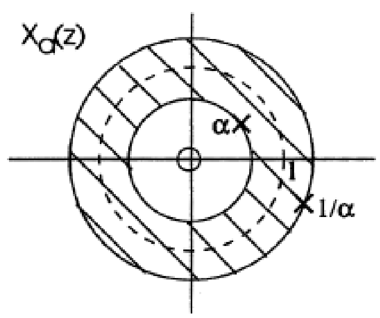
Chapter 3 Solution

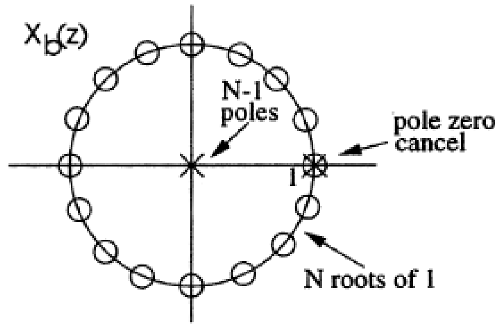
3.3

(a)

$$x_a[n] = \alpha^{|n|} \qquad 0 < |\alpha| < 1$$

$$\begin{aligned} X_a(z) &= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \qquad |\alpha| < |z| < \frac{1}{|\alpha|} \end{aligned}$$





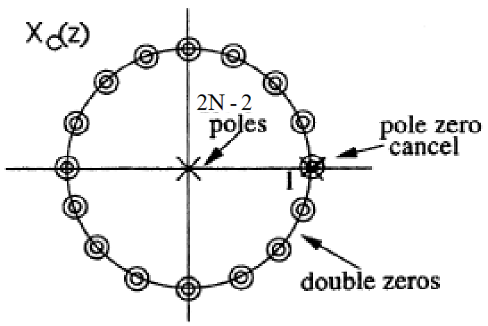
(b)

$$x_b = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & N \le n \\ 0, & n < 0 \end{cases} \Rightarrow X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)} \quad z \ne 0$$

(c)

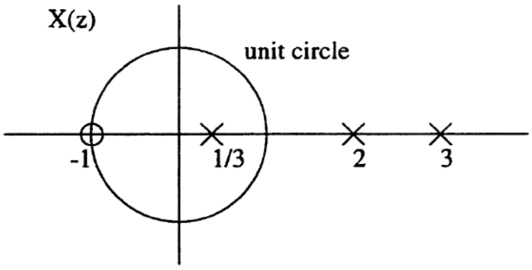
$$x_c[n] = x_b[n] * x_b[n] \Leftrightarrow X_c(z) = X_b(z) \cdot X_b(z)$$

$$X_c(z) = \left(\frac{z^N - 1}{z^{N-1}(z-1)}\right)^2 = \frac{1}{z^{2N-2}} \left(\frac{z^N - 1}{z-1}\right)^2 \qquad z \ne 0$$



3.4

The pole-zero plot of $X(z)$ appears below.



(a) For the Fourier transform of $x[n]$ to exist, the z-transform of $x[n]$ must have an ROC which includes the unit circle, therefore, $|\frac{1}{3}| < |z| < |2|$.
Since this ROC lies outside $\frac{1}{3}$, this pole contributes a right-sided sequence. Since the ROC lies inside 2 and 3, these poles contribute left-sided sequences. The overall $x[n]$ is therefore two-sided.

(b) Two-sided sequences have ROC's which look like washers. There are two possibilities. The ROC's corresponding to these are: $|\frac{1}{3}| < |z| < |2|$ and $|2| < |z| < |3|$.

(c) The ROC must be a connected region. For stability, the ROC must contain the unit circle. For causality the ROC must be outside the outermost pole. These conditions cannot be met by any of the possible ROC's of this pole-zero plot.

3.6

3.6. (a)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

Partial fractions: one pole $\rightarrow$ inspection, $x[n] = (-\frac{1}{2})^n u[n]$

Long division:

$$\begin{array}{r|llll} & 1 & -\frac{1}{2}z^{-1} & +\frac{1}{4}z^{-2} & + \cdots \\ \hline 1+\frac{1}{2}z^{-1} & 1 & & & \\ & 1 & +\frac{1}{2}z^{-1} & & \\ \hline & & -\frac{1}{2}z^{-1} & & \\ & & -\frac{1}{2}z^{-1} & -\frac{1}{4}z^{-2} & \\ \hline & & & +\frac{1}{4}z^{-2} & \\ & & & +\frac{1}{4}z^{-2} & +\frac{1}{8}z^{-3} \\ \hline \end{array}$$

$$\Longrightarrow x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

(b)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z| < \frac{1}{2}$$

Partial Fractions: one pole $\rightarrow$ inspection, $x[n] = -(-\frac{1}{2})^n u[-n-1]$

Long division:

$$\begin{array}{r|llll}
 & 2z & -\,4z^2 & +\,8z^3 & +\,\ldots \\
\hline
\frac{1}{2}z^{-1}+1 & 1 & & & \\
 & 1 & +\,2z & & \\
\hline
 & & -\,2z & & \\
 & & -\,2z & -\,4z^2 & \\
\hline
 & & & +\,4z^2 & \\
 & & & +\,4z^2 & +\,8z^3 \\
\hline
\end{array}$$

$$\Longrightarrow x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1]$$

(c)

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned}
X(z) &= \frac{-3}{1+\frac{1}{4}z^{-1}} + \frac{4}{1+\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\
x[n] &= \left[-3\left(-\frac{1}{4}\right)^n + 4\left(-\frac{1}{2}\right)^n\right] u[n]
\end{aligned}$$

Long division:

$$\begin{array}{r|llll}
 & 1 & +\,(-\frac{3}{4}-\frac{1}{2})z^{-1} & +\,(-\frac{3}{16}+1)z^{-2} & +\,\ldots \\
\hline
1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2} & 1 & -\,\frac{1}{2}z^{-1} & & \\
 & 1 & +\,\frac{3}{4}z^{-1} & +\,\frac{1}{8}z^{-2} & \\
\hline
 & & (-\frac{3}{4}-\frac{1}{2})z^{-1} & -\,\frac{1}{8}z^{-2} & \\
 & & (-\frac{3}{4}-\frac{1}{2})z^{-1} & +\,\frac{3}{4}(-\frac{3}{4}-\frac{1}{2})z^{-2} & +\,\frac{1}{8}(-\frac{3}{4}-\frac{1}{2})z^{-3} \\
\hline
 & & & [-\frac{1}{8}+\frac{3}{4}(\frac{3}{4}+\frac{1}{2})]z^{-2} & +\,\frac{1}{8}(\frac{3}{4}+\frac{1}{2})z^{-3}
\end{array}$$

$$\Longrightarrow x[n] = \left[-3(\left(-\frac{1}{4}\right)^n + \left(-\frac{1}{2}\right)^{n-2}\right] u[n]$$

(d)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned} X(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{1}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\ x[n] &= \left(-\frac{1}{2}\right)^n u[n] \end{aligned}$$

Long division: see part (i) above.

(e)

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a} \quad |z| > |a^{-1}|$$

Partial Fractions:

$$X(z) = -a - \frac{a^{-1}(1 - a^2)}{1 - a^{-1}z^{-1}} \qquad |z| > |a^{-1}|$$

$$x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

Long division:

$$\begin{array}{r|llll} & -\frac{1}{a} & -\left(\frac{a^{-1}-a}{a}\right)z^{-1} & -\left(\frac{a^{-1}-a}{a^2}\right)z^{-2} & + \ \ldots \\ \hline -a + z^{-1} & 1 & -\ az^{-1} & & \\ & 1 & -\ az^{-1} & & \\ \hline & & (a^{-1} - a)z^{-1} & \ldots & \end{array}$$

$$\Longrightarrow \qquad x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

3.9

3.9.

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{2}{(1-\frac{1}{2}z^{-1})} - \frac{1}{(1+\frac{1}{4}z^{-1})}$$

(a) $h[n]$ causal $\Rightarrow$ ROC outside $|z| = \frac{1}{2} \Rightarrow \; |z| > \frac{1}{2}$.

(b) ROC includes $|z| = 1 \;\Rightarrow$ stable.

(c)

$$\begin{aligned}
y[n] &= -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1] \\
Y(z) &= \frac{-\frac{1}{3}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} \\
&= \frac{1+z^{-1}}{(1+\frac{1}{4}z^{-1})(1-2z^{-1})} \qquad \frac{1}{4} < |z| < 2 \\
X(z) &= \frac{Y(z)}{H(z)} = \frac{(1-\frac{1}{2}z^{-1})}{(1-2z^{-1})} \qquad |z| < 2 \\
x[n] &= -(2)^n u[-n-1] + \frac{1}{2}(2)^{n-1}u[-n]
\end{aligned}$$

(d)

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

3.12

3.12. (a)

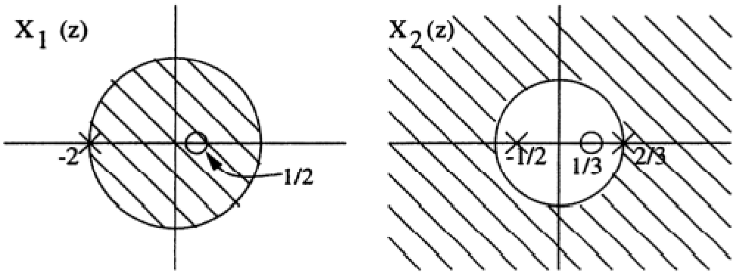
$$X_1(z) = \frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}}$$

The pole is at -2, and the zero is at 1/2.

(b)

$$X_2(z) = \frac{1-\frac{1}{3}z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})}$$

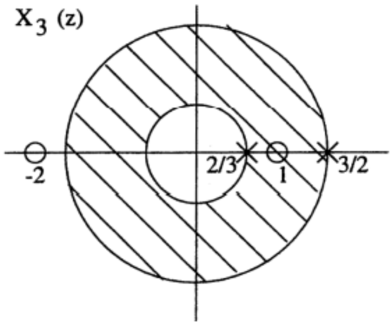
The poles are at -1/2 and 2/3, and the zero is at 1/3. Since $x_2[n]$ is causal, the ROC is extends from the outermost pole: $|z| > 2/3$.



(c)

$$X_3(z) = \frac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6}z^{-1} + z^{-2}}$$

The poles are at 3/2 and 2/3, and the zeros are at 1 and -2. Since $x_3[n]$ is absolutely summable, the ROC must include the unit circle: $2/3 < |z| < 3/2$.



3.16

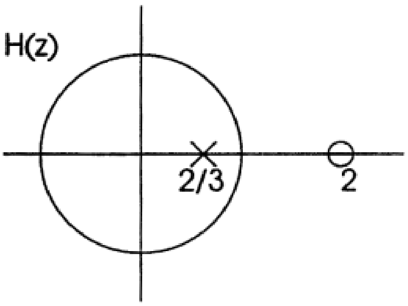
3.16. (a) To determine $H(z)$, we first find $X(z)$ and $Y(z)$:

$$\begin{aligned} X(z) &= \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - 2z^{-1}} \\ &= \frac{-\frac{5}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}, \qquad \frac{1}{3} < |z| < 2 \\ Y(z) &= \frac{5}{1 - \frac{1}{3}z^{-1}} - \frac{5}{1 - \frac{2}{3}z^{-1}} \\ &= \frac{-\frac{5}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - \frac{2}{3}z^{-1})}, \qquad |z| > \frac{2}{3} \end{aligned}$$

Now

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1 - 2z^{-1}}{1 - \frac{2}{3}z^{-1}} \qquad |z| > \frac{2}{3} \end{aligned}$$

The pole-zero plot of $H(z)$ is plotted below.



(b) Taking the inverse z-transform of $H(z)$, we get

$$\begin{aligned} h[n] &= (\frac{2}{3})^n u[n] - 2(\frac{2}{3})^{n-1} u[n-1] \\ &= (\frac{2}{3})^n (u[n] - 3u[n-1]) \end{aligned}$$

(c) Since

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - \frac{2}{3}z^{-1}},$$

we can write

$$Y(z)(1 - \frac{2}{3}z^{-1}) = X(z)(1 - 2z^{-1}),$$

whose inverse z-transform leads to

$$y[n] - \frac{2}{3}y[n-1] = x[n] - 2x[n-1]$$

(d) The system is stable because the ROC includes the unit circle. It is also causal since the impulse response $h[n] = 0$ for $n < 0$.

3.20

3.20. In both cases, the ROC of $H(z)$ has to be chosen such that ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$).

(a)

$$H(z) = \frac{1 - \frac{3}{4}z^{-1}}{1 + \frac{2}{3}z^{-1}}$$

The ROC is $|z| > \frac{2}{3}$.

(b)

$$H(z) = \frac{1}{1 - \frac{1}{6}z^{-1}}$$

The ROC is $|z| > \frac{1}{6}$.

3.30

(a) $x[n]$ is right-sided and

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

Long division:

$$\begin{array}{r|llll} & 1 & -\frac{2}{3}z^{-1} & +\frac{2}{9}z^{-2} & + \cdots \\ \hline 1 + \frac{1}{3}z^{-1} & 1 & -\frac{1}{3}z^{-1} & & \\ & 1 & -\frac{1}{3}z^{-1} & & \\ \hline & & -\frac{2}{3}z^{-1} & & \\ & & -\frac{2}{3}z^{-1} & -\frac{2}{9}z^{-2} & \\ \hline & & & +\frac{2}{9}z^{-2} & \end{array}$$

Therefore, $x[n] = 2(-\frac{1}{3})^n u[n] - \delta[n]$

(b)

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}} = \frac{3z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})} = \frac{4}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 - \frac{1}{4}z^{-1}}$$

Poles at $\frac{1}{2}$, and $-\frac{1}{4}$. $x[n]$ stable, $\Rightarrow |z| > \frac{1}{2}$ $\Rightarrow$ causal.
Therefore,

$$x[n] = 4\left(\frac{1}{2}\right)^n u[n] - 4\left(-\frac{1}{4}\right)^n u[n]$$

(c)

$$X(z) = \ln(1-4z) \qquad |z| < \frac{1}{4}$$
$$= -\sum_{i=1}^{\infty}\frac{(4z)^i}{i} = -\sum_{\ell=-\infty}^{-1}\frac{1}{\ell}(4z)^{-\ell}$$

Therefore,

$$x[n] = \frac{1}{n}(4)^{-n}u[-n-1]$$

(d)

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-3}} \qquad |z| > (3)^{-\frac{1}{3}} \Rightarrow \text{causal}$$

By long division:

$$\begin{array}{r|l} & 1 \quad +\tfrac{1}{3}z^{-3} \quad +\tfrac{1}{9}z^{-6} \quad + \ldots \\ \hline 1-\tfrac{1}{3}z^{-3} & 1 \\ & 1 \quad -\tfrac{1}{3}z^{-3} \\ & \quad +\tfrac{1}{3}z^{-3} \\ & \quad +\tfrac{1}{3}z^{-3} \quad -\tfrac{1}{9}z^{-6} \\ & \qquad +\tfrac{1}{9}z^{-6} \end{array}$$

$$\Longrightarrow \qquad x[n] = \begin{cases} \left(\frac{1}{3}\right)^{\frac{n}{3}}, & n = 0, 3, 6, \ldots \\ 0, & \text{otherwise} \end{cases}$$

3.31

(a)

$$X(z) = \frac{1}{(1+\frac{1}{2}z^{-1})^2(1-2z^{-1})(1-3z^{-1})} \qquad \frac{1}{2} < |z| < 2$$
$$= \frac{\frac{1}{35}}{(1+\frac{1}{2}z^{-1})^2} + \frac{\frac{58}{1225}}{(1+\frac{1}{2}z^{-1})} - \frac{\frac{1568}{1225}}{(1-2z^{-1})} + \frac{\frac{2700}{1225}}{(1-3z^{-1})}$$

Therefore,

$$x[n] = \frac{1}{35}(n+1)\left(\frac{-1}{2}\right)^n u[n+1] + \frac{58}{(35)^2}\left(\frac{-1}{2}\right)^n u[n] + \frac{1568}{(35)^2}(2)^n u[-n-1] - \frac{2700}{(35)^2}(3)^n u[-n-1]$$

(b)

$$X(z) = e^{z^{-1}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \ldots$$

Therefore, $x[n] = \frac{1}{n!}u[n]$.

(c)

$$X(z) = \frac{z^3-2z}{z-2} = z^2 + 2z + \frac{2}{1-2z^{-1}} \qquad |z| < 2$$

Therefore,

$$x[n] = \delta[n+2] + 2\delta[n+1] - 2(2)^n u[-n-1]$$

3.37

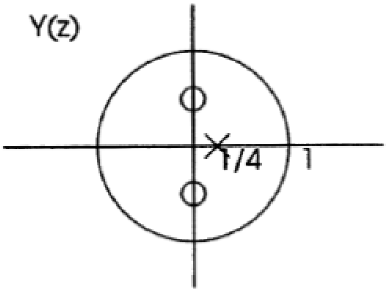
3.37 From pole-zero diagram

$$X(z) = \frac{z^2+1}{z-\frac{1}{2}}$$

(a)

$$y[n] = \left(\frac{1}{2}\right)^n x[n] \Rightarrow Y(z) = X(2z) = \frac{4z^2+1}{2z-\frac{1}{2}}$$

zeros $\pm\frac{1}{2}j$
poles $\frac{1}{4}, \infty$



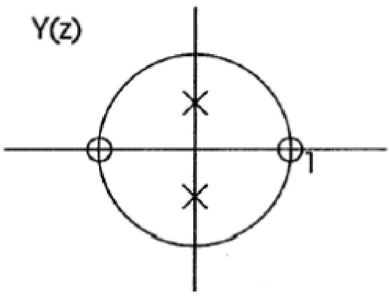
(b)

$$w[n] = \cos\left(\frac{\pi n}{2}\right)x[n] = \frac{1}{2}(e^{j\pi n/2} + e^{-j\pi n/2})x[n]$$

$$W(z) = \frac{1}{2}X(e^{-j\pi/2}z) + \frac{1}{2}X(e^{j\pi/2}z) = \frac{1}{2}X(-jz) + \frac{1}{2}X(jz)$$

$$W(z) = \frac{1}{2}\left(\frac{-z^2+1}{-jz-\frac{1}{2}}\right) + \frac{1}{2}\left(\frac{-z^2+1}{jz-\frac{1}{2}}\right) = \frac{z^2-1}{2(z^2+\frac{1}{4})}$$

poles at $\pm\frac{1}{2}j$
zeros at ± 1



3.41

3.41

$$X(z)=\frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}}+\frac{\frac{1}{4}}{1-2z^{-1}}$$

has poles at $z=\frac{1}{2}$ and $z=2$.

Since the unit circle is in the region of convergence $X(z)$ and $x[n]$ have both a causal and an anticausal part. The causal part is "outside" the pole at $\frac{1}{2}$. The anticausal part is "inside" the pole at 2, therefore, $x[0]$ is the sum of the two parts

$$x[0]=\lim_{z\to\infty}\frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}}+\lim_{z\to 0}\frac{\frac{1}{4}z}{z-2}=\frac{1}{3}+0=\frac{1}{3}$$

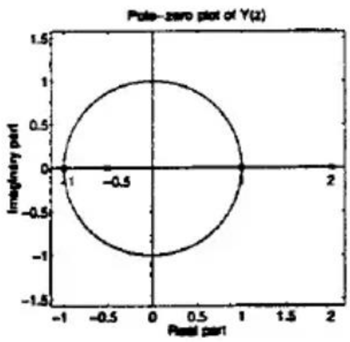
3.46

(a)

$$X(z)=\frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}}+\frac{\frac{4}{3}}{1-2z^{-1}}$$

The ROC is $\frac{1}{2}<|z|<2$.

(b) The following figure shows the pole-zero plot of $Y(z)$. Since $X(z)$ has poles at 0.5 and 2, the poles at 1 and -0.5 are due to $H(z)$. Since $H(z)$ is causal, its ROC is $|z|>1$. The ROC of $Y(z)$ must contain the intersection of the ROC of $X(z)$ and the ROC of $H(z)$. Hence the ROC of $Y(z)$ is $1<|z|<2$.



(c)

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{\frac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})}}{\frac{1}{(1-12z^{-1})(1-2z^{-1})}} \\
&= \frac{(1+z^{-1})(1-\frac{1}{2}z^{-1})}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} \\
&= 1 + \frac{\frac{2}{3}}{1-z^{-1}} + \frac{-\frac{2}{3}}{1+\frac{1}{2}z^{-1}}
\end{aligned}
$$

Taking the inverse z-transform, we find

$$h[n] = \delta[n] + \frac{2}{3}u[n] - \frac{2}{3}(-\frac{1}{2})^n u[n]$$

(d) Since $H(z)$ has a pole on the unit circle, the system is not stable.

3.48

3.48 (a) Since $y[n]$ is stable, its ROC contains the unit-circle. Hence, $Y(z)$ converges for $\frac{1}{2} < |z| < 2$.

(b) Since the ROC is a ring on the z-plane, $y[n]$ is a two-sided sequence.

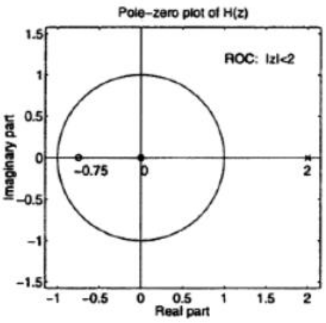
(c) $x[n]$ is stable, so its ROC contains the unit-circle. Also, it has a zero at ∞ so the ROC includes ∞. ROC: $|z| > \frac{3}{4}$.

(d) Since the ROC of $x[n]$ includes ∞, $X(z)$ contains no positive powers of z, and so $x[n] = 0$ for $n < 0$. Therefore $x[n]$ is causal.

(e)

$$
\begin{aligned}
x[0] &= X(z)|_{z=\infty} \\
&= \frac{A(1-\frac{1}{4}z^{-1})}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{2}z^{-1})}|_{z=\infty} \\
&= 0
\end{aligned}
$$

(f) $H(z)$ has zeros at -.75 and 0, and poles at 2 and ∞. Its ROC is $|z| < 2$.



(g) Since the ROC of $h[n]$ includes 0, $H(z)$ contains no negative powers of z, which implies that $h[n] = 0$ for $n > 0$. Therefore $h[n]$ is anti-causal.