An important part of the chapter was a discussion of some of the many properties of the z-transform that make it useful in analyzing discrete-time signals and systems. A variety of examples demonstrated how these properties can be used to find direct and inverse z-transforms.

Problems

Basic Problems with Answers

3.1. Determine the z-transform, including the ROC, for each of the following sequences:

(a) $\left(\frac{1}{2}\right)^n u[n]$

(b) $-\left(\frac{1}{2}\right)^n u[-n-1]$

(c) $\left(\frac{1}{2}\right)^n u[-n]$

(d) $\delta[n]$

(e) $\delta[n-1]$

(f) $\delta[n+1]$

(g) $\left(\frac{1}{2}\right)^n (u[n] - u[n-10])$.

3.2. Determine the z-transform of the sequence

$$x[n] = \begin{cases} n, & 0 \le n \le N-1, \\ N, & N \le n. \end{cases}$$

3.3. Determine the z-transform of each of the following sequences. Include with your answer the ROC in the z-plane and a sketch of the pole–zero plot. Express all sums in closed form; α can be complex.

(a) $x_a[n] = \alpha^{|n|}, \quad 0 < |\alpha| < 1.$

(b) $x_b[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$

(c) $x_c[n] = \begin{cases} n+1, & 0 \le n \le N-1, \\ 2N-1-n, & N \le n \le 2(N-1), \\ 0, & \text{otherwise.} \end{cases}$

Hint: Note that $x_b[n]$ is a rectangular sequence and $x_c[n]$ is a triangular sequence. First, express $x_c[n]$ in terms of $x_b[n]$.

3.4. Consider the z-transform $X(z)$ whose pole–zero plot is as shown in Figure P3.4.

(a) Determine the ROC of $X(z)$ if it is known that the Fourier transform exists. For this case, determine whether the corresponding sequence $x[n]$ is right sided, left sided, or two sided.

(b) How many possible two-sided sequences have the pole–zero plot shown in Figure P3.4?

(c) Is it possible for the pole–zero plot in Figure P3.4 to be associated with a sequence that is both stable and causal? If so, give the appropriate ROC.

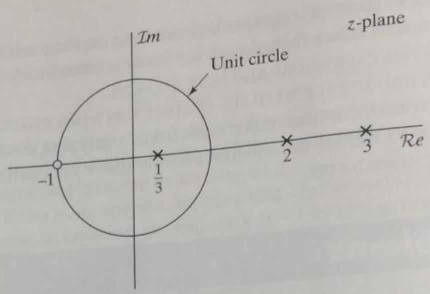

Figure P3.4

3.5. Determine the sequence $x[n]$ with z-transform

$$X(z) = (1 + 2z)(1 + 3z^{-1})(1 - z^{-1}).$$

3.6. Following are several z-transforms. For each, determine the inverse z-transform using both methods—partial fraction expansion and power series expansion—discussed in Section 3.3. In addition, indicate in each case whether the Fourier transform exists.

(a) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \quad |z| > \dfrac{1}{2}$

(b) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \quad |z| < \dfrac{1}{2}$

(c) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \quad |z| > \dfrac{1}{2}$

(d) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \quad |z| > \dfrac{1}{2}$

(e) $X(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}, \quad |z| > |1/a|$

3.7. The input to a causal LTI system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)}.$$

(a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the ROC.

(b) What is the ROC for $Y(z)$?

(c) Determine $y[n]$.

3.8. The system function of a causal LTI system is

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}.$$

The input to this system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1].$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output $y[n]$.
(c) Is the system stable? That is, is $h[n]$ absolutely summable?

3.9. A causal LTI system has impulse response $h[n]$, for which the z-transform is

$$H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}.$$

(a) What is the ROC of $H(z)$?
(b) Is the system stable? Explain.
(c) Find the z-transform $X(z)$ of an input $x[n]$ that will produce the output

$$y[n] = -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1].$$

(d) Find the impulse response $h[n]$ of the system.

3.10. Without explicitly solving for $X(z)$, find the ROC of the z-transform of each of the following sequences, and determine whether the Fourier transform converges:

(a) $x[n] = \left[\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right] u[n-10]$

(b) $x[n] = \begin{cases} 1, & -10 \le n \le 10, \\ 0, & \text{otherwise}, \end{cases}$

(c) $x[n] = 2^n u[-n]$

(d) $x[n] = \left[\left(\frac{1}{4}\right)^{n+4} - (e^{j\pi/3})^n\right] u[n-1]$

(e) $x[n] = u[n+10] - u[n+5]$

(f) $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n] + (2+3j)^{n-2} u[-n-1]$.

3.11. Following are four z-transforms. Determine which ones *could* be the z-transform of a *causal* sequence. Do not evaluate the inverse transform. You should be able to give the answer by inspection. Clearly state your reasons in each case.

(a) $\dfrac{(1-z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)}$

(b) $\dfrac{(z-1)^2}{\left(z-\frac{1}{2}\right)}$

(c) $\dfrac{\left(z-\frac{1}{4}\right)^5}{\left(z-\frac{1}{2}\right)^6}$

(d) $\dfrac{\left(z-\frac{1}{4}\right)^6}{\left(z-\frac{1}{2}\right)^5}$

3.12. Sketch the pole–zero plot for each of the following z-transforms and shade the ROC:

(a) $X_1(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + 2z^{-1}}$, ROC: $|z| < 2$

(b) $X_2(z) = \dfrac{1 - \frac{1}{3}z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{2}{3}z^{-1}\right)}$, $x_2[n]$ causal

(c) $X_3(z) = \dfrac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6}z^{-1} + z^{-2}}$, $x_3[n]$ absolutely summable.

3.13. A causal sequence $g[n]$ has the z-transform

$$G(z) = \sin(z^{-1})(1 + 3z^{-2} + 2z^{-4}).$$

Find $g[11]$.

3.14. If $H(z) = \dfrac{1}{1 - \frac{1}{4}z^{-2}}$ and $h[n] = A_1\alpha_1^n u[n] + A_2\alpha_2^n u[n]$, determine the values of A_1, A_2, α_1, and α_2.

3.15. If $H(z) = \dfrac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}}$ for $|z| > 0$, is the corresponding LTI system causal? Justify your answer.

3.16. When the input to an LTI system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1],$$

the corresponding output is

$$y[n] = 5\left(\frac{1}{3}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the pole(s) and zero(s) of $H(z)$ and indicate the ROC.
(b) Find the impulse response $h[n]$ of the system.
(c) Write a difference equation that is satisfied by the given input and output.
(d) Is the system stable? Is it causal?

3.17. Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1].$$

Determine all possible values for the system's impulse response $h[n]$ at $n = 0$.

3.18. A causal LTI system has the system function

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)(1 - z^{-1})}.$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output of this system, $y[n]$, for the input

$$x[n] = 2^n.$$

3.19. For each of the following pairs of input z-transform $X(z)$ and system function $H(z)$, determine the ROC for the output z-transform $Y(z)$:

(a)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

(b)

$$X(z) = \frac{1}{1-2z^{-1}}, \qquad |z| < 2$$

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

(c)

$$X(z) = \frac{1}{\left(1-\frac{1}{5}z^{-1}\right)\left(1+3z^{-1}\right)}, \qquad \frac{1}{5} < |z| < 3$$

$$H(z) = \frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

3.20. For each of the following pairs of input and output z-transforms $X(z)$ and $Y(z)$, determine the ROC for the system function $H(z)$:

(a)

$$X(z) = \frac{1}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$Y(z) = \frac{1}{1+\frac{2}{3}z^{-1}}, \qquad |z| > \frac{2}{3}$$

(b)

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| < \frac{1}{3}$$

$$Y(z) = \frac{1}{\left(1-\frac{1}{6}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}, \qquad \frac{1}{6} < |z| < \frac{1}{3}$$

Basic Problems

3.21. Consider an LTI system with impulse response

$$h[n] = \begin{cases} a^n, & n \geq 0, \\ 0, & n < 0, \end{cases}$$

and input

$$x[n] = \begin{cases} 1, & 0 \leq n \leq (N-1), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the output $y[n]$ by explicitly evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Determine the output $y[n]$ by computing the inverse z-transform of the product of the z-transforms of $x[n]$ and $h[n]$.

3.22. Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3}{1 + \frac{1}{3}z^{-1}}.$$

Suppose $x[n]$, the input to the system, is a unit step sequence.

(a) Determine the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.
(b) Determine the output $y[n]$ by computing the inverse z-transform of $Y(z)$.

3.23. An LTI system is characterized by the system function

$$H(z) = \frac{\left(1 - \frac{1}{2}z^{-2}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$

(a) Determine the impulse response of the system.
(b) Determine the difference equation relating the system input $x[n]$ and the system output $y[n]$.

3.24. Sketch each of the following sequences and determine their z-transforms, including the ROC:

(a) $\sum_{k=-\infty}^{\infty} \delta[n - 4k]$
(b) $\frac{1}{2}\left[e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right]u[n]$

3.25. Consider a right-sided sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})} = \frac{z^2}{(z - a)(z - b)}.$$

In Section 3.3, we considered the determination of $x[n]$ by carrying out a partial fraction expansion, with $X(z)$ considered as a ratio of polynomials in z^{-1}. Carry out a partial fraction expansion of $X(z)$, considered as a ratio of polynomials in z, and determine $x[n]$ from this expansion.

3.26. Determine the unilateral z-transform, including the ROC, for each of the following sequences:

(a) $\delta[n]$
(b) $\delta[n - 1]$
(c) $\delta[n + 1]$
(d) $\left(\frac{1}{2}\right)^n u[n]$
(e) $-\left(\frac{1}{2}\right)^n u[-n - 1]$
(f) $\left(\frac{1}{2}\right)^n u[-n]$
(g) $\{\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\}u[n]$
(h) $\left(\frac{1}{2}\right)^{n-1} u[n - 1]$.

3.27. If $\mathcal{X}(z)$ denotes the unilateral z-transform of $x[n]$, determine, in terms of $\mathcal{X}(z)$, the unilateral z-transform of the following:

(a) $x[n-2]$

(b) $x[n+1]$

(c) $\sum_{m=-\infty}^{n} x[m]$.

3.28. For each of the following difference equations and associated input and initial conditions, determine the response $y[n]$ for $n \geq 0$ by using the unilateral z-transform.

(a) $y[n] + 3y[n-1] = x[n]$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$
$$y[-1] = 1$$

(b) $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1]$

$$x[n] = u[n]$$
$$y[-1] = 0$$

(c) $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1]$

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$
$$y[-1] = 1$$

Advanced Problems

3.29. A causal LTI system has system function

$$H(z) = \frac{1 - z^{-1}}{1 - 0.25z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}.$$

(a) Determine the output of the system when the input is $x[n] = u[n]$.

(b) Determine the input $x[n]$ so that the corresponding output of the above system is $y[n] = \delta[n] - \delta[n-1]$.

(c) Determine the output $y[n]$ when the input is $x[n] = \cos(0.5\pi n)$ for $-\infty < n < \infty$. You may leave your answer in any convenient form.

3.30. Determine the inverse z-transform of each of the following. In parts (a)–(c), use the methods specified. (In part (d), use any method you prefer.)

(a) Long division:

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}, \qquad x[n] \text{ a right-sided sequence}$$

(b) Partial fraction:

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}, \qquad x[n] \text{ stable}$$

(c) Power series:

$$X(z) = \ln(1 - 4z), \qquad |z| < \frac{1}{4}$$

(d) $X(z) = \dfrac{1}{1 - \frac{1}{3}z^{-3}}, \qquad |z| > (3)^{-1/3}$

3.31. Using any method, determine the inverse z-transform for each of the following:

(a) $X(z) = \dfrac{1}{\left(1+\frac{1}{2}z^{-1}\right)^2 (1-2z^{-1})(1-3z^{-1})}$,

($x[n]$ is a stable sequence)

(b) $X(z) = e^{z^{-1}}$

(c) $X(z) = \dfrac{z^3 - 2z}{z-2}$, ($x[n]$ is a left-sided sequence)

3.32. Determine the inverse z-transform of each of the following. You should find the z-transform properties in Section 3.4 helpful.

(a) $X(z) = \dfrac{3z^{-3}}{\left(1-\frac{1}{4}z^{-1}\right)^2}$, $x[n]$ left sided

(b) $X(z) = \sin(z)$, ROC includes $|z| = 1$

(c) $X(z) = \dfrac{z^7 - 2}{1 - z^{-7}}$, $|z| > 1$

3.33. Determine a sequence $x[n]$ whose z-transform is $X(z) = e^z + e^{1/z}$, $z \neq 0$.

3.34. Determine the inverse z-transform of

$$X(z) = \log(1 - 2z), \qquad |z| < \frac{1}{2},$$

by

(a) using the power series

$$\log(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}, \qquad |x| < 1;$$

(b) first differentiating $X(z)$ and then using the derivative to recover $x[n]$.

3.35. For each of the following sequences, determine the z-transform and ROC, and sketch the pole–zero diagram:

(a) $x[n] = a^n u[n] + b^n u[n] + c^n u[-n-1]$, $|a| < |b| < |c|$

(b) $x[n] = n^2 a^n u[n]$

(c) $x[n] = e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n] - e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n-1]$

3.36. The pole–zero diagram in Figure P3.36 corresponds to the z-transform $X(z)$ of a causal sequence $x[n]$. Sketch the pole–zero diagram of $Y(z)$, where $y[n] = x[-n+3]$. Also, specify the ROC for $Y(z)$.

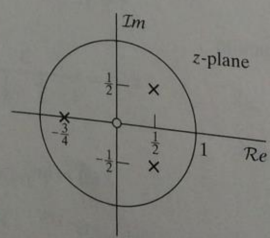

Figure P3.36

3.37. Let $x[n]$ be the sequence with the pole–zero plot shown in Figure P3.37. Sketch the pole–zero plot for:

(a) $y[n] = \left(\frac{1}{2}\right)^n x[n]$

(b) $w[n] = \cos\left(\frac{\pi n}{2}\right)x[n]$

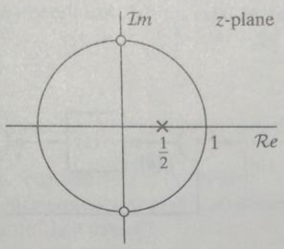

Figure P3.37

3.38. Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3 - 7z^{-1} + 5z^{-2}}{1 - \frac{5}{2}z^{-1} + z^{-2}}.$$

Suppose $x[n]$, the input to the system, is a unit step sequence.

(a) Find the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Find the output $y[n]$ by computing the inverse z-transform of $Y(z)$.

3.39. Determine the unit step response of the causal system for which the z-transform of the impulse response is

$$H(z) = \frac{1 - z^3}{1 - z^4}.$$

3.40. If the input $x[n]$ to an LTI system is $x[n] = u[n]$, the output is

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1].$$

(a) Find $H(z)$, the z-transform of the system impulse response, and plot its pole–zero diagram.

(b) Find the impulse response $h[n]$.

(c) Is the system stable?

(d) Is the system causal?

3.41. Consider a sequence $x[n]$ for which the z-transform is

$$X(z) = \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{4}}{1 - 2z^{-1}}$$

and for which the ROC includes the unit circle. Determine $x[0]$ using the initial-value theorem (see Problem 3.57).

3.42. In Figure P3.42, $H(z)$ is the system function of a causal LTI system.

(a) Using z-transforms of the signals shown in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.

(b) For the special case $H(z) = z^{-1}/(1 - z^{-1})$, determine $H_1(z)$ and $H_2(z)$.

(c) Is the system $H(z)$ stable? Are the systems $H_1(z)$ and $H_2(z)$ stable?

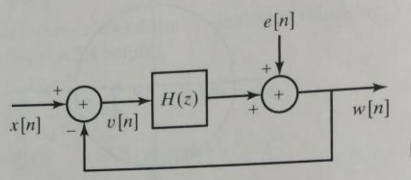

Figure P3.42

3.43. In Figure P3.43, $h[n]$ is the impulse response of the LTI system within the inner box. The input to system $h[n]$ is $v[n]$, and the output is $w[n]$. The z-transform of $h[n]$, $H(z)$, exists in the following ROC:

$$0 < r_{min} < |z| < r_{max} < \infty.$$

(a) Can the LTI system with impulse response $h[n]$ be bounded input, bounded output stable? If so, determine inequality constraints on r_{min} and r_{max} such that it is stable. If not, briefly explain why.

(b) Is the overall system (in the large box, with input $x[n]$ and output $y[n]$) LTI? If so, find its impulse response $g[n]$. If not, briefly explain why.

(c) Can the overall system be BIBO stable? If so, determine inequality constraints relating α, r_{min}, and r_{max} such that it is stable. If not, briefly explain why.

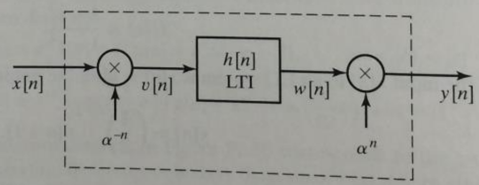

Figure P3.43

3.44. A causal and stable LTI system $\mathcal{S}$ has its input $x[n]$ and output $y[n]$ related by the linear constant-coefficient difference equation

$$y[n] + \sum_{k=1}^{10} \alpha_k y[n-k] = x[n] + \beta x[n-1].$$

Let the impulse response of $\mathcal{S}$ be the sequence $h[n]$.

(a) Show that $h[0]$ must be nonzero.

(b) Show that α_1 can be determined from knowledge of β, $h[0]$, and $h[1]$.

(c) If $h[n] = (0.9)^n \cos(\pi n/4)$ for $0 \le n \le 10$, sketch the pole–zero plot for the system function of $\mathcal{S}$, and indicate the ROC.

3.45. When the input to an LTI system is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1],$$

the output is

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the ROC.
(b) Find the impulse response $h[n]$ of the system.
(c) Write the difference equation that characterizes the system.
(d) Is the system stable? Is it causal?

3.46. When the input to a causal LTI system is

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}2^n u[-n-1],$$

the z-transform of the output is

$$Y(z) = \frac{1+z^{-1}}{(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)(1-2z^{-1})}.$$

(a) Find the z-transform of $x[n]$.
(b) What is the region of convergence of $Y(z)$?
(c) Find the impulse response of the system.
(d) Is the system stable?

3.47. Let $x[n]$ be a discrete-time signal with $x[n] = 0$ for $n \le 0$ and z-transform $X(z)$. Furthermore, given $x[n]$, let the discrete-time signal $y[n]$ be defined by

$$y[n] = \begin{cases} \frac{1}{n}x[n], & n > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $Y(z)$ in terms of $X(z)$.
(b) Using the result of part (a), find the z-transform of

$$w[n] = \frac{1}{n + \delta[n]}u[n-1].$$

3.48. The signal $y[n]$ is the output of an LTI system with impulse response $h[n]$ for a given input $x[n]$. Throughout the problem, assume that $y[n]$ is stable and has a z-transform $Y(z)$ with the pole–zero diagram shown in Figure P3.48-1. The signal $x[n]$ is stable and has the pole–zero diagram shown in Figure P3.48-2.

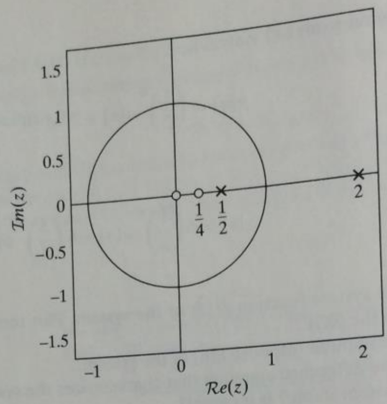

Figure P3.48-1

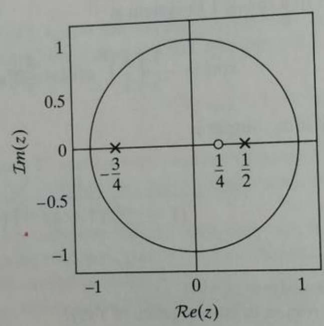

Figure P3.48-2

(a) What is the ROC, $Y(z)$?
(b) Is $y[n]$ left sided, right sided, or two sided?
(c) What is the ROC of $X(z)$?
(d) Is $x[n]$ a causal sequence? That is, does $x[n] = 0$ for $n < 0$?
(e) What is $x[0]$?
(f) Draw the pole–zero plot of $H(z)$, and specify its ROC.
(g) Is $h[n]$ anticausal? That is, does $h[n] = 0$ for $n > 0$?

3.49. Consider the difference equation of Eq. (3.66).

(a) Show that with nonzero initial conditions the unilateral z-transform of the output of the difference equation is

$$\mathcal{Y}(z) = -\frac{\sum_{k=1}^{N} a_k \left(\sum_{m=1}^{k} y[m-k-1] z^{-m+1} \right)}{\sum_{k=0}^{N} a_k z^{-k}} + \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \mathcal{X}(z).$$

(b) Use the result of (a) to show that the output has the form

$$y[n] = y_{ZIR}[n] + y_{ZICR}[n]$$

where $y_{ZIR}[n]$ is the output when the input is zero for all n and $y_{ZICR}[n]$ is the output when the initial conditions are all zero.

(c) Show that when the initial conditions are all zero, the result reduces to the result that is obtained with the bilateral z-transform.

Extension Problems

3.50. Let $x[n]$ denote a causal sequence; i.e., $x[n] = 0, n < 0$. Furthermore, assume that $x[0] \neq 0$ and that the z-transform is a rational function.

(a) Show that there are no poles or zeros of $X(z)$ at $z = \infty$, i.e., that $\lim_{z\to\infty} X(z)$ is nonzero and finite.

(b) Show that the number of poles in the finite z-plane equals the number of zeros in the finite z-plane. (The finite z-plane excludes $z = \infty$.)

3.51. Consider a sequence with z-transform $X(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials in z. If the sequence is absolutely summable and if all the roots of $Q(z)$ are inside the unit circle, is the sequence necessarily causal? If your answer is yes, clearly explain. If your answer is no, give a counterexample.

3.52. Let $x[n]$ be a causal stable sequence with z-transform $X(z)$. The *complex cepstrum* $\hat{x}[n]$ is defined as the inverse transform of the logarithm of $X(z)$; i.e.,

$$\hat{X}(z) = \log X(z) \overset{\mathcal{Z}}{\longleftrightarrow} \hat{x}[n],$$

where the ROC of $\hat{X}(z)$ includes the unit circle. (Strictly speaking, taking the logarithm of a complex number requires some careful considerations. Furthermore, the logarithm of a valid z-transform may not be a valid z-transform. For now, we assume that this operation is valid.)

Determine the complex cepstrum for the sequence

$$x[n] = \delta[n] + a\delta[n-N], \quad \text{where } |a| < 1.$$

3.53. Assume that $x[n]$ is real and even; i.e., $x[n] = x[-n]$. Further, assume that z_0 is a zero of $X(z)$; i.e., $X(z_0) = 0$.

(a) Show that $1/z_0$ is also a zero of $X(z)$.

(b) Are there other zeros of $X(z)$ implied by the information given?

3.54. Using the definition of the z-transform in Eq. (3.2), show that if $X(z)$ is the z-transform of $x[n] = x_R[n] + jx_I[n]$, then

(a) $x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*)$

(b) $x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z)$

(c) $x_R[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2}[X(z) + X^*(z^*)]$

(d) $x_I[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2j}[X(z) - X^*(z^*)]$.

3.55. Consider a *real* sequence $x[n]$ that has all the poles and zeros of its z-transform inside the unit circle. Determine, in terms of $x[n]$, a *real* sequence $x_1[n]$ not equal to $x[n]$, but for which $x_1[0] = x[0]$, $|x_1[n]| = |x[n]|$, and the z-transform of $x_1[n]$ has all its poles and zeros inside the unit circle.

3.56. A real finite-duration sequence whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle is uniquely specified to within a positive scale factor by its Fourier transform phase (Hayes et al., 1980).

An example of zeros at conjugate reciprocal pair locations is $z = a$ and $(a^*)^{-1}$. Even though we can generate sequences that do not satisfy the preceding set of conditions, almost any sequence of practical interest satisfies the conditions and therefore is uniquely specified to within a positive scale factor by the phase of its Fourier transform.

Consider a sequence $x[n]$ that is real, that is zero outside $0 \le n \le N-1$, and whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle. We wish to develop an algorithm that reconstructs $cx[n]$ from $\angle X(e^{j\omega})$, the Fourier transform phase of $x[n]$, where c is a positive scale factor.

(a) Specify a set of $(N-1)$ linear equations, the solution to which will provide the recovery of $x[n]$ to within a positive or negative scale factor from $\tan\{\angle X(e^{j\omega})\}$. You do not have to prove that the set of $(N-1)$ linear equations has a unique solution. Further, show that if we know $\angle X(e^{j\omega})$ rather than just $\tan\{\angle X(e^{j\omega})\}$, the sign of the scale factor can also be determined.

(b) Suppose

$$x[n] = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ 2, & n = 1, \\ 3, & n = 2, \\ 0, & n \ge 3. \end{cases}$$

Using the approach developed in part (a), demonstrate that $cx[n]$ can be determined from $\angle X(e^{j\omega})$, where c is a positive scale factor.

3.57. For a sequence $x[n]$ that is zero for $n < 0$, use Eq. (3.2) to show that

$$\lim_{z\to\infty} X(z) = x[0].$$

This result is called the *initial value theorem*. What is the corresponding theorem if the sequence is zero for $n > 0$?

3.58. The aperiodic autocorrelation function for a real-valued stable sequence $x[n]$ is defined as

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k].$$

(a) Show that the z-transform of $c_{xx}[n]$ is

$$C_{xx}(z) = X(z)X(z^{-1}).$$

Determine the ROC for $C_{xx}(z)$.

(b) Suppose that $x[n] = a^n u[n]$. Sketch the pole–zero plot for $C_{xx}(z)$, including the ROC. Also, find $c_{xx}[n]$ by evaluating the inverse z-transform of $C_{xx}(z)$.

(c) Specify another sequence, $x_1[n]$, that is not equal to $x[n]$ in part (b), but that has the same autocorrelation function, $c_{xx}[n]$, as $x[n]$ in part (b).

(d) Specify a third sequence, $x_2[n]$, that is not equal to $x[n]$ or $x_1[n]$, but that has the same autocorrelation function as $x[n]$ in part (b).

3.59. Determine whether or not the function $X(z) = z^*$ can correspond to the z-transform of a sequence. Clearly explain your reasoning.

3.60. Let $X(z)$ denote a ratio of polynomials in z; i.e.,

$$X(z) = \frac{B(z)}{A(z)}.$$

Show that if $X(z)$ has a 1^{st}-order pole at $z = z_0$, then the residue of $X(z)$ at $z = z_0$ is equal to

$$\frac{B(z_0)}{A'(z_0)},$$

where $A'(z_0)$ denotes the derivative of $A(z)$ evaluated at $z = z_0$.