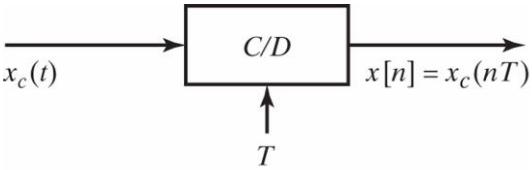
Sampling of Continuous-time Signals

- Advantages of digital signal processing, e.g., audio/video CD.
- Things to look at:
  - Continuous-to-discrete (C/D)
  - Discrete-to-continuous (D/C) – perfect reconstruction
  - Frequency-domain analysis of sampling process
  - Sampling rate conversion

✧ Periodic Sampling

- **Ideal continuous-to-discrete-time (C/D) converter**



Continuous-time signal: $x_c(t)$

Discrete-time signal: $x[n] = x_c(nT), -\infty < n < \infty$, T: sampling period

In theory, we break the C/D operation into two steps:

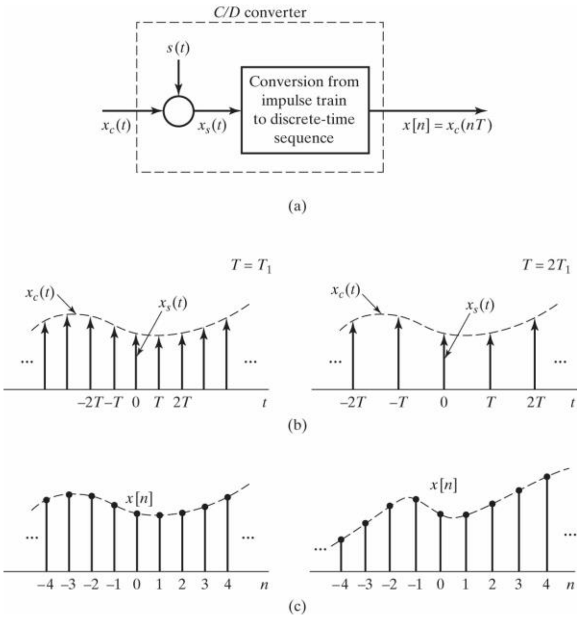
(1) Ideal sampling using "analog delta function (impulse)"

(2) Conversion from impulse train to discrete-time sequence

Step (1) can be modeled by mathematical equation.

Step (2) is a "concept", no mathematical model.

In reality, the electronic analog-to-digital (A/D) circuits can approximate the ideal C/D operation. This circuitry is one piece; it cannot be split into two steps.



- **Ideal sampling**

***Ideal* sampling signal**: impulse train (an analog signal)

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$ *T*: sampling period

Analog (continuous-time) signal: $x_c(t)$

Sampled (continuous-time) signal: $x_s(t)$

$$x_s(t) = x_c(t)s(t) = x_c(t)\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

✧ Frequency-domain Representation of Sampling

$$s(t) \leftrightarrow S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$, where $\Omega_s = 2\pi / T$

Remark: Ω: analog frequency (radians/s)

ω: discrete (normalized) frequency (radians/sample)

$$\Omega = \omega / T; \quad -\pi < \omega \le \pi, \quad -\frac{\pi}{T} < \Omega \le \frac{\pi}{T}$$

Step 1: *Ideal Sampling* (all in analog domain)

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{1}{T} X_c(j\Omega) * \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega) * \delta(\Omega - k\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

The sampled signal spectrum is the sum of shifted copies of the original.

Remark: In analog domain,

$$x(t)y(t) \quad \leftrightarrow \quad X(f) * Y(f)$$

$$= \frac{1}{2\pi} X(j\Omega) * Y(j\Omega)$$

Step 2: *Analog Impulses to Sequence* (analog to discrete-time)

No mathematical model. The spectrum of $x_s(t)$, $X_s(j\Omega)$, is the same as the spectrum of $x[n]$, $X(e^{j\Omega T})$. (See the Appendix at the end.)

Now, $X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

Thus, $X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right)$

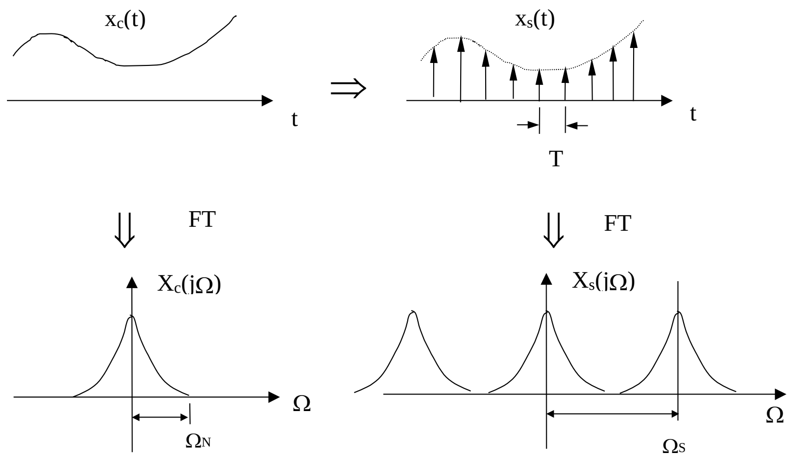
Remark: In time domain, $x_s(t)$ and $x[n]$ are two very different signals but they have the “same” spectra in frequency domain.

Two Cases**:**

(1) no aliasing: $\Omega_s > 2\Omega_N$, and

(2) aliasing: $\Omega_s < 2\Omega_N$, where Ω_N is the highest nonzero frequency component of $X_c(j\Omega)$.

After sampling, the replicas of $X_c(j\Omega)$ overlap (in frequency domain). That is, the higher frequency components of $X_c(j\Omega)$ overlap with the lower frequency components of $X_c\big(j(\Omega-\Omega_s)\big)$.



- **Nyquist Sampling Theorem**:

Let *x*(*t*) be a *bandlimited* signal with $X_c(j\Omega)=0$ for $|\Omega|\geq\Omega_N$. (i.e., no components at frequencies greater than Ω_N) Then $x_c(t)$ is uniquely determined by its samples $x[n]=x_c(nT),\ n=0,\pm1,\pm2,\ldots$, if $\Omega_s=\frac{2\pi}{T}\geq 2\Omega_N$. (Nyquist, Shannon)

-- **Nyquist frequency =** Ω_N , the bandwidth of signal.

-- **Nyquist rate** = $2\Omega_N$, the minimum sampling rate without distortion. (In some books, Nyquist frequency = Nyquist rate.)

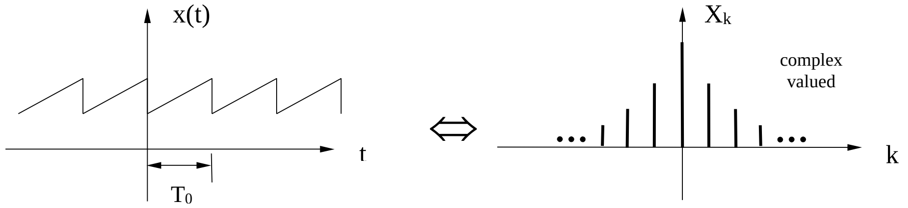
-- **Undersampling:** $\Omega_s<2\Omega_N$

-- **Overdampling:** $\Omega_s>2\Omega_N$

✧ Fourier Series, Fourier Transform, Discrete-Time Fourier Series & Discrete-Time Fourier Transform

- **Fourier Series**

x(t): periodic continuous-time signal with period T_0

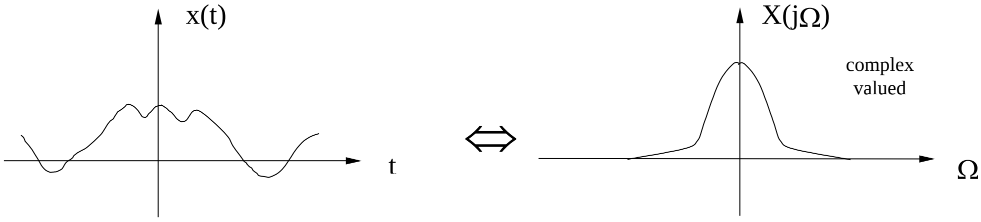


$$\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k \frac{t}{T_0}} = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \\ X_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\Omega_0 t} dt, \qquad \Omega_0 = \frac{2\pi}{T_0} \end{cases}$$

Power: $P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$

- **Fourier Transform**

x(t): continuous-time signal



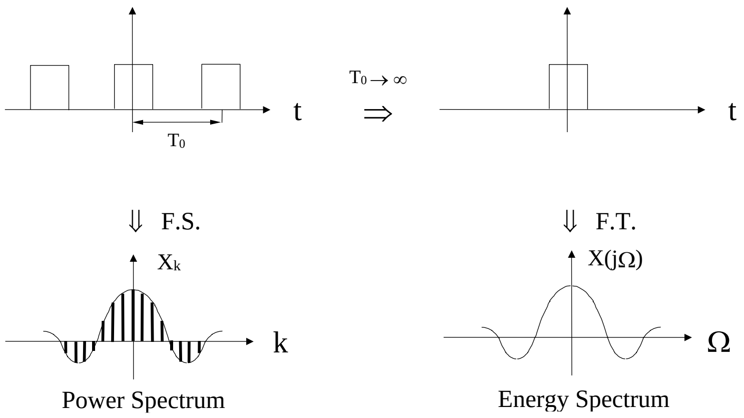
$$\begin{cases} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \\ X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \end{cases}$$

Energy: $P_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega$

Remark: (1) Other Notations

$$\begin{cases} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w) e^{jwt} dw \\ X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt \end{cases} \qquad \begin{cases} x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \end{cases}$$

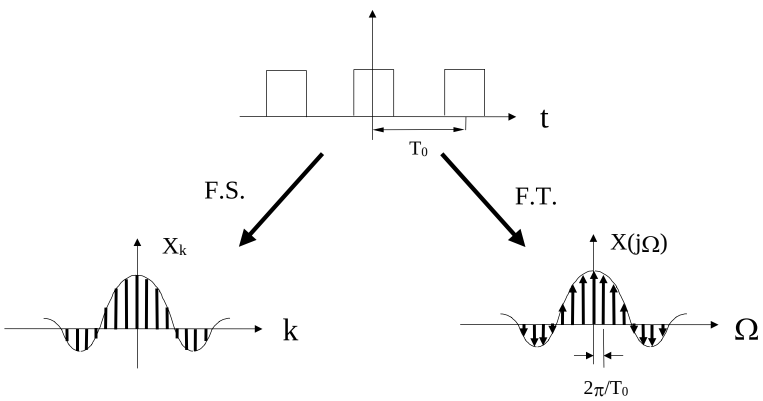
(2) Relationships between F.S. & F.T.



$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\Omega_0 t}dt \qquad X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

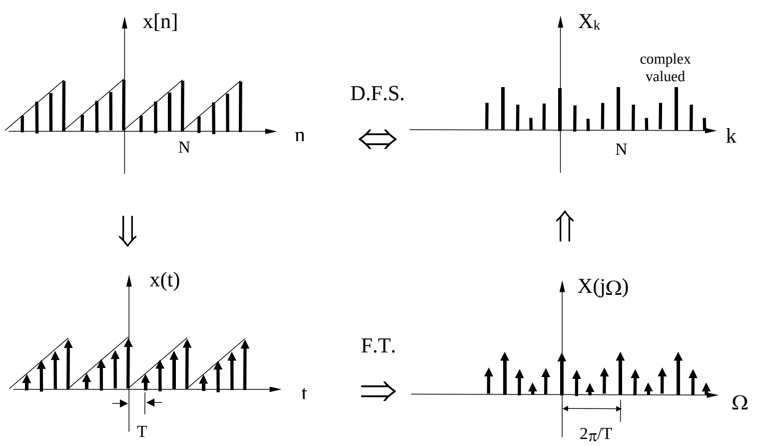
$$T_0 X_k \xrightarrow[k\Omega_0 \to \Omega]{T_0 \to \infty} X(j\Omega) = \lim_{T_0 \to \infty}(T_0 X_k)$$

(3) Periodic Signal



- **Discrete-Time Fourier Series**

  x[n]: periodic discrete-time signal with period N.

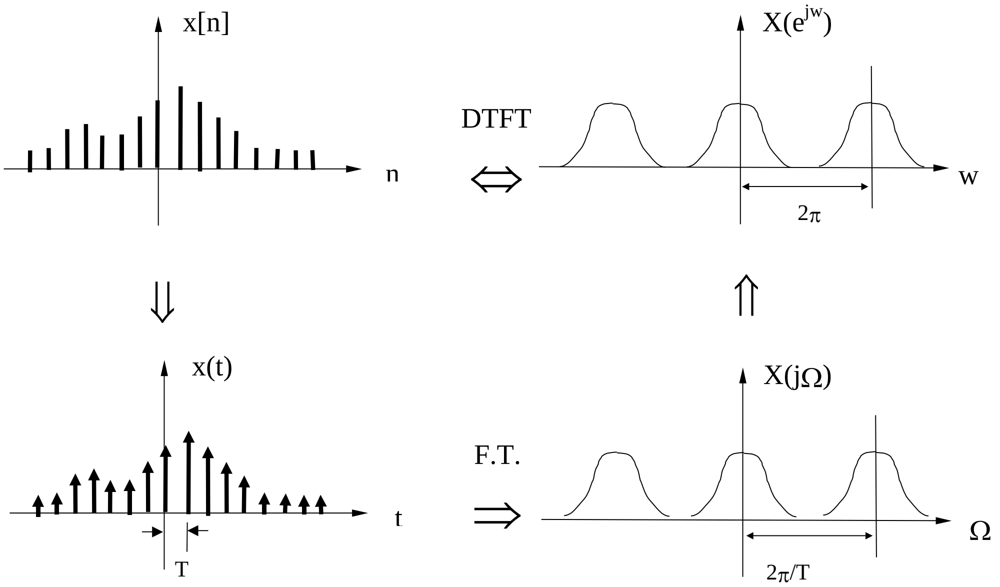


$$\begin{cases} x[n] = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi nk}{N}} \\ X_k = \displaystyle\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} \end{cases}$$

Power: $$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \sum_{k=0}^{N-1}|X_k|^2$$

- **Discrete-Time Fourier Transform**

  x[n]: discrete-time signal



$$\begin{cases} x[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw \\ X(e^{jw}) = \displaystyle\sum_{n=-\infty}^{\infty} x[n] e^{-jwn} \end{cases}$$

Energy: $$E_x = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{jw})|^2 dw$$

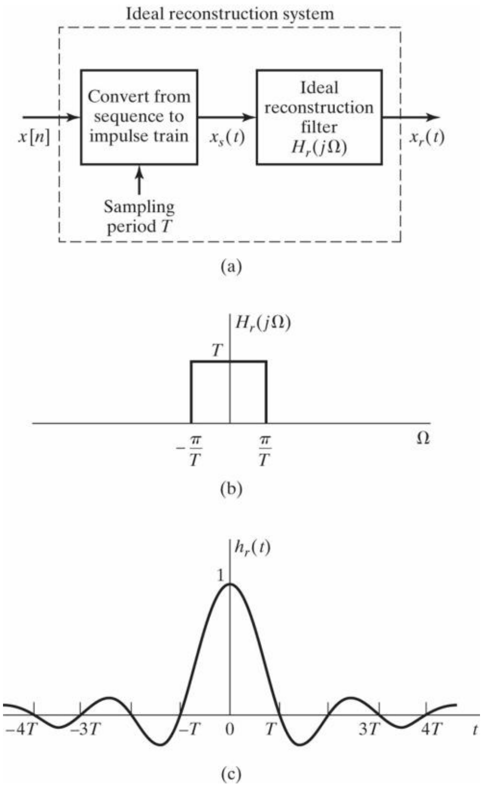
Remark: $X(e^{jw})$ *v.s.* $X(j\Omega)$

$X(e^{jw})$ is a frequency-scaled version of $X(j\Omega)$

$$X(j\Omega) = X(e^{jw})\Big|_{w=\Omega T}$$

✧ Reconstruction of a Band-limited Signal from Its Samples

-- **Perfect reconstruction:** recover the original continuous-time signal without distortion, e.g., ideal lowpass (bandpass) filter



Based on the frequency-domain analysis, if we can "clip" one copy of the original spectrum, $X_c(j\Omega)$, without distortion, we can achieve the perfect reconstruction. For example, we use the ideal low-pass filter as the reconstruction filter.

Remark: Note that $x_s(t)$ is an analog signal (impulses).

$$x_c(t) \to sampling \to x_s(t) = \sum x(nT)\delta(t-nT) \to seq.-convr. \to x[n]$$

$$x[n] \to impulse-convr. \to x_s(t) = \sum x[n]\delta(t-nT) \to recon. \to x_r(t)$$

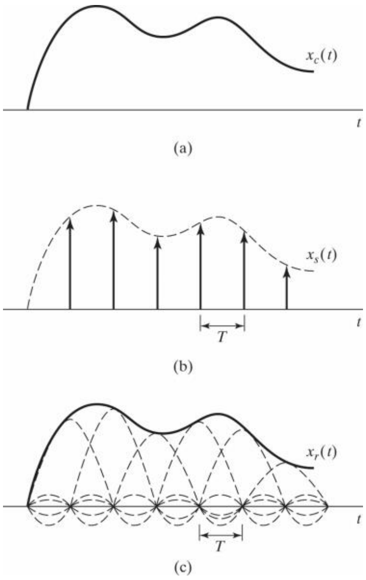
$$x_r(t) = x_s(t) * h_r(t) = \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} x[n]\delta(\lambda - nT) h_r(t-\lambda) \right\} d\lambda$$

$$= \sum_{n=-\infty}^{\infty} \left\{ x[n] \int_{-\infty}^{\infty} \delta(\lambda - nT) h_r(t-\lambda) d\lambda \right\} = \sum_{n=-\infty}^{\infty} x[n] h_r(t-nT)$$

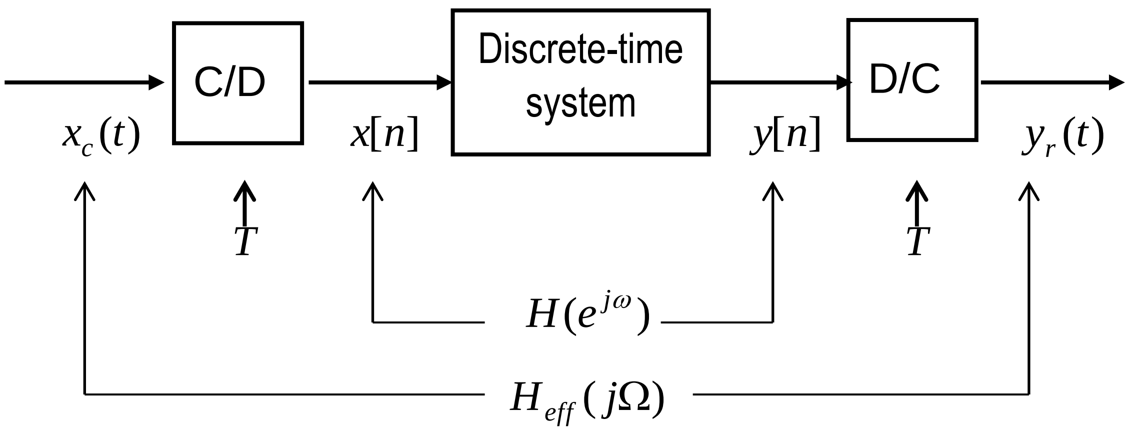
$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_r(j\Omega) e^{-j\Omega T n} = H_r(j\Omega)\{\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n}\}$$
$$= H_r(j\Omega) X(e^{jw})\Big|_{w=\Omega T} = H_r(j\Omega) X(e^{j\Omega T}) = H_r(j\Omega) X(j\Omega)$$

Ideal low-pass reconstruction filter:

$$H_r(j\Omega) = \begin{cases} T & -\pi/T < \Omega \le \pi/T \\ 0 & otherwise \end{cases} \qquad h_r(t) = \frac{\sin(\pi t/T)}{(\pi t/T)}$$



✧ Discrete-time Processing of Continuous-time Signals



- If this is an LTI system,

(1) $x[n] \to y[n]$: $Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$

(2) $x_c(t) \rightarrow x[n]$: $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$

(3) $y[n] \rightarrow y_r(t)$: $Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$

Note that "*T*" is included in the expression of $Y(e^{j\omega}), \quad \omega \leftarrow \Omega T$. This means "physical" frequency (not normalized).

(4) $x_c(t) \rightarrow \cdots \rightarrow y_r(t)$:

$$
\begin{aligned}
Y_r(j\Omega) &= H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) \\
&= H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\Omega - j\frac{2\pi k}{T}\right)
\end{aligned}
$$

If $H_r(j\Omega)$ is an ideal low-pass reconstruction filter, then

$$
Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T \\ 0, & otherwise \end{cases}
$$

In other words, if $x_c(t)$ is *band-limited* and is *ideally sampled* at a rate above the *Nyquist rate*, and the reconstruction filter is the *ideal low-pass* filter, then the *equivalent analog* filter has the same spectrum shape of the discrete-time filter.

$$
H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & otherwise \end{cases}
$$

Remark: In order to have the above equivalent relation between $H(e^{j\omega})$ and $H_{eff}(j\Omega)$, we need

(i) The system is LTI;

(ii) The input is bandlimited;

(iii) The input is sampled without aliasing and the ideal impulse train is used in sampling;

(iv) The ideal reconstruction filter is used to produce the analog output.

In practice, the above conditions are only approximately valid at best. However, there are methods in designing the sampling and the reconstruction processes to make the approximation better.

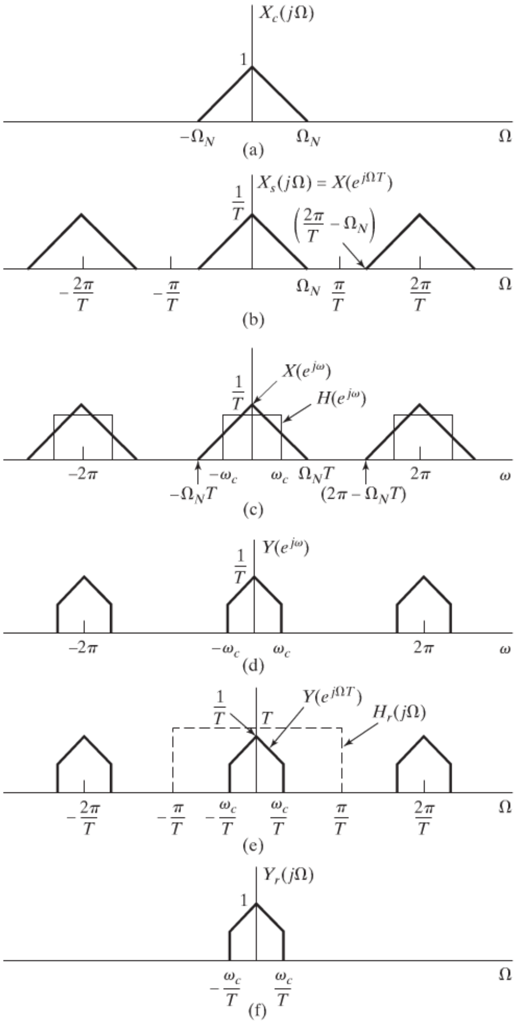


Figure 4.12 (a) Fourier transform of a bandlimited input signal. (b) Fourier transform of sampled input plotted as a function of continuous-time frequency Ω. (c) Fourier transform $X(e^{j\omega})$ of sequence of samples and frequency response $H(e^{j\omega})$ of discrete-time system plotted versus ω. (d) Fourier transform of output of discrete-time system. (e) Fourier transform of output of discrete-time system and frequency response of ideal reconstruction filter plotted versus Ω. (f) Fourier transform of output.

$$H_{eff}(j\Omega) = H_c(j\Omega)$$

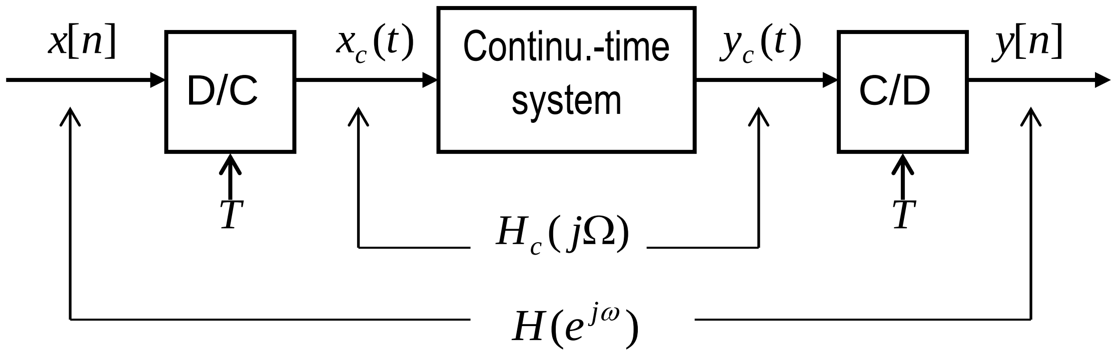
$$H(e^{jw}) = H_c(j\frac{w}{T}) \qquad |w| < \pi$$

T is chosen s.t. $H_c(j\Omega) = 0$, for $|\Omega| \ge \frac{\pi}{T}$

$$\Rightarrow \quad h[n] = Th_c(nT)$$

The impulse response of the discrete-time system is a scaled, sampled version of $h_c(t)$.

✧ Continuous-time Processing of Discrete-time Signals



$$X_c(j\Omega) = TX(e^{j\Omega T}), \qquad |\Omega| < \frac{\pi}{T}$$

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega), \qquad |\Omega| < \frac{\pi}{T}$$

$$Y(e^{jw}) = \frac{1}{T}Y_c(j\frac{w}{T}), \qquad |w| < \pi$$

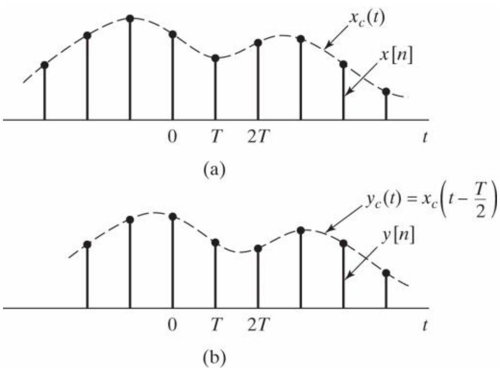
$$\Rightarrow \quad Y(e^{jw}) = \frac{1}{T}H_c(j\frac{w}{T})X_c(j\frac{w}{T}) = H_c(j\frac{w}{T})X(e^{jw}) \qquad |w| < \pi$$

$$\Rightarrow \quad H(e^{jw}) = H_c(j\frac{w}{T}) \qquad |w| < \pi$$

or, equivalently, $H(e^{j\Omega T}) = H_c(j\Omega) \qquad |\Omega| < \frac{\pi}{T}$

Example: Noninteger Delay

$$H(e^{jw}) = e^{-jw\Delta} \qquad |\mathrm{w}| < \pi$$



(a)

(b)

✧ Change the Sampling Rate Using Discrete-time Processing

$$x_c(t)\ \begin{cases} \to T \to & x[n]=x_c(nT) \\ \to T' \to & x'[n]=x_c(nT') \end{cases}$$

Original sampling period: T

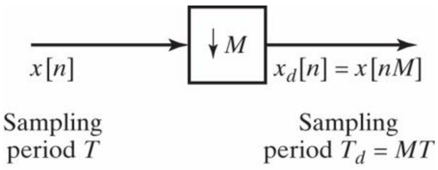
New sampling period: T' $\qquad\qquad T \neq T'$

• Sampling rate reduction by an integer factor

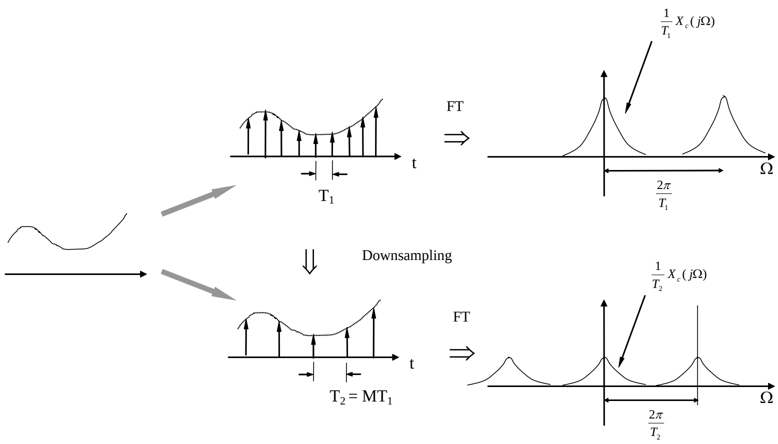
■ Sampling rate compressor:

$T' = MT$, where M is an integer

$x_d[n] = x[nM] = x_c(nMT)$



Compressor



Aliasing: If the original signal BW is not small enough to meet the Nyquist rate requirement, prefiltering is needed.

The Original **The Downsampled**

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T}-\frac{2\pi k}{T}\right)\right) \qquad X_d(e^{j\omega}) = \frac{1}{T'}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'}-\frac{2\pi r}{T'}\right)\right)$$

r	$\cdots$	-2	-1	0	1	2	$\cdots$	$(M-1)$	M	$(M+1)$	$\cdots$
i	$\cdots$	$(M-2)$	$(M-1)$	0	1	2	$\cdots$	$(M-1)$	1	2	$\cdots$
k	$\cdots$	-1	-1	0	0	0	$\cdots$	0	1	1	$\cdots$

Old and new index: $r = i + kM$ $\qquad r, k = -\infty, \cdots, -2, -1, 0, 1, 2, \cdots, \infty,$

$i = 0, 1, 2, \cdots, M-1$

$$X_d(e^{j\omega}) = \frac{1}{T'} \sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'} - \frac{2\pi r}{T'} \right) \right)$$

$$= \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT} \right) \right)$$

$$= \frac{1}{MT} \sum_{k=-\infty}^{\infty} \sum_{i=0}^{M-1} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi kM}{MT} - \frac{2\pi i}{MT} \right) \right)$$

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi i}{MT} - j\frac{2\pi k}{T} \right) \right]$$

$$= \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M} \right)} \right)$$

The down-sampled spectrum = sum of shifted replica of the original

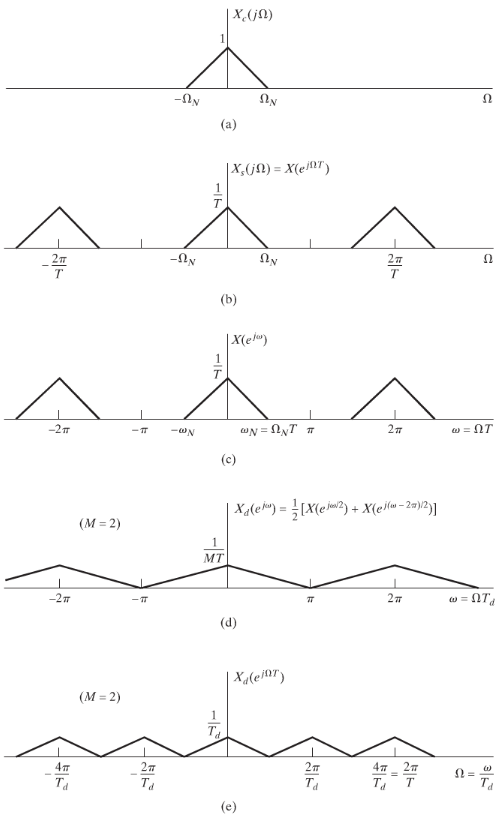


Figure 4.20 Frequency-domain illustration of downsampling.

Downsampling with aliasing

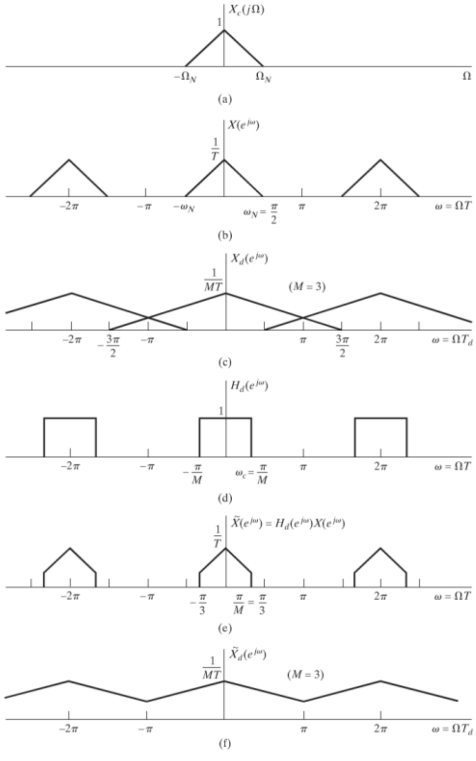
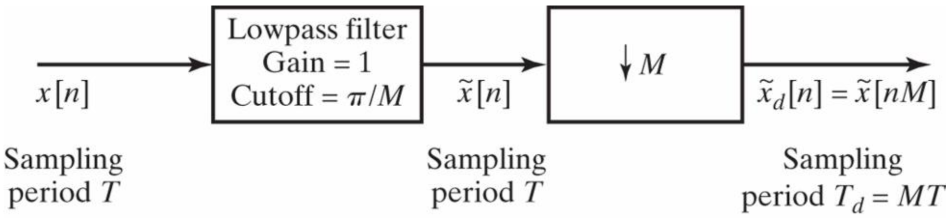


Figure 4.21 (a)–(c) Downsampling with aliasing. (d)–(f) Downsampling with prefiltering to avoid aliasing.

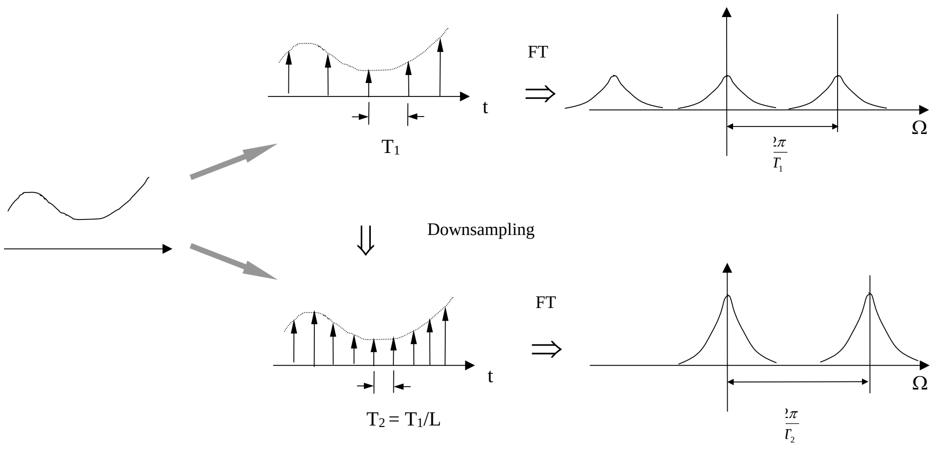
To avoid aliasing

$\Rightarrow \quad w_N M < \pi$



General System for Sampling Rate Reduction by M

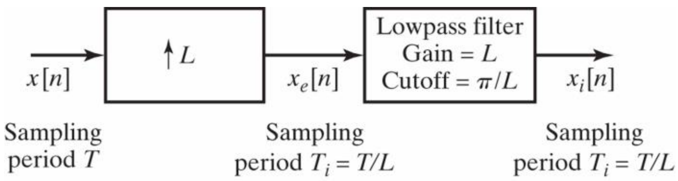
- **Increasing sampling rate by an integer factor**



■ **Sampling rate expander**

$T' = T/L$, where L is an integer

$$x_i[n] = x_c\left(n\frac{T}{L}\right)$$



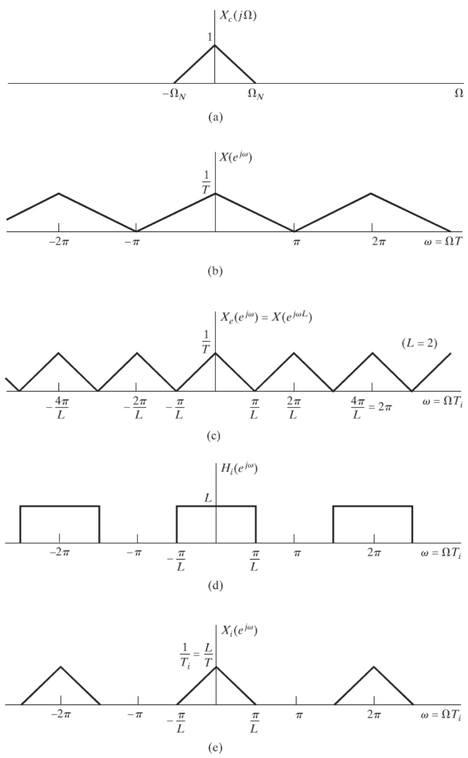


Figure 4.24 Frequency-domain illustration of interpolation.

(1) shape is compressed; (2) replicas are removed

(1) Increase samples

<Time-domain>

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & otherwise \end{cases} = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

<Frequency-domain>

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right)e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\left(\sum_{n=-\infty}^{\infty}\delta[n-kL]e^{-j\omega n}\right) = X(e^{j\omega L})$$

Note that $\sum_{n=-\infty}^{\infty}\delta[n-kL]e^{-j\omega n} = e^{-j\omega Lk}$

Remark: Essentially, the horizontal frequency axis is compressed.

The shape of the spectrum is not changed.

$$old\_\omega = \Omega T,\ new\_\omega = \Omega T' = \Omega T/L,\ old\_\omega = new\_\omega \cdot L$$

Remark: At this point, we only insert zeros into the original signal. In time domain, this signal doesn't look like the original.

(2) Ideal lowpass filtering

<Frequency-domain>

$$H_i(j\Omega) = \begin{cases} 1 & -\pi/(TL) < \Omega \le \pi/(TL) \\ 0 & otherwise \end{cases}$$

<Time-domain>

$h_i[n] = \dfrac{\sin(\pi n/L)}{(\pi n/L)}$, an interpolator!

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}$$

■ **Linear interpolation**

$$h_{lin}[n] = \begin{cases} 1 - |n|/L, & |n| \le L \\ 0, & otherwise \end{cases}$$

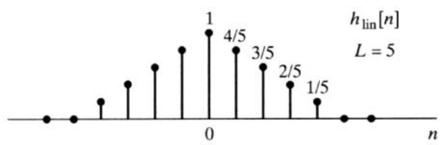
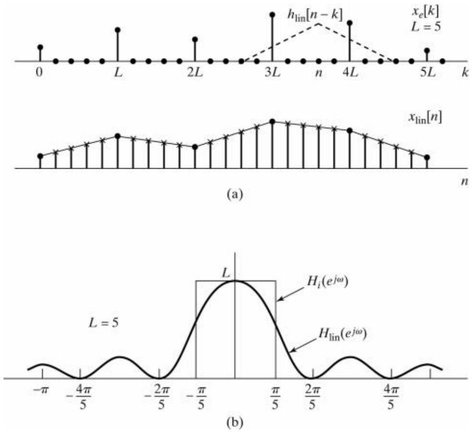


Figure 4.26 Impulse response for linear interpolation.

$$H_{lin}(e^{j\omega}) = \frac{1}{L}\left[\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right]^2$$

$$x_{lin}[n] = \sum_{k=-\infty}^{\infty} x[k]h_{lin}[n-kL]$$



• Changing sampling rate by a rational factor

Idea: **Sampling period** $T \xrightarrow{\text{interpolation}} \frac{T}{L} \xrightarrow{\text{decimation}} \frac{M}{L}T$

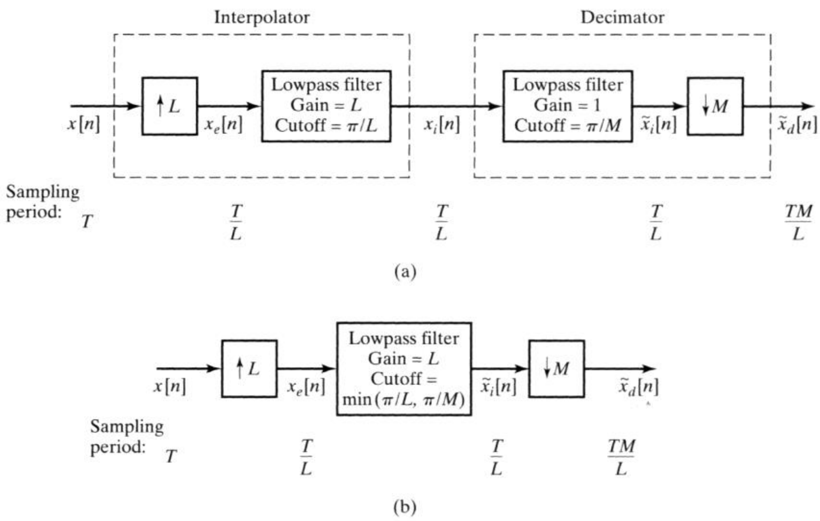


Figure 4.28 (a) System for changing the sampling rate by a noninteger factor. (b) Simplified system in which the decimation and interpolation filters are combined.

Remark: In general, if the factor is not rational, go back to the continuous signals.

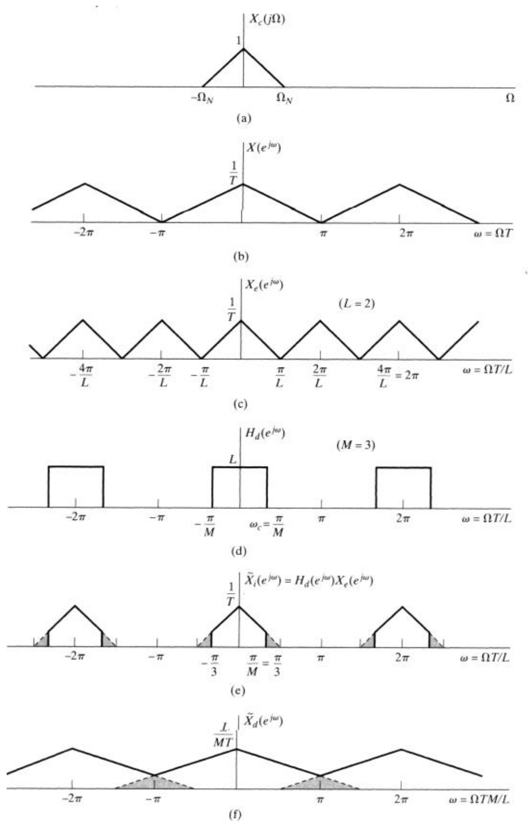


Figure 4.29 Illustration of changing the sampling rate by a noninteger factor.

- **In summary:**

-- Sampling

Time-domain	Frequency -domain
Prefiltering	Limit bandwidth $\Omega_s > 2\Omega_N$
Analog sampling (impulse train)	Duplicate and shift (Ω)
Analog to discrete $\delta(t) \to \delta[n]$	$\Omega \to \omega$

-- Reconstruction

Time-domain	Frequency -domain
Discrete to analog $\delta[n] \to \delta(t)$	$\omega \to \Omega$
Interpolation	Remove extra copies (Ω)

-- Down-sampling

Time-domain	Frequency -domain
Prefiltering	Limit bandwidth
Drop samples (rearrange index)	Expand (by a factor of M) and duplicate (insert (M-1) copies)

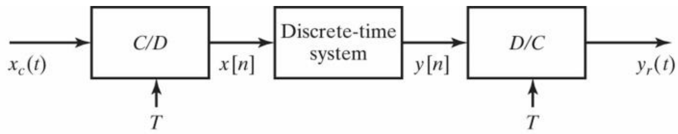
-- Up-sampling

Time-domain	Frequency -domain
Insert zeros	Shrink (by a factor of L)
Interpolation	Remove extra copies in a 2π period

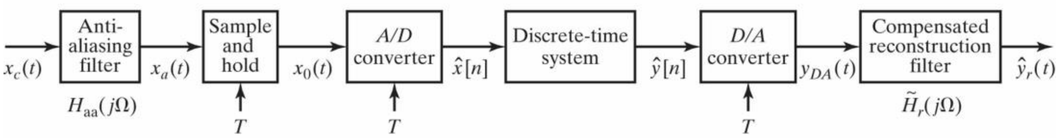
✧ Digital Processing of Analog Signals

Ideal C/D converter → (approximation) analog-to-digital (A/D) converter

Ideal D/C converter → (approximation) digital-to-analog (D/A) converter



(a)

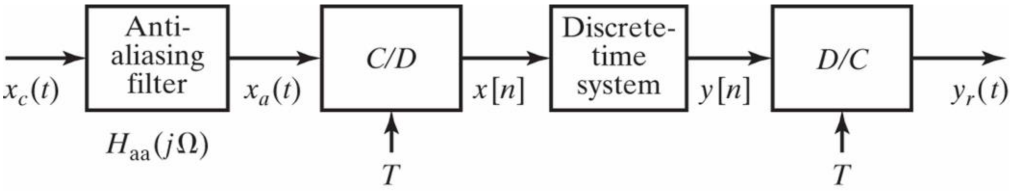


(b)

Prefiltering to Avoid Aliasing

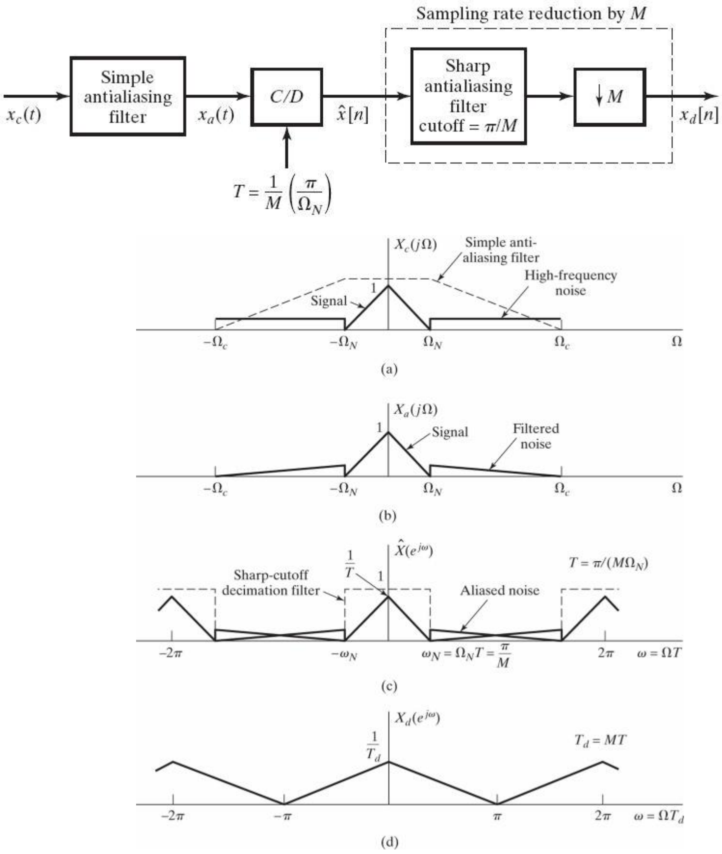
Ideal antialiasing filter: Ideal low-pass filter (difficult to implement sharp-cutoff analog filters).

← A solution: simple prefilter and oversampling followed by sharp antialiasing filters in discrete-time domain.



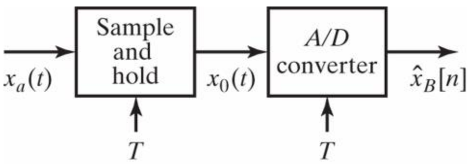
Remark: Sharp cutoff analog filters are expensive and difficult to implement.

A/D conversion ⇒ the input continuous-time signal is sampled at a very high sampling rate.



● A/D Conversion

Digital: discrete in *time* and discrete in *amplitude*

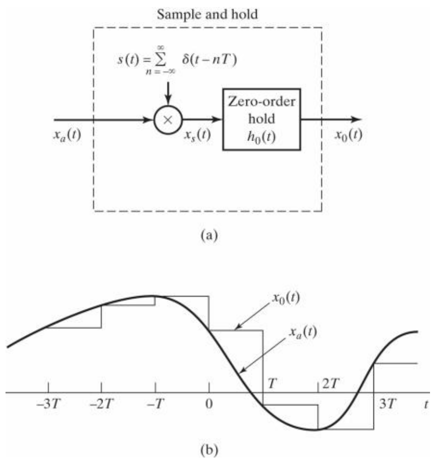


Ideal sample-and-hold: Sample the (input) analog signal and hold its value for T secnds.

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT)$$

$$h_0(t) = \begin{cases} 1, & 0 < t < T \\ 0, & otherwise \end{cases}$$

$$x_0(t) = \sum_{n=-\infty}^{\infty} x_a(nT) h_0(t - nT) = \{ \sum_{n=-\infty}^{\infty} x_a(nT) \delta(t - nT) \} * h_0(t)$$



Quantization: Transform the input sample $x[n]$ into one of a finite set of prescribed values.

$\hat{x}[n] = Q(x[n])$, $\hat{x}[n]$ is the quantized sample

Note: Quantization is a non-linear operation.

(i) Uniform quantizer – uniformly spaced quantization levels; very popular (also called *linear* quantizer)

(ii) Nonuniform quantizer – may be more efficient for certain applicaitons

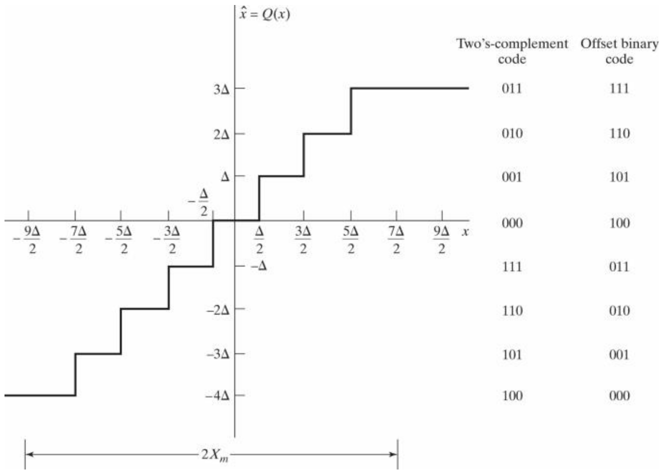
- Parameters in a quantizer

(1) Decision levels – partition the dynamic range of input signal

(2) Quantization (representation) levels – the output values of a quantizer; a quantization level represents all samples between two nearby decision levels

(3) Full-scale level – the quantizer input dynamic range

Note: Typically, when the decision levels are first chosen, then the best quantization levels are decided (for a given input probability distribution). On the other hand, when the quantization levels are chosen, the best decision levels are decided.



- Quantization error analysis

For a **uniform** quantizer**,** there are two key parameters:

(i)step size Δ, and (ii) full-scale level ($\pm X_m$)

Assume (B+1) bits are used to represent the quantized values.

$$\Delta = \frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B}$$

Quantization error: $e[n] = \hat{x}[n] - x[n]$ = quanitized value – true value

It is clear that $-\frac{\Delta}{2} < e[n] < \frac{\Delta}{2}$.

Statistical characteristics of $e[n]$:

(1) $e[n]$ is stationary (probability distribution unchanged)

(2) $e[n]$ is uncorrelated with $x[n]$

(3) $e[n]$, $e[n+1]$, … are uncorrelated (white)

(4) $e[n]$ has a uniform distribution

The preceding assumptions are (approximately) valid if the signal is sufficiently *complex* and the quantization steps are sufficiently *small*, …

Mean square error (MSE) of $e[n]$ (= variance if zero mean)

$$(\sigma_e)^2 = E\{(e-\bar{e})^2\} = \int_{-\Delta/2}^{\Delta/2} e^2 \frac{1}{\Delta} de = \frac{\Delta^2}{12}$$

-- Expressed in terms of 2^B and X_m

$$\sigma_e^2 = \frac{2^{-2B} X_m^2}{12}$$

-- SNR (signal-to-noise ratio) due to quantization

$$SNR = 10\log_{10}\frac{\sigma_x^2}{\sigma_e^2} = 10\log_{10}\frac{12\cdot 2^{2B}\sigma_x^2}{X_m^2} = 10.8 + 6.02B - 20\log_{10}\frac{X_m}{\sigma_x}$$

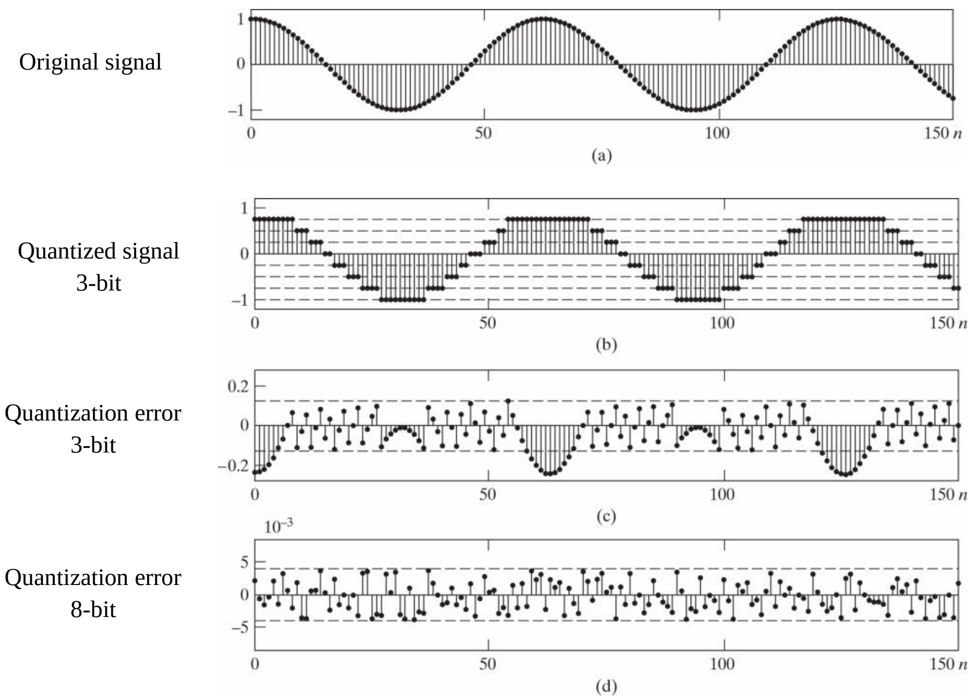
Remarks:

(1) One bit buys a 6dB SNR improvement.

(2) If the input is Gaussian, a small percentage of the input samples would have an amplitude greater than $4\sigma_x$.

If we choose $X_m = 4\sigma_x$, $SNR \approx 6B - 1.25dB$

For example, a 96dB SNR requires a 16-bit quantizer.

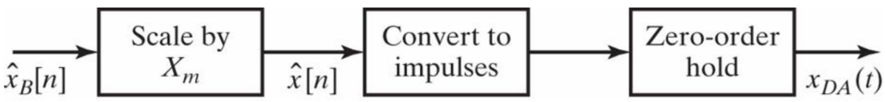


- **D/A Conversion**

The ideal lowpass filter is replaced by a "practical" filter.

Examples of practical filters: *zero-order hold* and *first-order hold*.

Mathematical model:



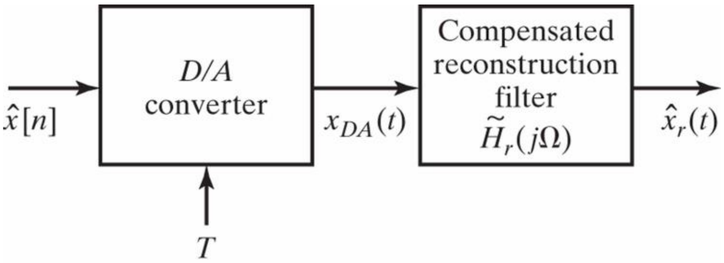
$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n] h_0(t - nT)$$

= quantized input * impulse response of "practical" interpolation filter

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x[n] h_0(t - nT) + \sum_{n=-\infty}^{\infty} e[n] h_0(t - nT)$$

$$= x_0(t) \quad + \quad e_0(t)$$

Purpose: Find a compensation filter $\tilde{h}_r(t)$ to compensate for the distortion caused by the non-ideal $h_0(t)$ so that its output $\hat{x}_r(t)$ is close to the analog original $x_a(t)$.



In frequency domain:

$$X_0(j\Omega) = F_t\left\{\sum_{n=-\infty}^{\infty} x[n] h_0(t - nT)\right\} = \sum_{n=-\infty}^{\infty} x[n] H_0(j\Omega) e^{-j\Omega nT}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega nT}\right) H_0(j\Omega) = X(e^{j\Omega T}) H_0(j\Omega)$$

Because $X\left(e^{j\Omega T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - k\frac{2\pi k}{T}\right)\right)$,

$$X_0(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - k\frac{2\pi k}{T}\right)\right) H_0(j\Omega)$$

[The interpolation filter $H_0(j\Omega)$ is used to remove the replicas.]

If $H_0(j\Omega)$ is not an ideal lowpass filter, we design a compensated reconstruction filter, $\tilde{H}_r(j\Omega) = \frac{H_r(j\Omega)}{H_0(j\Omega)}$, where $H_r(j\Omega)$ is the ideal lowpass filter.

(1) ***Zero-order hold***

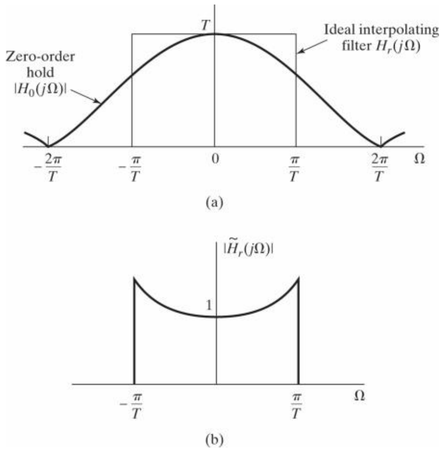
$$h_0(t) = \begin{cases} 1, & 0 < t < T \\ 0, & otherwise \end{cases}$$

or $H_0(j\Omega) = \frac{2\sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}$

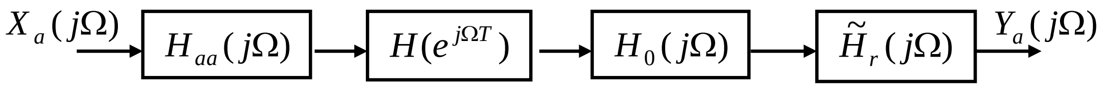
Thus, the compensated reconstruction filter is

$$\tilde{H}_r(j\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0, & |\Omega| > \pi/T \end{cases}$$

Remark: A "practical" filter cannot achieve this approximation.



Overall system:



Anti-aliasing Processing Zero-order-hold Compensated reconstruction.

$$H_{eff}(j\Omega) = \tilde{H}_r(j\Omega) \cdot H_0(j\Omega) \cdot H(e^{j\Omega T}) \cdot H_{aa}(j\Omega)$$

$$P_{e_a}(j\Omega) = \left|\tilde{H}_r(j\Omega) \cdot H_0(j\Omega) \cdot H(e^{j\Omega T})\right|^2 \sigma_e^2 \quad \text{where} \quad \sigma_e^2 = \frac{\Delta^2}{12}$$