Chapter 7 Solution

7.1

7.1. Using the partial fraction technique, we see

$$H_c(s) = \frac{s+a}{(s+a)^2+b^2} = \frac{0.5}{s+a+jb} + \frac{0.5}{s+a-jb}$$

Now we can use the Laplace transform pair

$$e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{s+\alpha}$$

to get

$$h_c(t) = \frac{1}{2}\left(e^{-(a+jb)t} + e^{-(a-jb)t}\right)u(t).$$

(a) Therefore,

$$\begin{aligned} h_1[n] &= h_c(nT) = \frac{1}{2}\left[e^{-(a+jb)nT} + e^{-(a-jb)nT}\right]u[n] \\ H_1(z) &= \frac{0.5}{1-e^{-(a+jb)T}z^{-1}} + \frac{0.5}{1-e^{-(a-jb)T}z^{-1}}, \quad |z| > e^{-aT} \end{aligned}$$

(b) Since

$$s_c(t) = \int_{-\infty}^{t} h_c(\tau)d\tau \longleftrightarrow \frac{H_c(s)}{s} = S_c(s)$$

we get

$$S_c(s) = \frac{s+a}{s(s+a+jb)(s+a-jb)} = \frac{A_1}{s} + \frac{A_2}{s+a+jb} + \frac{A_2^*}{s+a-jb}$$

where

$$A_1 = \frac{a}{a^2+b^2}, \qquad A_2 = -\frac{0.5}{a+jb}$$

Though the system $h_2[n]$ is related by step invariance to $h_c(t)$, the signal $s_2[n]$ is related to $s_c(t)$ by impulse invariance. Therefore, we know the poles of the partial fraction expansion of $S_c(s)$ above must transform as $z_k = e^{s_k T}$, and we can find

$$S_2(z) = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-e^{-(a+jb)T}z^{-1}} + \frac{A_2^*}{1-e^{-(a-jb)T}z^{-1}}$$

Now, since the relationship between the step response and the impulse response is

$$s_2[n] = \sum_{k=-\infty}^{n} h_2[k] = \sum_{k=-\infty}^{\infty} h_2[k]u[n-k] = h_2[n] * u[n]$$

$$S_2(z) = \frac{H_2(z)}{1-z^{-1}}$$

We can finally solve for $H_2(z)$

$$\begin{aligned} H_2(z) &= S_2(z)(1-z^{-1}) \\ &= A_1 + A_2\frac{1-z^{-1}}{1-e^{-(a+jb)T}z^{-1}} + A_2^*\frac{1-z^{-1}}{1-e^{-(a-jb)T}z^{-1}}, \quad |z| > e^{-aT} \end{aligned}$$

where A_1 and A_2 are as given above.

(c)

$$\begin{aligned} s_1[n] &= \sum_{k=-\infty}^{n} h_1[k] = \frac{1}{2}\sum_{k=0}^{n}\left(e^{-(a+jb)kT} + e^{-(a-jb)kT}\right) \\ &= \frac{1}{2}\left[\frac{1-e^{-(a+jb)(n+1)T}}{1-e^{-(a+jb)T}} + \frac{1-e^{-(a-jb)(n+1)T}}{1-e^{-(a-jb)T}}\right]u[n] \\ &= \left[B_1 + B_2 e^{-(a+jb)Tn} + B_2^* e^{-(a-jb)Tn}\right]u[n] \end{aligned}$$

where

$$B_1 = \frac{1-e^{-aT}\cos bT}{1-2e^{-aT}\cos bT + e^{-2aT}}, \qquad B_2 = -\frac{e^{-(a+jb)T}}{1-e^{-(a+jb)T}}$$

¿From this we can see that

$$\begin{aligned} S_1(z) &= \frac{B_1}{1-z^{-1}} + \frac{B_2}{1-e^{-(a+jb)T}z^{-1}} + \frac{B_2^*}{1-e^{-(a-jb)T}z^{-1}} \\ &\neq S_2(z) \end{aligned}$$

since the partial fraction constants are different. Therefore, $s_1[n] \neq s_2[n]$, the two step responses are not equal.

Taking the inverse z-transform of $H_2(z)$

$$\begin{aligned} h_2[n] &= A_1\delta[n] + A_2\left[e^{-(a+jb)Tn}u[n] - e^{-(a+jb)T(n-1)}u[n-1]\right] \\ &\quad + A_2^*\left[e^{-(a-jb)Tn}u[n] - e^{-(a-jb)T(n-1)}u[n-1]\right] \end{aligned}$$

where A_1 and A_2 are as defined earlier. By comparing $h_1[n]$ and $h_2[n]$ one sees that $h_1[n] \neq h_2[n]$.

The overall idea this problem illustrates is that a filter designed with impulse invariance is different from a filter designed with step invariance.

7.4

7.4. (a) In the impulse invariance design, the poles transform as $z_k = e^{s_k T_d}$ and we have the relationship

$$\frac{1}{s+a} \longleftrightarrow \frac{T_d}{1-e^{-aT_d}z^{-1}}$$

Therefore,

$$\begin{aligned} H_c(s) &= \frac{2/T_d}{s+0.1} - \frac{1/T_d}{s+0.2} \\ &= \frac{1}{s+0.1} - \frac{0.5}{s+0.2} \end{aligned}$$

The above solution is not unique due to the periodicity of $z = e^{j\omega}$. A more general answer is

$$H_c(s) = \frac{2/T_d}{s+\left(0.1+j\frac{2\pi k}{T_d}\right)} - \frac{1/T_d}{s+\left(0.2+j\frac{2\pi l}{T_d}\right)}$$

where k and l are integers.

(b) Using the inverse relationship for the bilinear transform,

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$$

we get

$$\begin{aligned} H_c(s) &= \frac{2}{1-e^{-0.2}\left(\frac{1-s}{1+s}\right)} - \frac{1}{1-e^{-0.4}\left(\frac{1-s}{1+s}\right)} \\ &= \frac{2(s+1)}{s(1+e^{-0.2})+(1-e^{-0.2})} - \frac{(s+1)}{s(1+e^{-0.4})+(1-e^{-0.4})} \\ &= \left(\frac{2}{1+e^{-0.2}}\right)\left(\frac{s+1}{s+\frac{1-e^{-0.2}}{1+e^{-0.2}}}\right) - \left(\frac{1}{1+e^{-0.4}}\right)\left(\frac{s+1}{s+\frac{1-e^{-0.4}}{1+e^{-0.4}}}\right) \end{aligned}$$

Since the bilinear transform does not introduce any ambiguity, the representation is unique.

7.15

7.15. This filter requires a maximal passband error of $\delta_p = 0.05$, and a maximal stopband error of $\delta_s = 0.1$. Converting these values to dB gives

$$\delta p = -26 \text{ dB}$$
$$\delta s = -20 \text{ dB}$$

This requires a window with a peak approximation error less than -26 dB. Looking in Table 7.1, the Hanning, Hamming, and Blackman windows meet this criterion.

Next, the minimum length L required for each of these filters can be found using the "approximate width of mainlobe" column in the table since the mainlobe width is about equal to the transition width. Note that the actual length of the filter is $L = M + 1$.

Hanning:

$$\begin{aligned} 0.1\pi &= \frac{8\pi}{M} \\ M &= 80 \end{aligned}$$

Hamming:

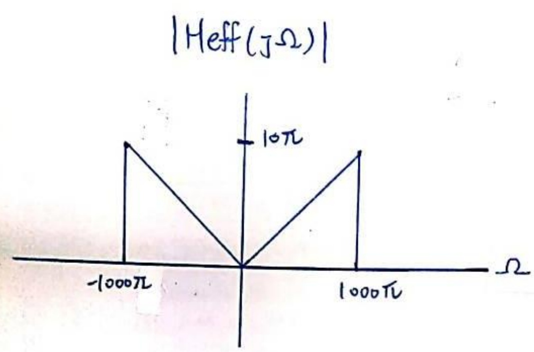
$$\begin{aligned} 0.1\pi &= \frac{8\pi}{M} \\ M &= 80 \end{aligned}$$

Blackman:

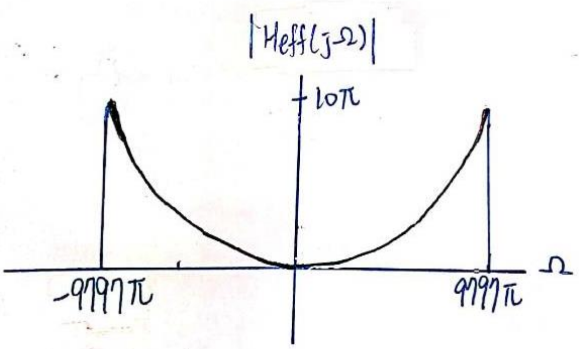
$$\begin{aligned} 0.1\pi &= \frac{12\pi}{M} \\ M &= 120 \end{aligned}$$

7.24

(a)



(b)



7.25

7.25. (a) By using Parseval's theorem,

$$\begin{aligned}\epsilon^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega \\ &= \sum_{n=-\infty}^{\infty} |e[n]|^2\end{aligned}$$

where

$$e[n] = \begin{cases} h_d[n], & n < 0, \\ h_d[n] - h[n], & 0 \le n \le M, \\ h_d[n], & n > M \end{cases}$$

(b) Since we only have control over $e[n]$ for $0 \le m \le M$, we get that ϵ^2 is minimized if $h[n] = h_d[n]$ for $0 \le n \le M$.

(c)

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

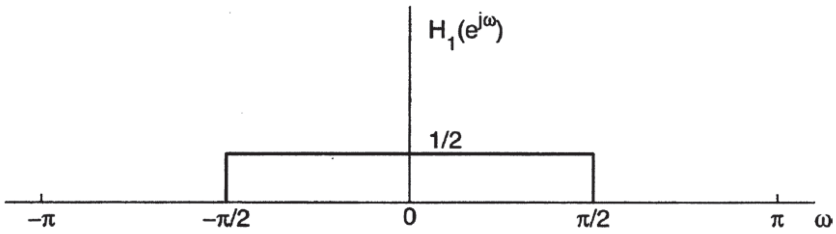
which is a rectangular window.

7.28

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \le \pi \end{cases}$$

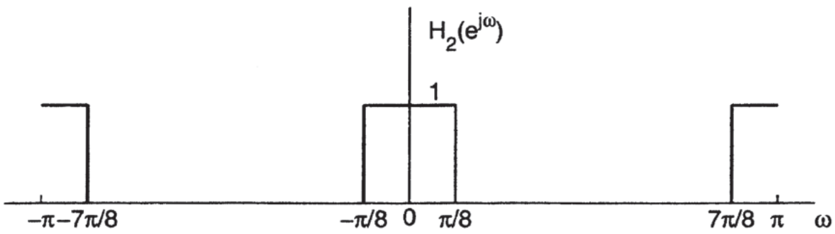
(a)

$$\begin{aligned} h_1[n] &= h[2n] \\ H_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[2n]e^{j\omega n} \\ &= \sum_{n \text{ even}} h[n]e^{\frac{j\omega n}{2}} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[h[n] + (-1)^n h[n]\right] e^{j\frac{\omega n}{2}} \\ &= \frac{1}{2}H(e^{j\frac{\omega}{2}}) + \frac{1}{2}H\left(e^{j\frac{\omega+2\pi}{2}}\right) \end{aligned}$$



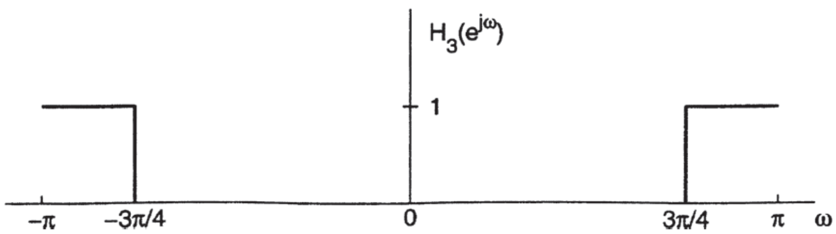
(b)

$$\begin{aligned} H_2(e^{j\omega}) &= \sum_{n \text{ even}} h[n/2]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega 2n} \\ &= H\left(e^{j2\omega}\right) \end{aligned}$$



(c)

$$H_3(e^{j\omega}) = H\left(e^{j(\omega+\pi)}\right)$$



7.35

(a) From the figure, $H(e^{j\omega})$ exhibits eight alternations of the error on the interval $0 \le \omega \le \pi$ as an approximation to an ideal lowpass filter with the given parameters. Because a lowpass filter designed with the Parks-McClellan algorithm has either $L+2$ or $L+3$ alternations and because we are told that there is another filter out there that meets the specs with $N_2 > N_1$, we should consider the $L+3$ case to find the smaller value of L.

With $L+3=8$ alternations, $L=L_1=5$.

(b) Since there are 8 alternations, L can be no greater than 6. Since the only other possible value of L for a lowpass filter was found in (a), we have $L_2=6$ as the only possible value.

(c) Yes. Since both filters have identical frequency responses, they must have identical impulse responses.

(d) While the alternation theorem states that *for a given* r there is a unique r th degree polynomial that satisfies it, the theorem makes no claim about how this polynomial may or may not relate to a polynomial satisfying the alternation theorem for a different value of r.

It turns out, in this case, that the single 5^{th} degree polynomial satisfying the alternation theorem for $r_1 = L_1 = 5$ is identical to the single 6^{th} degree polynomial satisfying the alternation theorem for $r_2 = L_2 = 6$.

7.44

(a) A Type-I lowpass filter that is optimal in the Parks-McClellan can have either $L+2$ or $L+3$ alternations. The second case is true only when an alternation occurs at all band edges. Since this filter does not have an alternation at $\omega = 0$ it only has $L+2$ alternations. From the figure we see there are 9 alternations so $L = 7$. Thus, $M = 2L = 2(7) = 14$.

(b) We have

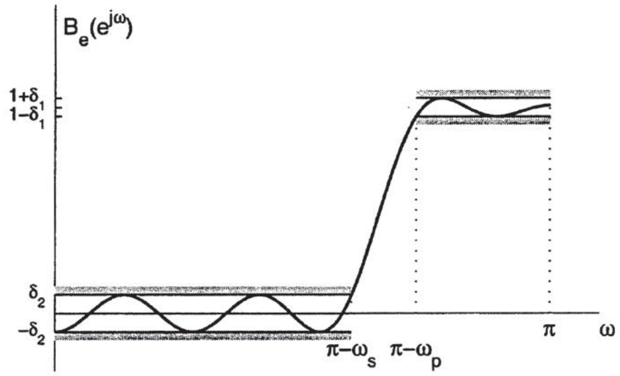
$$\begin{aligned} h_{HP}[n] &= -e^{j\pi n} h_{LP}[n] \\ H_{HP}(e^{j\omega}) &= -H_{LP}(e^{j(\omega-\pi)}) \\ &= -A_e(e^{j(\omega-\pi)})e^{-j(\omega-\pi)\frac{M}{2}} \\ &= A_e(e^{j(\omega-\pi)})e^{-j\omega\frac{M}{2}} \\ &= B_e(e^{j\omega})e^{-j\omega\frac{M}{2}} \end{aligned}$$

where

$$B_e(e^{j\omega}) = A_e(e^{j(\omega-\pi)})$$

The fact that $M = 14$ is used to simplify the exponential term in the third line above.

(c)



(d) *The assertion is correct.* The original amplitude function was optimal in the Parks-McClellan sense. The method used to create the new filter did not change the filter length, transition width, or relative ripple sizes. All it did was slide the frequency response along the frequency axis creating a new error function $E'(\omega) = E(\omega - \pi)$. Since translation does not change the Chebyshev error ($\max|E(\omega)|$) the new filter is still optimal.