.

- **Very long sequence (FIR filtering)**

- Block convolution

- ⊙ Method 1 – **overlap and add**

Partition the long sequence into sections of shorter length.

For example, the filter impulse response $h[n]$ has finite length P and the input data $x[n]$ is nearly “infinite”.

$$\text{Let } x[n] = \sum_{r=0}^{\infty} x_r[n - rL] \text{ where } x_r[n] = \begin{cases} x[n + rL], & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

The system (filter) output is a linear convolution:

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL] \text{ where } y_r[n] = x_r[n] * h[n]$$

Remark: The convolution length is $L + P - 1$. That is, the $L + P - 1$ point DFT is used. $y_r[n]$ has $L + P - 1$ data points; among them, $(P-1)$ points should be added to the next section.

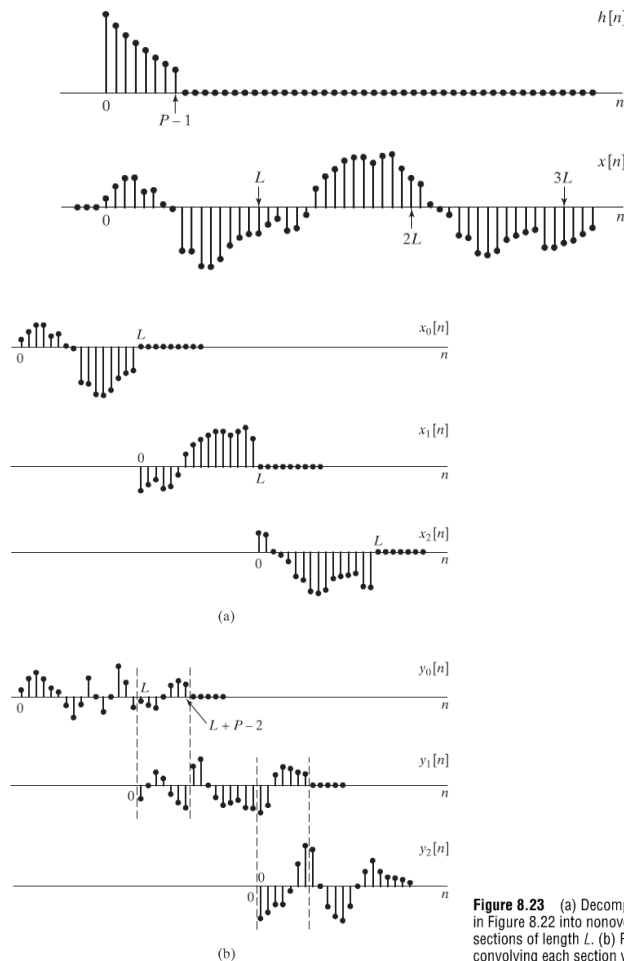


Figure 8.23 (a) Decomposition of $x[n]$ in Figure 8.22 into nonoverlapping sections of length L . (b) Result of convolving each section with $h[n]$.

This is called **overlap-add method**.

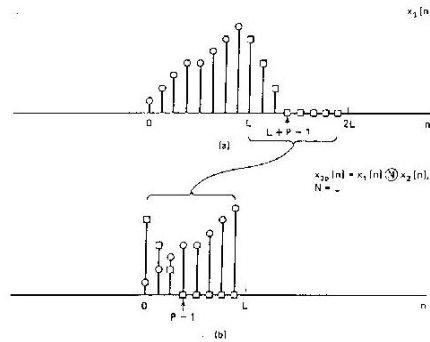
(*Key:* The input data are partitioned into *nonoverlapping* sections \rightarrow the section outputs are overlapped and added together.)

⊙ Method 2 – **overlap and save**

Partition the long sequence into overlapping sections.

After computing DFT and IDFT, throw away some (incorrect) outputs.

For each section (length L , which is also the DFT size), we want to retain the correct data of length $(L - (P - 1))$ points



Let $h[n]$, length P

$x_r[n]$, length L ($L > P$)

Then, $y_r[n]$ contains $(P-1)$ incorrect points at the beginning.

Therefore, we divide into sections of length L but each section overlaps the preceding section by $(P-1)$ points.

$$x_r[n] = x[n + r(L - P + 1) - (P - 1)], \quad 0 \leq n \leq L - 1$$

This is called **overlap-save method**.

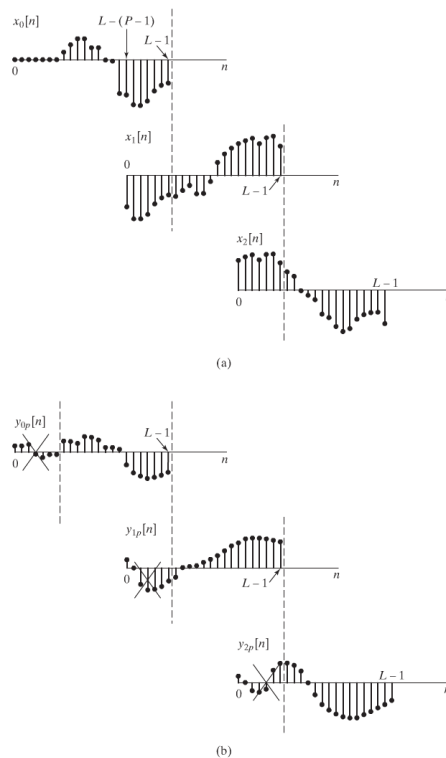


Figure 8.24 (a) Decomposition of $x[n]$ in Figure 8.22 into overlapping sections of length L . (b) Result of convolving each section with $h[n]$. The portions of each filtered section to be discarded in forming the linear convolution are indicated.