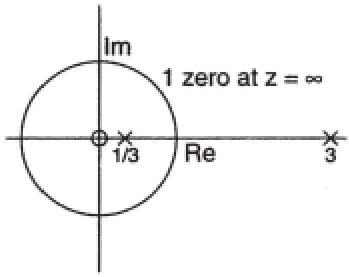
DSP HW4 sol

5.2. We have $y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$ or $z^{-1}Y(z) - \frac{10}{3}Y(z) + zY(z) = X(z)$. So,

$$\begin{aligned} H(z) &= \frac{1}{z^{-1} - \frac{10}{3} + z} \\ &= \frac{z}{(z-\frac{1}{3})(z-3)} \\ &= \frac{-\frac{1}{8}}{z-\frac{1}{3}} + \frac{\frac{9}{8}}{z-3} \end{aligned}$$

(a)



(b)

$$H(z) = \frac{-\frac{1}{8}z^{-1}}{1-\frac{1}{3}z^{-1}} + \frac{\frac{9}{8}z^{-1}}{1-3z^{-1}}$$

Stable ⇒ ROC is $\frac{1}{3} \le |z| \le 3$. Therefore,

$$h[n] = -\frac{1}{8}\left(\frac{1}{3}\right)^{n-1} u[n-1] - \frac{9}{8}(3)^{n-1}u[-n]$$

5.6. (a)

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^{n} u[n] - \frac{4}{3}(2)^n u[-n-1]$$

$$X(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \qquad \frac{1}{2} < |z| < 2$$

(b)

$$Y(z) = \frac{1-z^{-2}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

This has the same poles as the input, therefore the ROC is still $\frac{1}{2} < |z| < 2$.

(c)

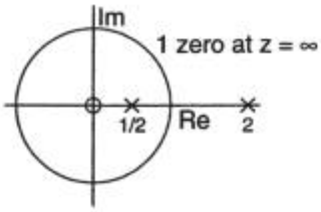
$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \Leftrightarrow h[n] = \delta[n] - \delta[n-2]$$

5.9.

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n]$$

$$z^{-1}Y(z) - \frac{5}{2}Y(z) + zY(z) = X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \\ &= \frac{z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= \frac{\frac{2}{3}}{1-2z^{-1}} - \frac{\frac{2}{3}}{1-\frac{1}{2}z^{-1}} \end{aligned}$$



Three regions of convergence:

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

Includes $|z| = 1$, so this is stable.

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}(2)^n u[n] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

ROC outside of largest pole, so this is causal.

5.11. (a) *It cannot be determined.* The ROC might or might not include the unit circle.

(b) *It cannot be determined.* The ROC might or might not include $z = \infty$.

(c) *False.* Given that the system is causal, we know that the ROC must be outside the outermost pole. Since the outermost pole is outside the unit circle, the ROC will not include the unit circle, and thus the system is not stable.

(d) *True.* If the system is stable, the ROC must include the unit circle. Because there are poles both inside and outside the unit circle, any ROC including the unit circle must be a ring. A ring-shaped ROC means that we have a two-sided system.

5.12. (a) *Yes.* The poles $z = \pm j(0.9)$ are inside the unit circle so the system is stable.

(b) First, factor $H(z)$ into two parts. The first should be minimum phase and therefore have all its poles and zeros inside the unit circle. The second part should contain the remaining poles and zeros.

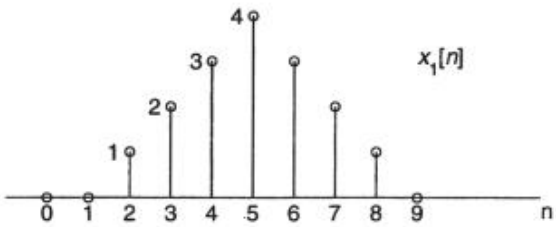
$$H(z) = \underbrace{\frac{1+0.2z^{-1}}{1+0.81z^{-2}}}_{\text{minimum phase}} \qquad \underbrace{\frac{1-9z^{-2}}{1}}_{\substack{\text{poles \& zeros} \\ \text{outside unit circle}}}$$

Allpass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include the factor $(1 - \frac{1}{9}z^{-2})$ in both parts of the equation above the first part will remain minimum phase and the second will become allpass.

$$\begin{aligned} H(z) &= \frac{(1+0.2z^{-1})(1-\frac{1}{9}z^{-2})}{1+0.81z^{-2}} \cdot \frac{1-9z^{-2}}{1-\frac{1}{9}z^{-2}} \\ &= H_1(z)H_{ap}(z) \end{aligned}$$

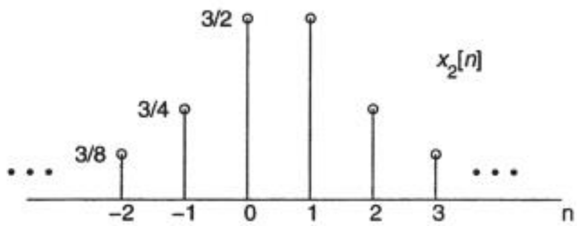
5.14. (a) By the symmetry of $x_1[n]$ we know it has linear phase. The symmetry is around $n = 5$ so the continuous phase of $X_1(e^{j\omega})$ is $\arg[X_1(e^{j\omega})] = -5\omega$. Thus,

$$\text{grd}[X_1(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[X_1(e^{j\omega})]\} = -\frac{d}{d\omega}\{-5\omega\} = 5$$



(b) By the symmetry of $x_2[n]$ we know it has linear phase. The symmetry is around $n = 1/2$ so we know the phase of $X_2(e^{j\omega})$ is $\arg[X_2(e^{j\omega})] = -\omega/2$. Thus,

$$\text{grd}[X_2(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[X_2(e^{j\omega})]\} = -\frac{d}{d\omega}\left\{-\frac{\omega}{2}\right\} = \frac{1}{2}$$



5.18. A minimum phase system with an equivalent magnitude spectrum can be found by analyzing the system function, and reflecting all poles are zeros that are outside the unit circle to their conjugate reciprocal locations. This will move them inside the unit circle. Then, all poles and zeros for $H_{min}(z)$ will be inside the unit circle. Note that a scale factor may be introduced when the pole or zero is reflected inside the unit circle.

(a) Simply reflect the zero at $z = 2$ to its conjugate reciprocal location at $z = \frac{1}{2}$. Then, determine the scale factor.

$$H_{min}(z) = 2\left(\frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{3}z^{-1}}\right)$$

(b) First, simply reflect the zero at $z = -3$ to its conjugate reciprocal location at $z = -\frac{1}{3}$. Then, determine the scale factor. This results in

$$H_{min}(z) = 3\frac{\left(1 + \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}{z^{-1}\left(1 + \frac{1}{3}z^{-1}\right)}$$

The $\left(1 + \frac{1}{3}z^{-1}\right)$ terms cancel, leaving

$$H_{min}(z) = 3\frac{\left(1 - \frac{1}{2}z^{-1}\right)}{z^{-1}}$$

Note that the term $\frac{1}{z^{-1}}$ does not affect the frequency response magnitude of the system. Consequently, it can be removed. Thus, the remaining term has a zero inside the unit circle, and is therefore minimum phase. As a result, we are left with the system

$$H_{min}(z) = 3\left(1 - \frac{1}{2}z^{-1}\right)$$

(c) Simply reflect the zero at 3 to its conjugate reciprocal location at $\frac{1}{3}$ and reflect the pole at $\frac{4}{3}$ to its conjugate reciprocal location at $\frac{3}{4}$. Then, determine the scale factor.

$$H_{min}(z) = \frac{9}{4}\frac{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 - \frac{3}{4}z^{-1}\right)^2}$$

5.19. Due to the symmetry of the impulse responses, all the systems have generalized linear phase of $\arg[H(e^{j\omega})] = \beta - n_o\omega$ where n_o is the point of symmetry in the impulse response graphs. The group delay is

$$\text{grd}\left[H_i(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\arg\left[H_i(e^{j\omega})\right]\right\} = -\frac{d}{d\omega}\left\{\beta - n_o\omega\right\} = n_o$$

To find each system's group delay we need only find the point of symmetry n_o in each system's impulse response.

$$\begin{aligned} \text{grd}\left[H_1(e^{j\omega})\right] &= 2 & \qquad \text{grd}\left[H_4(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_2(e^{j\omega})\right] &= 1.5 & \qquad \text{grd}\left[H_5(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_3(e^{j\omega})\right] &= 2 & \qquad \text{grd}\left[H_6(e^{j\omega})\right] &= 3.5 \end{aligned}$$

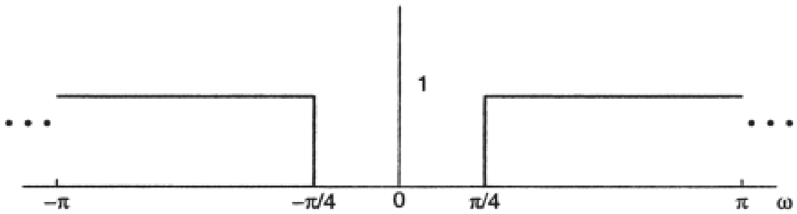
5.20. (a) *Yes.* The system function could be a generalized linear phase system implemented by a linear constant-coefficient differential equation (LCCDE) with real coefficients. The zeros come in a set of four: a zero, its conjugate, and the two conjugate reciprocals. The pole-zero plot could correspond to a Type I FIR linear phase system.

(b) *No.* This system function could not be a generalized linear phase system implemented by a LCCDE with real coefficients. Since the LCCDE has real coefficients, its poles and zeros must come in conjugate pairs. However, the zeros in this pole-zero plot do not have corresponding conjugate zeros.

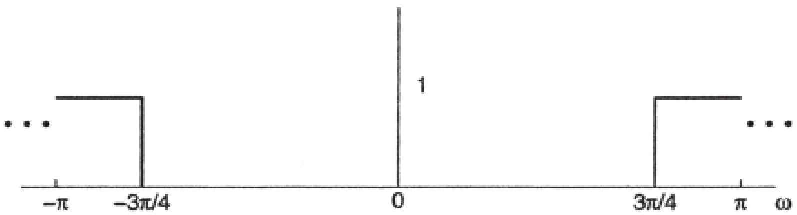
(c) *Yes.* The system function could be a generalized linear phase system implemented by a LCCDE with real coefficients. The pole-zero plot could correspond to a Type II FIR linear phase system.

5.21. $h_{lp}[n]$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$

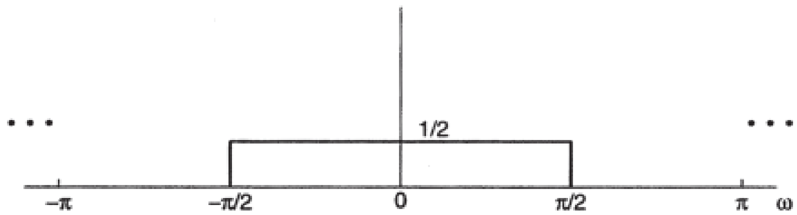
(a) $y[n] = x[n] - x[n] * h_{lp}[n] \Rightarrow H(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$
This is a highpass filter.



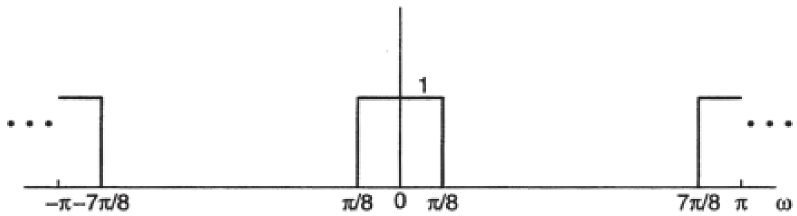
(b) $x[n]$ is first modulated by π, lowpass filtered, and demodulated by π. Therefore, $H_{lp}(e^{j\omega})$ filters the high frequency components of $X(e^{j\omega})$.
This is a highpass filter.



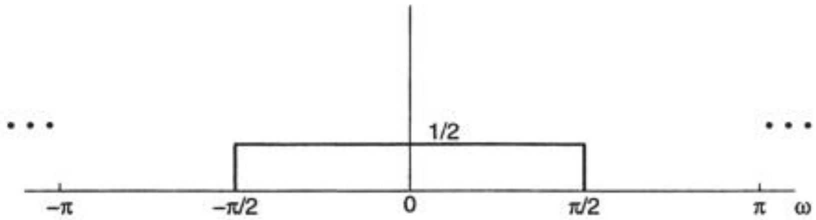
(c) $h_{lp}[2n]$ is a downsampled version of the filter. Therefore, the frequency response will be "spread out" by a factor of two, with a gain of $\frac{1}{2}$.
This is a lowpass filter.



(d) This system upsamples $h_{lp}[n]$ by a factor of two. Therefore, the frequency axis will be compressed by a factor of two.
This is a bandstop filter.



(e) This system upsamples the input before passing it through $h_{lp}[n]$. This effectively doubles the frequency bandwidth of $H_{lp}(e^{j\omega})$.
This is a lowpass filter.



5.22.

(i) Real-valued impulse response:

Poles that aren't real must be in complex conjugate pairs. Zeros that aren't real must be in complex conjugate pairs.

(ii) Finite impulse response:

All poles are at the origin. The ROC is the entire z-plane, except possibly $z = 0$.

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

Causality combined with the given symmetry property implies a finite-length $h[n]$ that can only be nonzero between time zero and time 2α. Thus we must have all poles at the origin and at most 2α zeros. The z transform of $h[2\alpha - n]$ is $z^{-2\alpha}H(1/z)$, so any zero of $H(z)$ at $c \neq 0$ must be paired with a zero at $1/c$.

(iv) Minimum phase:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

(v) All-pass:

Each pole is paired with a zero at the conjugate reciprocal location.

5.23.

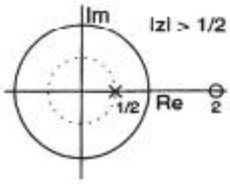
$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = \frac{Y(z)}{X(z)}, \quad \text{causal, so ROC is } |z| > a$$

(a) Cross multiplying and taking the inverse transform

$$y[n] - ay[n-1] = x[n] - \frac{1}{a}x[n-1]$$

(b) Since $H(z)$ is causal, we know that the ROC is $|z| > a$. For stability, the ROC must include the unit circle. So, $H(z)$ is stable for $|a| < 1$.

(c) $a = \frac{1}{2}$



(d)

$$H(z) = \frac{1}{1 - az^{-1}} - \frac{a^{-1}z^{-1}}{1 - az^{-1}}, \quad |z| > a$$

$$h[n] = (a)^n u[n] - \frac{1}{a}(a)^{n-1}u[n-1]$$

(e)

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1 - a^{-1}e^{-j\omega}}{1 - ae^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{1 - a^{-1}e^{-j\omega}}{1 - ae^{-j\omega}} \cdot \frac{1 - a^{-1}e^{j\omega}}{1 - ae^{j\omega}}$$

$$\begin{aligned} |H(e^{j\omega})| &= \left(\frac{1 + \frac{1}{a^2} - \frac{2}{a}\cos\omega}{1 + a^2 - 2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a}\left(\frac{a^2 + 1 - 2a\cos\omega}{1 + a^2 - 2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a} \end{aligned}$$

5.29

A.

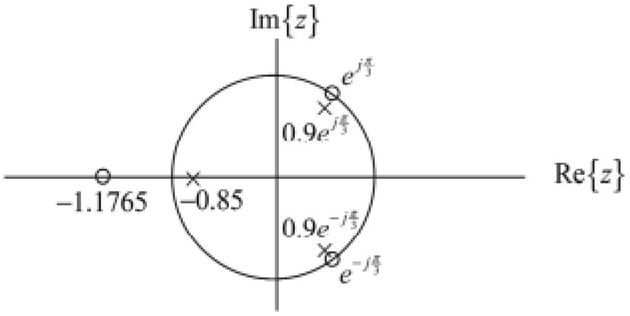
$$H(z) = \frac{\left(1 - e^{j\frac{\pi}{3}} z^{-1}\right)\left(1 - e^{-j\frac{\pi}{3}} z^{-1}\right)\left(1 + 1.1765 z^{-1}\right)}{\left(1 - 0.9 e^{j\frac{\pi}{3}} z^{-1}\right)\left(1 - 0.9 e^{-j\frac{\pi}{3}} z^{-1}\right)\left(1 + 0.85 z^{-1}\right)}$$

$$= \frac{1 + 0.1765 z^{-1} - 0.1765 z^{-2} + 1.1765 z^{-3}}{1 - 0.05 z^{-1} + 0.045 z^{-2} + 0.6885 z^{-3}}$$

$$= \frac{Y(z)}{X(z)}.$$

$$y[n] = 0.05 y[n-1] - 0.45 y[n-2] - 0.6885 y[n-3]$$
$$+ x[n] + 0.1765 x[n-1] - 0.1765 x[n-2] + 1.1765 x[n-3].$$

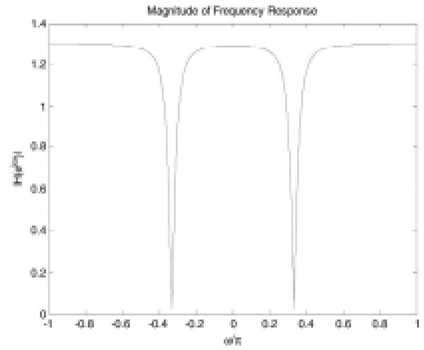
B.



Since the system is causal, the ROC is the region outside the outermost pole.

$|z| > 0.9$.

C.



The zeros on the unit circle null the frequency response at $\omega = \pm\pi/3$. The sharpness of the nulls depend on how close the nearby poles are to the zeros. The factor $\frac{1+1.1765z^{-1}}{1+0.85z^{-1}} = 1.1765\frac{z^{-1}+0.85}{1+0.85z^{-1}}$ is allpass and does not affect the magnitude response.

D. 1. True. The system is stable because the ROC contains the unit circle.

2. False. The impulse response must approach zero for large n because the system is stable.

3. False. The system function has a zero on the unit circle at $\omega = \pi/3$. This negates the effect of the pole, and since the pole is not on the unit circle, the pole does not cancel the zero. Instead, the sharpness of the notch depends on how close the pole is to the zero.

4. False. There is a zero outside the unit circle.

5. False. The system is not a minimum-phase system so it does not have a causal and stable inverse.

5.32. (a)

$$H(z) = \frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}, \quad \text{stable, so the ROC is } \tfrac{1}{2} < |z| < 3$$

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$Y(z) = X(z)H(z) = \frac{\frac{4}{5}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{5}}{1-3z^{-1}} - \frac{1}{1-z^{-1}}, \quad 1 < |z| < 3$$

$$y[n] = \frac{4}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{5}(3)^n u[-n-1] - u[n]$$

(b) ROC includes $z = \infty$ so $h[n]$ is causal. Since both $h[n]$ and $x[n]$ are 0 for $n < 0$, we know that $y[n]$ is also 0 for $n < 0$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}$$

$$Y(z) - \frac{7}{2}z^{-1}Y(z) + \frac{3}{2}z^{-2}Y(z) = z^{-2}X(z)$$

$$y[n] = x[n-2] + \frac{7}{2}y[n-1] - \frac{3}{2}y[n-2]$$

Since $y[n] = 0$ for $n < 0$, recursion can be done:

$$y[0] = 0, \quad y[1] = 0, \quad y[2] = 1$$

(c)

$$H_i(z) = \frac{1}{H(z)} = z^2 - \frac{7}{2}z + \frac{3}{2}, \quad \text{ROC: entire z-plane}$$

$$h_i[n] = \delta[n+2] - \frac{7}{2}\delta[n+1] + \frac{3}{2}\delta[n]$$

5.37.

Convolving two symmetric sequences yields another symmetric sequence. A symmetric sequence convolved with an antisymmetric sequence gives an antisymmetric sequence. If you convolve two antisymmetric sequences, you will get a symmetric sequence.

$$A:\ h_1[n] * h_2[n] * h_3[n] = (h_1[n] * h_2[n]) * h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, $(-1 \le n \le 7)$
$(h_1[n] * h_2[n]) * h_3[n]$ is antisymmetric about $n = 3$, $(-3 \le n \le 9)$

Thus, system A has generalized linear phase

$$B:\ (h_1[n] * h_2[n]) + h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, as we noted above. Adding $h_3[n]$ to this sequence will destroy all symmtery, so this does not have generalized linear phase.

5.40.

(a) To find the poles and zeros of $H(z)$, rewrite it as

$$H(z) = \frac{(z^2-9)\left(z+\frac{1}{3}\right)}{z^2\left(z-\frac{1}{3}\right)}$$

Zeros: $z = 3, -3, -1/3$

Poles: $z = 0, 0, 1/3$

The zeros at 3 and -3 can't be in the minimum phase system, so they must go in the all-pass system. In order to make the latter all-pass, it must also have poles at $1/3$ and $-1/3$. Since these were not part of $H(z)$, they must be cancelled by zeros in the minimum phase system. Inserting the zero at $1/3$ in the minimum phase system cancels the pole that was there.

$$\begin{aligned} H_{min}(z) &= \frac{1}{K}\left(1+\frac{1}{3}z^{-1}\right)^2 \\ H_A(z) &= K\frac{\left(1-3z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} \end{aligned}$$

The product of these two functions is the original $H(z)$ given in the problem. Since we want the all-pass system to have unity gain, $|H_A(z)| = 1$ for any z on the unit circle, e.g. $z = 1$. This yields $|K| = 1/9$.

Decompositions into minimum phase and all-pass systems are unique up to a scale factor.

(b) Yes, $H_{min}(z)$ is FIR. All its poles are at the origin.

(c) The phase of $H_{min}\left(e^{j\omega}\right)$ is

$$-\arctan\left(\frac{-\frac{2}{3}\sin(\omega)-\frac{1}{9}\sin(2\omega)}{1+\frac{2}{3}\cos(\omega)+\frac{1}{9}\cos(2\omega)}\right)$$

This is not a linear or affine function of ω. However, we can rewrite $H(z)$ as the product of the following two systems:

$$\begin{aligned} H_{lp}(z) &= \left(1+3z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right) \\ H_{A2}(z) &= \frac{1-3z^{-1}}{1-\frac{1}{3}z^{-1}} \end{aligned}$$

This is equivalent to $H_{lp}(z) = 1 + (10/3)z^{-1} + z^{-2}$. The impulse response has even symmetry and the system is linear phase.

5.41

	$H_a(z)$	$H_b(z)$	$H_c(z)$
stable	X	O	O
IIR	O	X	O
FIR	X	O	X
min. phase	X	X	X
all-pass	X	X	O
generalized linear phase	X	O	X
positive group delay	X	O	O