Chapter 2 Solution

2.23

2.23. (a) Since $\cos(\pi n)$ only takes on values of +1 or -1, this transformation outputs the current value of $x[n]$ multiplied by either ± 1. $T(x[n]) = (-1)^n x[n]$.

- Hence, it is stable, because it doesn't change the magnitude of $x[n]$ and hence satisfies bounded-in/bounded-out stability.
- It is causal, because each output depends only on the current value of $x[n]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = \cos(\pi n)x_1[n]$, and $y_2[n] = T(x_2[n]) = \cos(\pi n)x_2[n]$. Now

$$T(ax_1[n] + bx_2[n]) = \cos(\pi n)(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = (-1)^n x[n]$, then $T(x[n-1]) = (-1)^n x[n-1] \neq y[n-1]$.

(b) This transformation simply "samples" $x[n]$ at location which can be expressed as k^2.

- The system is stable, since if $x[n]$ is bounded, $x[n^2]$ is also bounded.
- It is not causal. For example, $Tx[4] = x[16]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = x_1[n^2]$, and $y_2[n] = T(x_2[n]) = x_2[n^2]$. Now

$$T(ax_1[n] + bx_2[n]) = ax_1[n^2] + bx_2[n^2]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = x[n^2]$, then $T(x[n-1]) = x[n^2 - 1] \neq y[n-1]$.

(c) First notice that

$$\sum_{k=0}^{\infty} \delta[n-k] = u[n]$$

So $T(x[n]) = x[n]u[n]$. This transformation is therefore stable, causal, linear, but not time-invariant.

To see that it is not time invariant, notice that $T(\delta[n]) = \delta[n]$, but $T(\delta[n+1]) = 0$.

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

- This is not stable. For example, $T(u[n]) = \infty$ for all n ¿ 1.
- It is not causal, since it sums *forward* in time.
- It is linear, since

$$\sum_{k=n-1}^{\infty} ax_1[k] + bx_2[k] = a \sum_{k=n-1}^{\infty} x_1[k] + b \sum_{k=n-1}^{\infty} x_2[k]$$

- It is time-invariant. Let

$$y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k],$$

then

$$T(x[n-n_0]) = \sum_{k=n-n_0-1}^{\infty} x[k] = y[n-n_0]$$

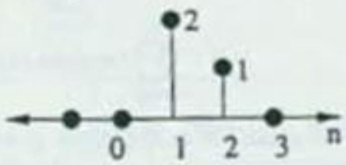
2.25

We use the graphical approach to compute the convolution:

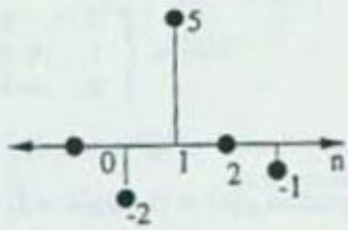
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

(a) $y[n] = x[n] * h[n]$

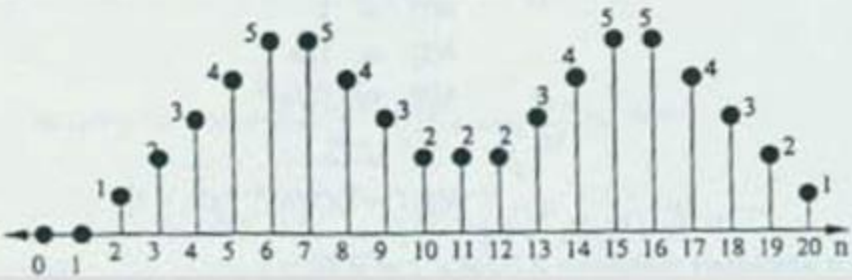
$$y[n] = \delta[n-1] * h[n] = h[n-1]$$



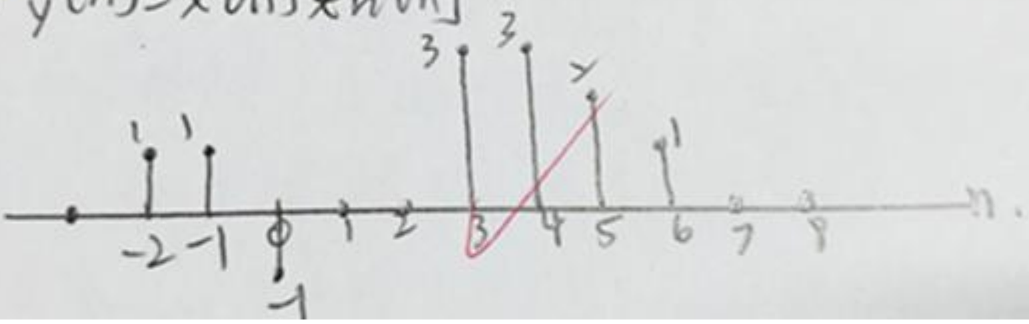
(b) $y[n] = x[n] * h[n]$



(c) $y[n] = x[n] * h[n]$



(d) $y[n] = x[n] * h[n]$



2.29

- System A:

$$x[n] = (\frac{1}{2})^n$$

This input is an eigenfunction of an LTI system. That is, if the system is linear, the output will be a replica of the input, scaled by a complex constant.
Since $y[n] = (\frac{1}{4})^n$, System A is NOT LTI.

- System B:

$$x[n] = e^{jn/8}u[n]$$

The Fourier transform of $x[n]$ is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} e^{jn/8}u[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} e^{-j(\omega-\frac{1}{8})n} \\ &= \frac{1}{1-e^{-j(\omega-\frac{1}{8})}}. \end{aligned}$$

The output is $y[n] = 2x[n]$, thus

$$Y(e^{j\omega}) = \frac{2}{1-e^{-j(\omega-\frac{1}{8})}}.$$

Therefore, the frequency response of the system is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= 2. \end{aligned}$$

Hence, the system is a linear amplifier. We conclude that System B is LTI, and unique.

- System C: Since $x[n] = e^{jn/8}$ is an eigenfunction of an LTI system, we would expect the output to be given by

$$y[n] = \gamma e^{jn/8},$$

where γ is some complex constant, if System C were indeed LTI. The given output, $y[n] = 2e^{jn/8}$, indicates that this is so.
Hence, System C is LTI. However, it is not unique, since the only constraint is that

$$H(e^{j\omega})|_{\omega=1/8} = 2.$$

2.45

$$x[n] = -b^n u[-n-1] = \begin{cases} -b^n, & n \le -1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} -b^n e^{-j\omega n}.$$

Let $k = -n$. Then

$$X\left(e^{j\omega}\right) = \sum_{k=1}^{\infty} -b^{-k} e^{j\omega k}$$

$$= -\left\{\left(\sum_{k=0}^{\infty} b^{-k} e^{j\omega k}\right) - 1\right\}$$

$$= 1 - \sum_{k=0}^{\infty} \left(b^{-1} e^{j\omega}\right)^k$$

$$= 1 - \frac{1}{1 - \frac{e^{j\omega}}{b}},$$

where the last step is true only for $\left|b^{-1} e^{j\omega}\right| < 1$, or $\left|b^{-1}\right| < 1$, or $|b| > 1$. Now we have

$$X\left(e^{j\omega}\right) = \frac{1 - \frac{e^{-j\omega}}{b} - 1}{1 - \frac{e^{-j\omega}}{b}}$$

$$= \frac{-be^{-j\omega}\left(-\frac{1}{b} e^{j\omega}\right)}{-be^{-j\omega}\left(1 - \frac{1}{b} e^{j\omega}\right)}$$

$$X\left(e^{j\omega}\right) = \frac{1}{1 - be^{-j\omega}}$$

only when $|b| > 1$.

Now suppose

$$Y\left(e^{j\omega}\right) = \frac{2e^{-j\omega}}{1 + 2e^{-j\omega}} = 2\frac{1}{1 - (-2)e^{-j\omega}} e^{-j\omega}$$

Using the above transform pair and then shifting to the right by one,

$$y[n] = 2\left[-(-2)^{n-1} u[-(n-1)-1]\right] = -2(-2)^{n-1} u[-n]$$

$$= (-2)^n u[-n].$$

2.47

(a) Notice that $x_1[n] = x_2[n] + x_3[n+4]$, so if $T\{\cdot\}$ is linear,

$$\begin{aligned} T\{x_1[n]\} &= T\{x_2[n]\} + T\{x_3[n+4]\} \\ &= y_2[n] + y_3[n+4] \end{aligned}$$

From Fig P2.4, the above equality is not true. Hence, the system is NOT LINEAR.
(b) To find the impulse response of the system, we note that

$$\delta[n] = x_3[n+4]$$

Therefore,

$$\begin{aligned} T\{\delta[n]\} &= y_3[n+4] \\ &= 3\delta[n+6] + 2\delta[n+5] \end{aligned}$$

(c) Since the system is known to be time-invariant and not linear, we cannot use choices such as:

$$\delta[n] = x_1[n] - x_2[n]$$

and

$$\delta[n] = \frac{1}{2}x_2[n+1]$$

to determine the impulse response. With the given information, we can only use shifted inputs.

2.54

(a) From the figure,

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= (x[n] * (\delta[n] + h_1[n])) * h_2[n]. \end{aligned}$$

Let $h[n]$ be the impulse response of the overall system,

$$y[n] = x[n] * h[n].$$

Comparing with the above expression,

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \beta\alpha^{(n-1)}u[n-1] \end{aligned}$$

(b) Taking the Fourier transform of $h[n]$ from part (a),

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \alpha^n u[n]e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} \alpha^{(n-1)}u[n-1]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} + \beta \sum_{\ell=0}^{\infty} \alpha^{(\ell-1)} e^{-j\omega \ell}, \end{aligned}$$

where we have used $\ell = (n-1)$ in the second sum.

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \text{ for } |\alpha| < 1. \end{aligned}$$

Note that the Fourier transform of $\alpha^n u[n]$ is well known, and the second term of $h[n]$ (see part (a)) is just a scaled and shifted version of $\alpha^n u[n]$. So, we could have used the properties of the Fourier transform to reduce the algebra.

(c) We have

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}},$$

cross multiplying,

$$Y(e^{j\omega})[1-\alpha e^{-j\omega}] = X(e^{j\omega})[1+\beta e^{-j\omega}]$$

taking the inverse Fourier transform, we have

$$y[n] - \alpha y[n-1] = x[n] + \beta x[n-1].$$

(d) From part (a):

$$h[n] = 0, \text{ for } n < 0.$$

This implies that the system is CAUSAL.
If the system is stable, its Fourier transform exists. Therefore, the condition for stability is the same as the condition imposed on the frequency response of part (b). That is, STABLE, if $|\alpha| < 1$.

2.56

2.56. Let $x[n] = \delta[n]$, then

$$X(e^{j\omega}) = 1$$

The output of the ideal lowpass filter:

$$W(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = H(e^{j\omega})$$

The multiplier:

$$(-1)^n w[n] = e^{-j\pi n} w[n]$$

causes a shift in the frequency domain:

$$W(e^{j(\omega-\pi)}) = H(e^{j(\omega-\pi)})$$

The overall output:

$$y[n] = e^{-j\pi n} w[n] + w[n]$$

$$Y(e^{j\omega}) = H(e^{j(\omega-\pi)}) + H(e^{j\omega})$$

Noting that:

$$H(e^{j(\omega-\pi)}) = \begin{cases} 1, & \frac{\pi}{2} \leq |\omega| \leq \pi \\ 0, & |\omega| < \frac{\pi}{2} \end{cases}$$

$Y(e^{j\omega}) = 1$, thus $y[n] = \delta[n]$.

2.60

2.60. (a) We start by interpreting each clue.

(i) The system is causal implies

$$h[n] = 0 \text{ for } n \leq 0.$$

(ii) The Fourier transform is conjugate symmetric implies $h[n]$ is real.

(iii) The DTFT of the sequence $h[n+1]$ is real implies $h[n+1]$ is even.

From the above observations, we deduce that $h[n]$ has length 3, therefore it has finite duration.

(b) From part (a) we know that $h[n]$ is length 3 with even symmetry around $h[1]$. Let $h[0] = h[2] = a$ and $h[1] = b$, from (iv) and using Parseval's theorem, we have

$$2a^2 + b^2 = 2.$$

From (v), we also have

$$2a - b = 0.$$

Solving the above equations, we get

$$\begin{aligned} h[0] &= \frac{1}{\sqrt{3}} \\ h[1] &= \frac{2}{\sqrt{3}} \\ h[2] &= \frac{1}{\sqrt{3}} \end{aligned}$$

or

$$\begin{aligned} h[0] &= -\frac{1}{\sqrt{3}} \\ h[1] &= -\frac{2}{\sqrt{3}} \\ h[2] &= -\frac{1}{\sqrt{3}}. \end{aligned}$$

2.70

2.70. The system could be LTI. A possible impulse response is:

$$\begin{aligned} h[n] &= (\delta[n] - \frac{1}{4}\delta[n-1]) * (\frac{1}{2})^n \\ &= (\frac{1}{2})^n - \frac{1}{4}(\frac{1}{2})^{n-1}. \end{aligned}$$