8.2

8.2. (a) Using the analysis equation of Eq. (8.11)

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Since $\tilde{x}[n]$ is also periodic with period $3N$,

$$\begin{aligned}\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=N}^{2N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=2N}^{3N-1} \tilde{x}[n] W_{3N}^{kn}\end{aligned}$$

Performing a change of variables in the second and third summations of $\tilde{X_3}[k]$,

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + W_{3N}^{kN} \sum_{n=0}^{N-1} \tilde{x}[n+N] W_{3N}^{kn} + W_{3N}^{2kN} \sum_{n=0}^{N-1} \tilde{x}[n+2N] W_{3N}^{kn}$$

Since $\tilde{x}[n]$ is periodic with period N, and $W_{3N}^{kn} = W_N^{(\frac{k}{3})n}$,

$$\begin{aligned}\tilde{X}_3[k] &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(\frac{k}{3}n)} \\ &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \tilde{X}[k] \\ &= \begin{cases} 3\tilde{X}[k/3], & k = 3\ell \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

(b) Using $N = 2$ and $\tilde{x}[n]$ as in Fig P8.2-1:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\ &= \sum_{n=0}^{1} \tilde{x}[n] e^{-j\frac{2\pi}{2}kn} \\ &= \tilde{x}[0] + \tilde{x}[1] e^{-j\pi k} \\ &= 1 + 2(-1)^k \\ &= \begin{cases} 3, & k = 0 \\ -1, & k = 1 \end{cases}\end{aligned}$$

Observe, from Fig. P8.2-1, that $\tilde{x}[n]$ is also periodic with period $3N = 6$:

$$\begin{aligned}\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{5} \tilde{x}[n] e^{-j\frac{\pi}{3}kn} \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k})(1 + 2(-1)^{\frac{k}{3}}) \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k}) \tilde{X}[k/3] \\ &= \begin{cases} 9, & k = 0 \\ -3, & k = 3 \\ 0, & k = 1, 2, 4, 5. \end{cases}\end{aligned}$$

8.7

8.7. We have a six-point uniform sequence, $x[n]$, which is nonzero for $0 \le n \le 5$. We sample the Z-transform of $x[n]$ at four equally-spaced points on the unit circle.

$$X[k] = X(z)|_{z=e^{(2\pi k/4)}}$$

We seek the sequence $x_1[n]$ which is the inverse DFT of $X[k]$. Recall the definition of the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is zero for all n outside $0 \le n \le 5$, we may replace the infinite summation with a finite summation. Furthermore, after substituting $z = e^{j(2\pi k/4)}$, we obtain

$$X[k] = \sum_{n=0}^{5} x[n]W_4^{kn}, \quad 0 \le k \le 4$$

Note that we have taken a 4-point DFT, as specified by the sampling of the Z-transform; however, the original sequence was of length 6. As a result, we can expect some aliasing when we return to the time domain via the inverse DFT.

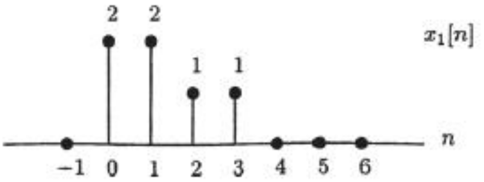
Performing the DFT,

$$X[k] = W_4^{0k} + W_4^{k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k}, \quad 0 \le k \le 4$$

Taking the inverse DFT by inspection, we note that there are six impulses (one for each value of n above). However,

$$W_4^{4k} = W_4^{0k} \text{ and } W_4^{5k} = W_4^{k},$$

so two points are aliased. The resulting time-domain signal is



8.9

8.9. Given a 20-pt finite-duration sequence $x[n]$:

(a) We wish to obtain $X(e^{j\omega})|_{\omega=4\pi/5}$ using the smallest DFT possible. A possible size of the DFT is evident by the periodicity of $e^{j\omega}|_{\omega=4\pi/5}$. Suppose we choose the size of the DFT to be $M = 5$. The data sequence is 20 points long, so we use the time-aliasing technique derived in the previous problem. Specifically, we alias $x[n]$ as:

$$x_1[n] = \sum_{r=-\infty}^{\infty} x[n+5r]$$

This aliased version of $x[n]$ is periodic with period 5 now. The 5-pt DFT is computed. The desired value occurs at a frequency corresponding to:

$$\frac{2\pi k}{N} = \frac{4\pi}{5}$$

For $N = 5$, $k = 2$, so the desired value may be obtained as $X[k]|_{k=2}$.

(b) Next, we wish to obtain $X(e^{j\omega})|_{\omega=10\pi/27}$.

The smallest DFT is of size $L = 27$. Since the DFT is larger than the data block size, we pad $x[n]$ with 7 zeros as follows:

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le 19 \\ 0, & 20 \le n \le 26 \end{cases}$$

We take the 27-pt DFT, and the desired value corresponds to $X[k]$ evaluated at $k = 5$.

8.12

8.12. (a)

$$x[n] = \cos(\frac{\pi n}{2}), \quad 0 \le n \le 3$$

transforms to

$$X[k] = \sum_{n=0}^{3} \cos(\frac{\pi n}{2}) W_4^{kn}, \quad 0 \le k \le 3$$

The cosine term contributes only two non-zero values to the summation, giving:

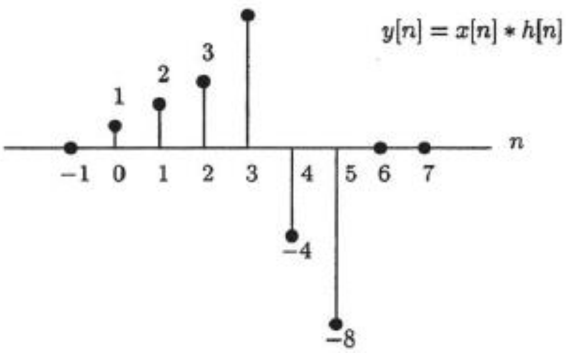
$$\begin{aligned} X[k] &= 1 - e^{-j\pi k}, \quad 0 \le k \le 3 \\ &= 1 - W_4^{2k} \end{aligned}$$

(b)

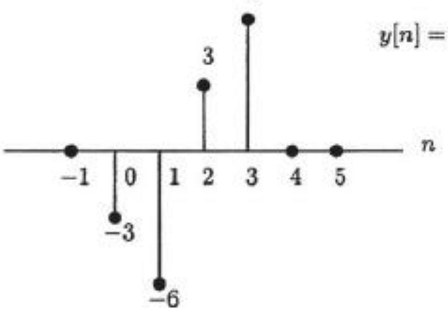
$$h[n] = 2^n, \quad 0 \le n \le 3$$

$$\begin{aligned} H[k] &= \sum_{n=0}^{3} 2^n W_4^{kn}, \quad 0 \le k \le 3 \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} \end{aligned}$$

(c) Remember, circular convolution equals linear convolution plus aliasing. We need $N \ge 3 + 4 - 1 = 6$ to avoid aliasing. Since $N = 4$,we expect to get aliasing here. First, find $y[n] = x[n] * h[n]$:



For this problem, aliasing means the last three points ($n = 4, 5, 6$) will wrap-around on top of the first three points, giving $y[n] = x[n] ④ h[n]$:



(d) Using the DFT values we calculated in parts (a) and (b):

$$\begin{aligned} Y[k] &= X[k]H[k] \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k} - 2W_4^{3k} - 4W_4^{4k} - 8W_4^{5k} \end{aligned}$$

Since $W_4^{4k} = W_4^{0k}$ and $W_4^{5k} = W_4^k$

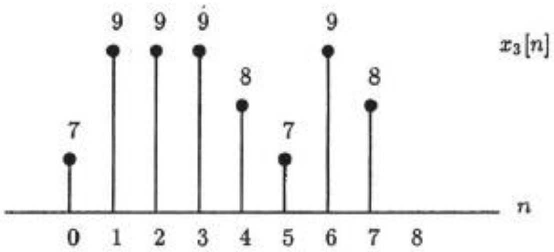
$$Y[k] = -3 - 6W_4^k + 3W_4^{2k} + 6W_4^{3k}, \quad 0 \le k \le 3$$

Taking the inverse DFT:

$$y[n] = -3\delta[n] - 6\delta[n-1] + 3\delta[n-2] + 6\delta[n-3], \quad 0 \le n \le 3$$

8.14

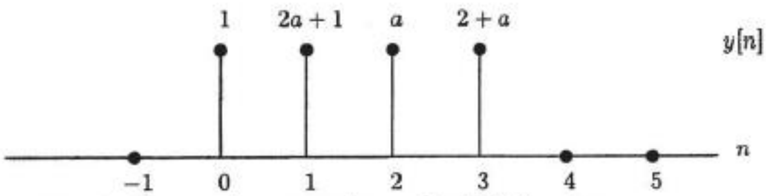
8.14. $x_3[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 8$. Carrying out the 8-point circular convolution, we get:



We thus conclude $x_3[2] = 9$.

8.15

8.15. $y[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 4$. Carrying out the 4-point circular convolution, we get:



Matching the above sequence to the one given, we get $a = -1$, which is unique.

8.21

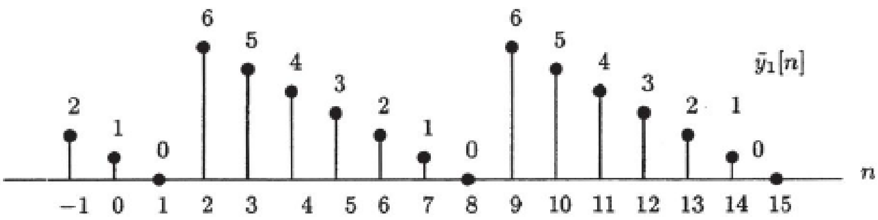
8.21. (a) We seek a sequence $\tilde{y}_1[n]$ such that

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

From the discussion of Section 8.2.5, $\tilde{y}[n]$ is the result of the periodic convolution between $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

$$\tilde{y}_1[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

Since $\tilde{x}_2[n]$ is a periodic impulse, shifted by two, the resultant sequence will be a shifted (by two) replica of $\tilde{x}_1[n]$.



Using the analysis equation of Eq. (8.11), we may rigorously derive $\tilde{y}_1[n]$:

$$\begin{aligned}
\tilde{X}_1[k] &= \sum_{n=0}^{6} \tilde{x}_1[n]W_7^{kn} \\
&= 6 + 5W_7^k + 4W_7^{2k} + 3W_7^{3k} + 2W_7^{4k} + W_7^{5k} \\
\tilde{X}_2[k] &= \sum_{n=0}^{6} \tilde{x}_2[n]W_7^{kn} \\
&= W_7^{2k} \\
\tilde{Y}_1[k] &= \tilde{X}_1[k]\tilde{X}_2[k] \\
&= 6W_7^{2k} + 5W_7^{3k} + 4W_7^{4k} + 3W_7^{5k} + 2W_7^{6k} + W_7^{7k}
\end{aligned}$$

Noting that $W_7^{7k} = e^{j\frac{2\pi}{7}(7k)} = 1 = W_7^{0k}$, we use the synthesis equation of Eq. (8.12) to construct $\tilde{y}_1[n]$. The result is identical to the sequence depicted above.

(b) The DFS of the signal illustrated in Fig. P8.21-2.is given by:

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{6} \tilde{x}_3[n]W_7^{kn} \\
&= 1 + W_7^{4k}
\end{aligned}$$

Therefore:

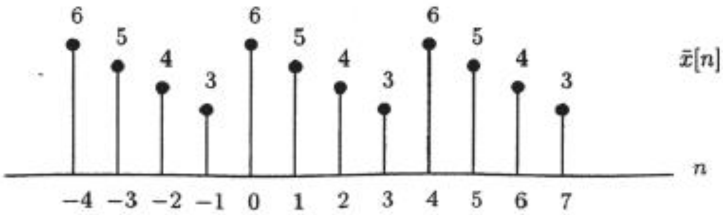
$$\begin{aligned}
\tilde{Y}_2[k] &= \tilde{X}_1[k]\tilde{X}_3[k] \\
&= \tilde{X}_1[k] + W_7^{4k}\tilde{X}_1[k]
\end{aligned}$$

Since the DFS is linear, the inverse DFS of $\tilde{Y}_2[k]$ is given by:

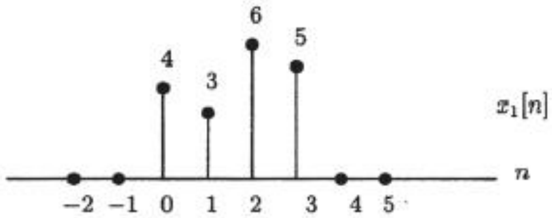
$$\tilde{y}_2[n] = \tilde{x}_1[n] + \tilde{x}_1[n-4].$$

8.24

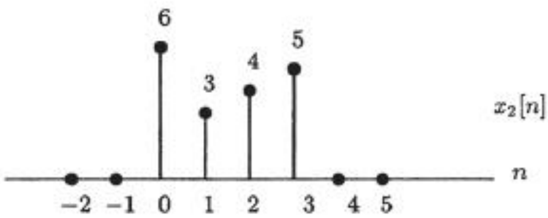
8.24. We may approach this problem in two ways. First, the notion of modulo arithmetic may be simplified if we utilize the implied periodic extension. That is, we redraw the original signal as if it were periodic with period $N = 4$. A few periods are sufficient:



To obtain $x_1[n] = x[((n-2))_4]$, we shift by two (to the right) and only keep those points which lie in the original domain of the signal (i.e. $0 \le n \le 3$):



To obtain $x_2[n] = x[((-n))_4]$, we fold the pseudo-periodic version of $x[n]$ over the origin (time-reversal), and again we set all points outside $0 \le n \le 3$ equal to zero. Hence,



Note that $x[((0))_4] = x[0]$, etc.

In the second approach, we work with the given signal. The signal is confined to
$0 \le n \le 3$; therefore, the circular nature must be maintained by picturing the signal on the circumference of a cylinder.

8.28

8.28. **(a)** Using the analysis equation

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}$$
$$=\sum_{n=0}^{5}x[n]W_6^{kn}$$
$$=6W_6^0+5W_6^k+4W_6^{2k}+3W_6^{3k}+2W_6^{4k}+W_6^{5k}.$$

(b)

$$W[k]=W_6^{-2k}X[k]$$
$$=6W_6^{-2k}+5W_6^{-k}+4+3W_6^k+2W_6^{2k}+W_6^{3k}.$$

Using the fact that $W_6^k=e^{-j\frac{2\pi k}{6}}$,

$$W_6^{-2k}=e^{j\frac{4\pi k}{6}}=e^{j\frac{4\pi k}{6}}\times e^{-j2\pi k}\ (\text{since } e^{-j2\pi k}=1)$$
$$=e^{-j\frac{8\pi k}{6}}=W_6^{4k},$$

and similarly

$$W_6^{-k}=W_6^{5k}.$$

Then

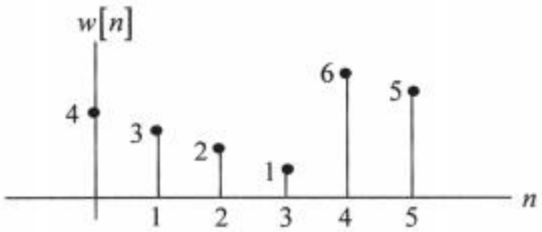
$$W[k]=4+3W_6^k+2W_6^{2k}+W_6^{3k}+6W_6^{4k}+5W_6^{5k}.$$

Using the synthesis equation,

$$w[n]=\frac{1}{6}\sum_{k=0}^{5}W[k]W_6^{-kn}.$$

We could go ahead and solve the problem in this "brute force" method, but notice that each $\delta[n-k]\xrightarrow{DFT}W_N^k$. Then,

$$w[n]=4\delta[n]+3\delta[n-1]+2\delta[n-2]+\delta[n-3]+6\delta[n-4]+5\delta[n-5].$$



Notice that multiplying by W_6^{-2k} in frequency has the effect of a shift of 2 in time, but modulo 6.

(c) One way to do this is to compute the linear convolution and then add copies of it shifted by N (6 in this case). Another method is to use the DFT, find the product $H[k]X[k]$, and then take an inverse DFT. We know

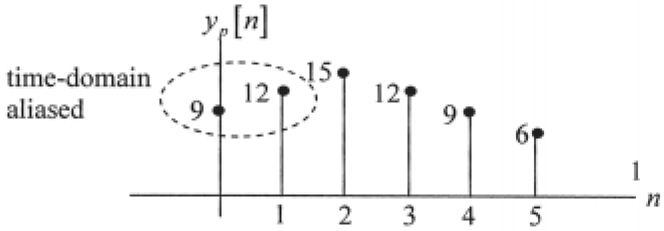
$$X[k]=6W_6^0+5W_6^k+4W_6^{2k}+3W_6^{3k}+2W_6^{4k}+W_6^{5k}$$
$$H[k]=1+W_6^k+W_6^{2k}$$

Then

$$Y_p[k] = 6 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$
$$+ 6W_6^k + 5W_6^{2k} + 4W_6^{3k} + 3W_6^{4k} + 2W_6^{5k} + W_6^{6k}$$
$$+ 6W_6^{2k} + 5W_6^{3k} + 4W_6^{4k} + 3W_6^{5k} + 2W_6^{6k} + W_6^{7k}$$
$$= 9 + 12W_6^k + 15W_6^{2k} + 12W_6^{3k} + 9W_6^{4k} + 6W_6^{5k},$$

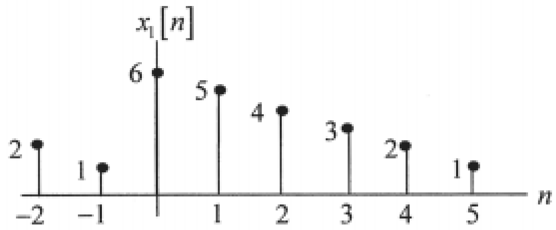
where we have used $W_6^{6k} = 1$ and $W_6^{7k} = W_6^k$. Now we have

$$y_p[n] = 9\delta[n] + 12\delta[n-1] + 15\delta[n-2] + 12\delta[n-3] + 9\delta[n-4] + 6\delta[n-5].$$



(d) To ensure that no time-domain aliasing occurs in the output, N should be large enough to accommodate the length of the linear convolution. That is, $N \geq 6 + 3 - 1 = 8$

(e)



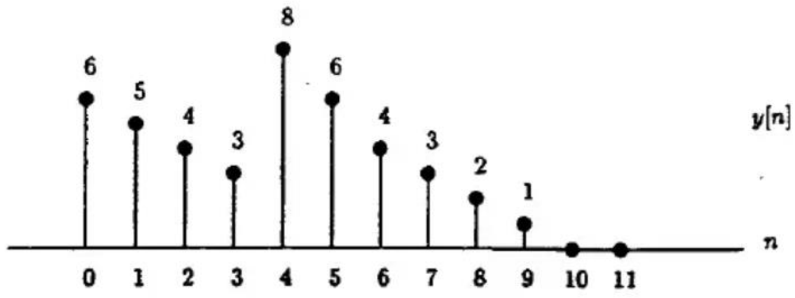
This new input when convolved with $h[n]$ will give the circular convolution found in (c)

We merely extend $x[n]$ as a periodic signal with period 6 samples.

(f) In general $x_1[n]$ is constructed by extending $x[n]$ periodically for $n = -M$ to $L - 1$

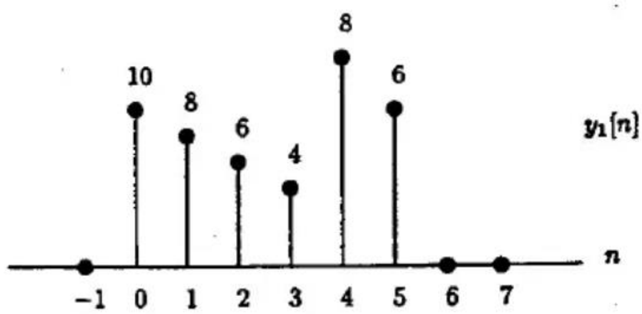
8.32

Circular convolution equals linear convolution plus aliasing. First, we find $y[n] = x_1[n] * x_2[n]$:



Note that $y[n]$ is a ten point sequence ($N = 6 + 5 - 1$).

(a) For $N = 6$, the last four non-zero point ($6 \le n \le 9$) will alias to the first four points, giving us $y_1[n] = x_1[n] \circledsix x_2[n]$



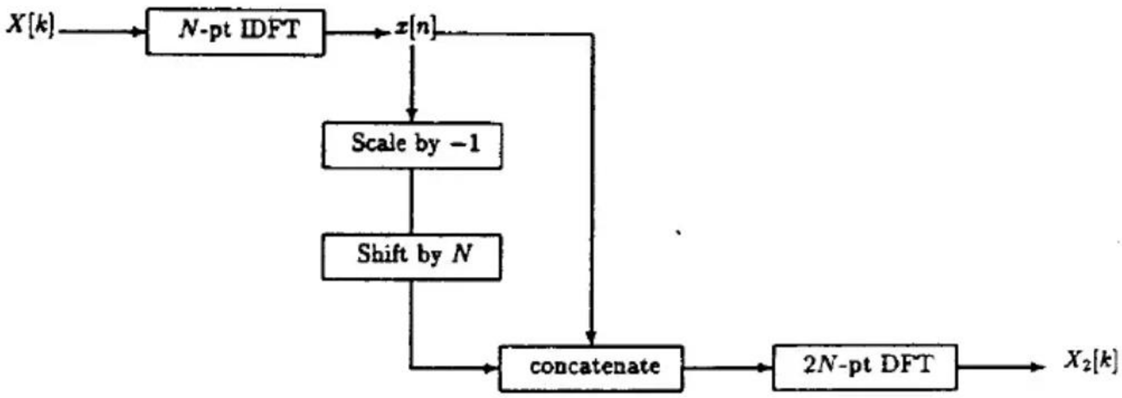
(b) For $N = 10, N \ge 6 + 5 - 1$, so no aliasing occurs, and circular convolution is identical to linear convolution.

8.35

(a) Since

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ -x[n-N], & N \le n \le 2N-1 \\ 0, & \text{otherwise} \end{cases}$$

If $X[k]$ is known, $x_2[n]$ can be constructed by :

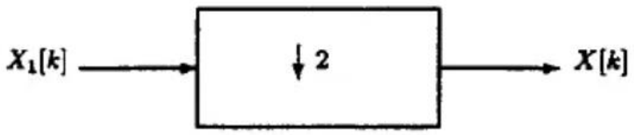


(b) To obtain $X[k]$ from $X_1[k]$, we might try to take the inverse DFT (2N-pt) of $X_1[k]$, then take the N-pt DFT of $x_1[n]$ to get $X[k]$.

However, the above approach is highly inefficient. A more reasonable approach may be achieved if we examine the DFT analysis equations involved. First,

$$\begin{aligned} X_1[k] &= \sum_{n=0}^{2N-1} x_1[n] W_{2N}^{kn}, \qquad 0 \le k \le (2N-1) \\ &= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{(k/2)n}, \quad 0 \le k \le (N-1) \\ X_1[k] &= X[k/2], \quad 0 \le k \le (N-1) \end{aligned}$$

Thus, an easier way to obtain $X[k]$ from $X_1[k]$ is simply to decimate $X_1[k]$ by two.



8.37

(a) Overlap Add:

If we divide the input into sections of length L, each section will have an output length:

$$L + 100 - 1 = L + 99.$$

Thus, the required length is,

$$L = 256 - 99 = 157.$$

If we had 63 sections, $63 \times 157 = 9891$, there will be a remainder of 109 points. Hence, we must pad the remaining data to 256 and use one more DFT computation.

Therefore, we require 64 DFTs and 64 IDFTs. Since $h[n]$ also requires a DFT, the total is:

$$\text{65 DFTs and 64 IDFTs.}$$

(b) Overlap save:

We require 99 zeros to be padded in front of the sequence. The first 99 points of the output of each section will be discarded. Thus the length after padding is 10099 points. The length of each section overlap is $256 - 99 = 157 = L$.

We require $65 \times 157 = 10205$ to get all 10099 points. Because $h[n]$ also requires a DFT:

$$\text{66 DFTs and 65 IDFTs.}$$

(c) Ignoring the transients at the beginning and end of the direct convolution, each output point requires 100 multiplications and 99 additions.

Overlap add:

$$\begin{aligned} \#\text{multiplications} &= 129(1024) &= 132096, \\ \#\text{additions} &= 129(2048) &= 264192. \end{aligned}$$

Overlap save:

$$\begin{aligned} \#\text{multiplications} &= 131(1024) &= 134144, \\ \#\text{additions} &= 131(2048) &= 268288. \end{aligned}$$

Direct convolution:

$$\begin{aligned} \#\text{multiplications} &= 100(100000) &= 1000,000, \\ \#\text{additions} &= 99(100000) &= 990,000. \end{aligned}$$

Note: The number of additions can be accurately estimated, but it does not causes much difference in the answers. Give yourself full points if your answer is within 5%.

Note: Give yourself full points even if you missed to notice that $h[n]$ should contribute only 1 to the number of FFT. But be careful about this mistake in the future.