The z-Transform

✧ Introduction

- Why do we study them?
  - A generalization of DTFT.

    Some sequences that do not converge for DTFT have valid z-transforms.
  - Better notation (compared to FT) in analytical problems (complex variable theory)
  - Solving difference equation. → algebraic equation.

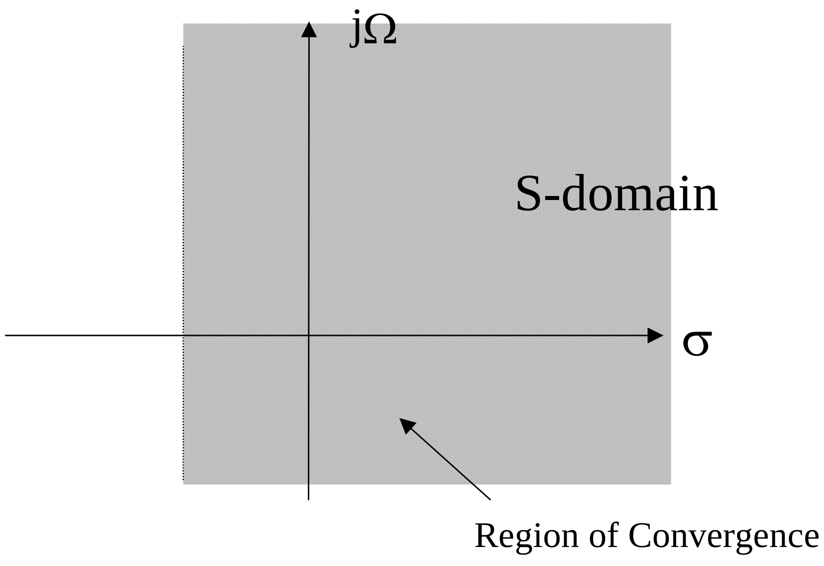
- **Fourier Transform, Laplace Transform, DTFT, & z-Transform**

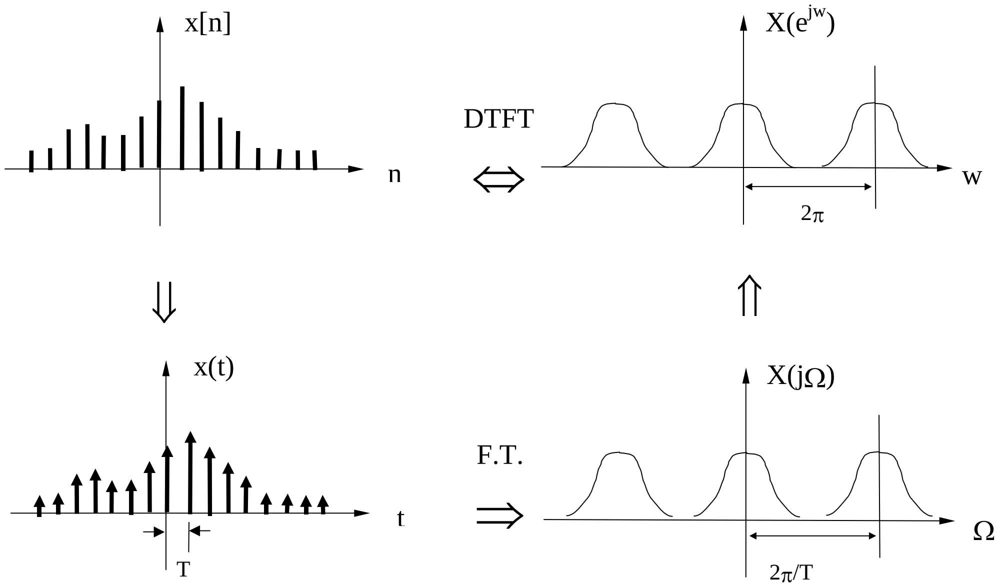
  Fourier Transform

$$\Im\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

To encompass a broader class of signals:

$$\int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\Omega t}dt \equiv \int_{-\infty}^{\infty} x(t)e^{-st}dt \equiv L\{x(t)\} \quad \text{Laplace Transform}$$





$$x(t) = \sum_{k=-\infty}^{\infty} x[k]\delta(t-kT)$$

Similarly,

$$L\{x(t)\} = L\{\sum_{k=-\infty}^{\infty} x[k]\delta(t-kT)\} = \int_{-\infty}^{\infty}\{\sum_{k=-\infty}^{\infty} x[k]\delta(t-kT)\}e^{-st}dt = \sum_{k=-\infty}^{\infty} x[k]\int_{-\infty}^{\infty}\delta(t-kT)e^{-st}dt$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-skT} \equiv \sum_{k=-\infty}^{\infty} x[k]z^{-k} \equiv \mathrm{Z}\{\mathrm{x}[n]\} \equiv X(z)$$

z-Transform

- Eigenfunctions of discrete-time LTI systems

If $x[n] = z_0^n$ z_0^n : some complex constant

$$y[n] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} z_0^{n-k}h[k] = \{\sum_{k=-\infty}^{\infty} h[k]z_0^{-k}\}z_0^n = H(z_0)z_0^n$$

Remark:

$$X(z)\Big|_{z=e^{jw}} = \sum_{n=-\infty}^{\infty} x[n]e^{-jnw}$$

DTFT can be viewed as a special case: $z = e^{j\omega}$

✧ z-Transform

- (**Two-sided) *z*-Transform** (bilateral *z*-Transform)

Forward: $Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \equiv X(z)$

From DTFT viewpoint: $Z\{x[n]\} = F\{r^{-n}x[n]\}\big|_{re^{j\omega}=z}$

(Or, DTFT is a special case of *z*-T when $z = e^{j\omega}$, *unit circle*.)

Inverse: $x[n] = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz \equiv Z^{-1}[X(z)]$

Note: The integration is evaluated along a counterclockwise circle on the complex *z* plane with a radius *r*. (A proof of this formula requires the complex variable theory.)

- **Single-sided z-Transform** (unilateral) – for causal sequences

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

- **Region of Convergence** (ROC)

The set of values of *z* for which the *z*-transform converges.

  - *Uniform convergence*

    If $z = re^{j\omega}$ (polar form), the *z*-transform converges uniformly if $x[n]r^{-n}$ is *absolutely summable*; that is,

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

  - In general, if some value of *z*, say $z = z_1$, is in the ROC, then all values of *z* on the circle defined by $|z| = |z_1|$ are also in the ROC. ➔ ROC is a "ring".
  - If ROC contains the unit circle, |*z*| =1, then the FT of this sequence converges.
  - By its definition, *X*(*z*) is a Laurent series (complex variable)

    ➔ *X*(*z*) is an *analytic function* in its ROC

    ➔ *All* its derivatives are continuous (in *z*) within its ROC.

- DTFT v.s. *z*-Transform

-- $x_1[n] = \dfrac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$

Not absolutely summable; but square summable

→ *z*-transform does not exist; DTFT (in m.s. sense) exists.

-- $x_2[n] = \cos \omega_0 n, \quad -\infty < n < \infty$

Not absolutely summable; not square summable

→ *z*-transform does not exist; “useful” DTFT (impulses) exists.

-- $x_3[n] = a^n u[n], \quad |a| > 1, \quad -\infty < n < \infty$

→ *z*-transform exists (a certain ROC); DTFT does not exist.

- **Some Common *Z*-T Pairs**

TABLE 3.1 SOME COMMON *z*-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

✧ Properties of ROC for z-Transform

- **Rational functions**

$$X(z)=\frac{P(z)}{Q(z)}$$

Poles – Roots of the denominator; the z such that $X(z)\to\infty$

Zeros – Roots of the numerator; the z such that $X(z)=0$

- Properties of ROC

(1) The ROC is a ring or disk in the *z*-plane centered at the origin.

(2) The F.T. of $x[n]$ converges absolutely ⇔ its ROC includes the unit circle.

(3) The ROC cannot contain any poles.

(4) If $x[n]$ is *finite-duration*, then the ROC is the entire *z*-plane except possibly $z=0$ or $z=\infty$.

(5) If $x[n]$ is *right-sided*, the ROC, if exists, must be of the form $|z|>r_{\max}$ except possibly $z=\infty$, where $r_{\max}$ is the magnitude of the largest pole.

(6) If $x[n]$ is *left-sided*, the ROC, if exists, must be of the form $|z|<r_{\min}$ except possibly $z=0$, where $r_{\min}$ is the magnitude of the smallest pole.

(7) If $x[n]$ is *two-sided*, the ROC must be of the form $r_1<|z|<r_2$ if exists, where r_1 and r_2 are the magnitudes of the interior and exterior poles.

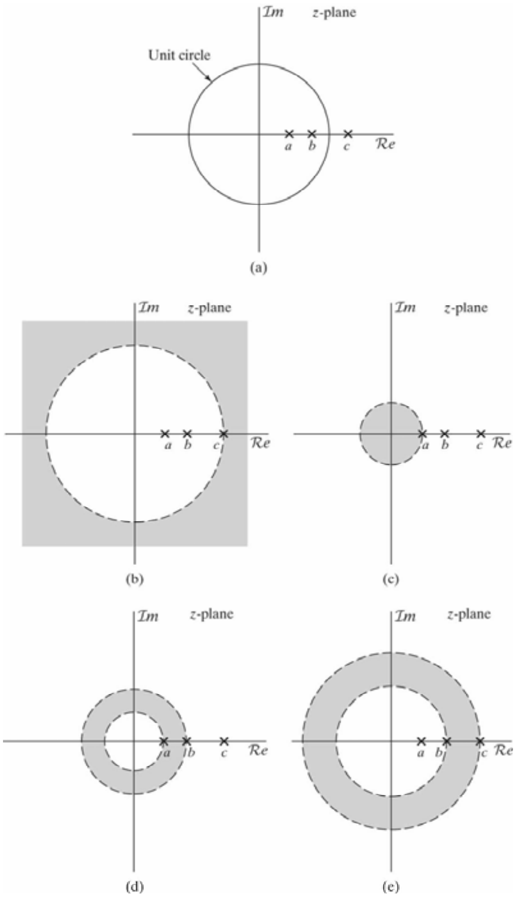
(8) The ROC must be a connected region.

In general, if $X(z)$ is rational, its inverse has the following form (assuming *N* poles: $\{d_k\}$)

$x[n]=\sum_{k=1}^{N}A_k(d_k)^n$. For a right-sided sequence, it means $n\ge N_1$, where N_1 is the first nonzero sample.

The *n*th term in the *z*-transform is $x[n]r^{-n}=\sum_{k=1}^{N}A_k(d_k r^{-1})^n$.

This sequence converges if $\sum_{n=N_1}^{\infty}|d_k r^{-1}|^n<\infty$ for every pole $k=1,\ldots,N$. In order to be so, $|r|>|d_k|,\ k=1,\ldots,N$.



✧ Pole Location and Time-Domain Behavior for Causal Signals

Reference: Digital Signal Processing by Proakis & Manolakis

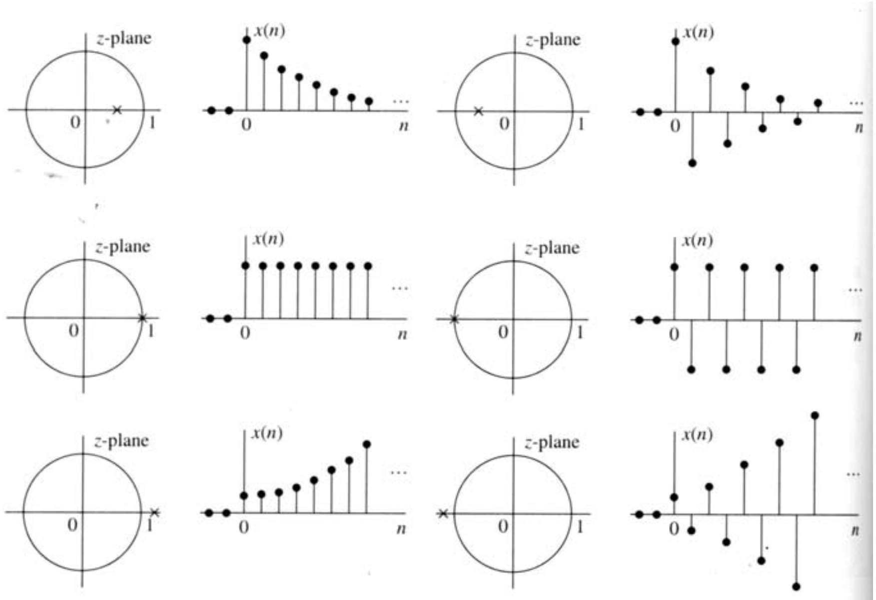


Figure 3.11 Time-domain behavior of a single-real pole causal signal as a function of the location of the pole with respect to the unit circle.

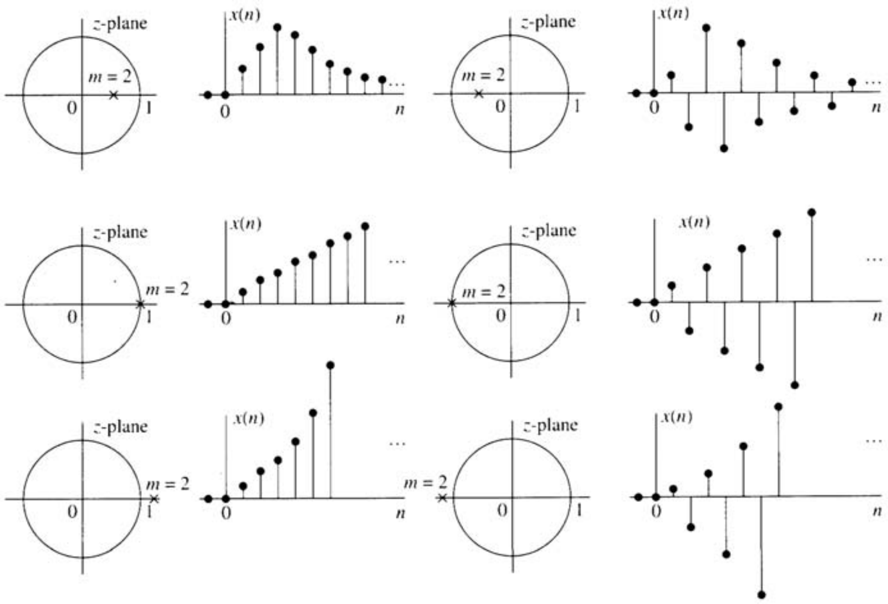


Figure 3.12 Time-domain behavior of causal signals corresponding to a double ($m = 2$) real pole, as a function of the pole location.

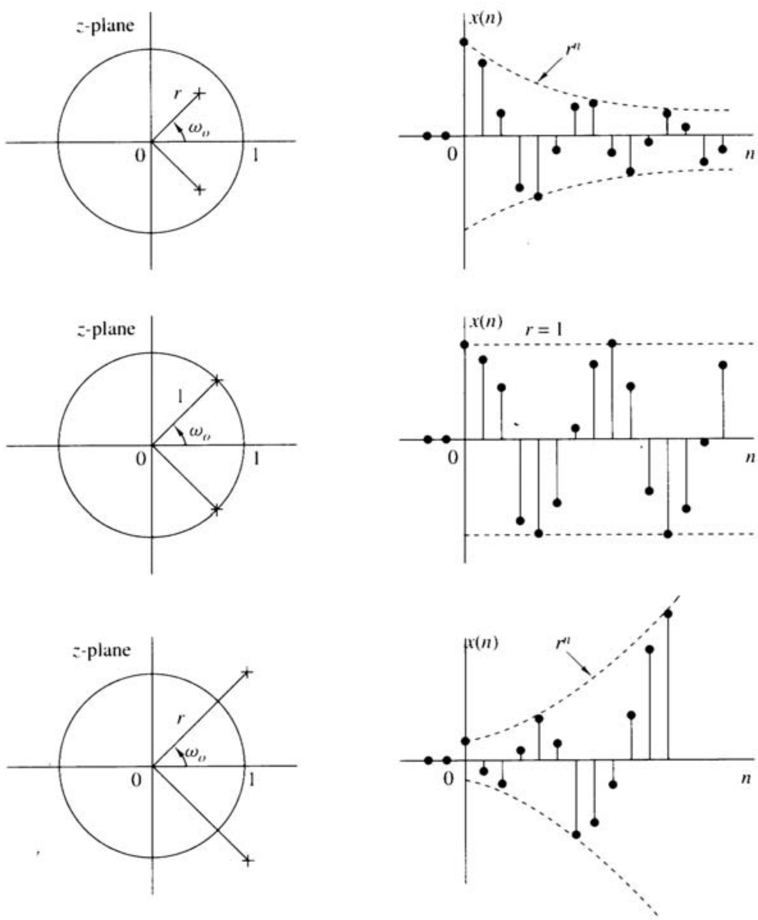


Figure 3.13 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

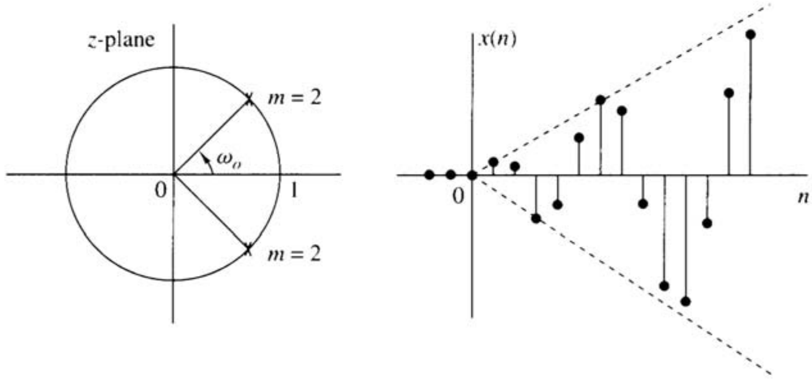


Figure 3.14 Causal signal corresponding to a double pair of complex-conjugate poles on the unit circle.

✧ The Inverse z-Transform

Inverse formula: $x[n] = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz$

This formula can be proved using Cauchy integral theorem (complex variable theory).

- Methods of evaluating the inverse *z*-transform

  (1) Table lookup or inspection

  (2) Partial fraction expansion

  (3) Power series expansion

- **Inspection** (transform pairs in the table) – memorized them
- **Partial Fraction Expansion**

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} \rightarrow X(z) = \frac{z^N (b_0 z^M + \cdots + b_M)}{z^M (a_0 z^N + \cdots + a_N)}$$

Hence, it has *M* zeros (roots of $\sum b_k z^{M-k}$), *N* poles (roots of $\sum a_k z^{N-k}$), and (*M*-*N*) poles at zero if *M*>*N* (or (*N*-*M*) zeros at zero if *N*>*M*).

→ $X(z) = \frac{b_0 (1 - c_1 z^{-1}) \cdots (1 - c_M z^{-1})}{a_0 (1 - d_1 z^{-1}) \cdots (1 - d_N z^{-1})}$; c_k, nonzero zeros; d_k, nonzero poles.

■ **Case 1:** $M < N$, strictly proper

Simple (single) poles:

$$X(z) = \frac{A_1}{(1 - d_1 z^{-1})} + \frac{A_2}{(1 - d_2 z^{-1})} + \cdots + \frac{A_N}{(1 - d_N z^{-1})}$$

where $A_k = (1-d_k z^{-1})X(z)|_{z=d_k}$

Multiple poles: Assume d_i is the *s*th order pole. (Repeated *s* times)

$$X(z) = \sum_{k=1,k\neq i}^{N} \frac{A_k}{(1-d_k z^{-1})} + \frac{C_1}{(1-d_i z^{-1})} + \frac{C_2}{(1-d_i z^{-1})^2} + \cdots + \frac{C_s}{(1-d_i z^{-1})^s}$$

single-pole terms *multiple-pole terms*

where $C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1-d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}$

- **Case 2**: $M \geq N$

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{(1-d_k z^{-1})} + \sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}$$

impulses *single-poles* *multiple-pole*

- **Power Series Expansion**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Case 1: Right-sided sequence, ROC: $|z| > r_{\max}$

It is expanded in powers of z^{-1}.

Ex. $X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$

- Case 2: Left-sided sequence, ROC: $|z| < r_{\min}$

It is expanded in powers of z.

Ex. $X(z) = \frac{1}{1-az^{-1}}, \quad |z| < |a|$

- Case 3: Two-sided sequence, ROC: $r_1 < |z| < r_2$

$X(z) = X_+(z) + X_-(z)$

converges for $|z| > r_1$ converges for $|z| < r_2$

→ $x[n] = x_+[n] + x_-[n]$

causal sequence anti-causal sequence

✧ z-Transform Properties

If $x[n] \leftrightarrow X[z]$ and $y[n] \leftrightarrow Y[z]$, ROC: R_X, R_Y

- **Linearity:** $ax[n]+by[n] \leftrightarrow aX(z)+bY(z)$

  ROC: $R' \supset R_X \cap R_Y$ -- At least as large as their intersection; larger if pole/zero cancellation occurs

- **Time Shifting:** $x[n-n_0] \leftrightarrow z^{-n_0} X(z)$ ROC: $R' = R_X \pm \{0 \; or \; \infty\}$

- **Multiplication by an exponential seqence:**

  $a^n x[n] \leftrightarrow X(z/a)$ ROC: $R' = |a| R_X$ -- expands or contracts

- **Differentiation of X(z):** $nx[n] \leftrightarrow -z\dfrac{dX(z)}{dz}$, ROC: $R' = R_X$

- **Conjugation of a complex sequence:** $x^*[n] \leftrightarrow X^*(z^*)$, ROC: $R' = R_X$

- **Time reversal:** $x^*[-n] \leftrightarrow X^*(1/z^*)$,

  ROC: $R' = 1/R_X$ (Meaning: If $R_X : r_R < |z| < r_L$, then $R' : 1/r_L < |z| < 1/r_R$.

  Corollary: $x[-n] \leftrightarrow X(1/z)$

- **Convolution:** $x[n] * y[n] \leftrightarrow X(z)Y(z)$

  ROC: $R' \supset R_X \cap R_Y$ (=, if no pole/zero cancellation)

- **Initial Value Theorem:**

  If $x[n]=0$, $n<0$, ***then*** $x[0] = \lim_{z\to\infty} X(z)$

- **Final Value Theorem:**

If (1) $x[n]=0$, $n<0$, and

(2) all singularities of $(1-z^{-1})X(z)$ are inside the unit circle,

then $x[\infty]=\lim_{z\to 1}(1-z^{-1})X(z)$

Remarks: (1) If all poles of $X(z)$ are inside unit circle, $x[n]\to 0$ as $n\to\infty$

(2) If there are multiple poles at "1", $x[n]\to\infty$ as $n\to\infty$

(3) If poles are on the unit circle but not at "1", $x[n]\approx\cos\omega_0 n$

<Supplementary>

z-Transform Solutions of Linear Difference Equations

Use ***single-sided*** z-transform:

$$Z\{y[n-1]\}=z^{-1}Y(z)+y[-1]$$
$$Z\{y[n-2]\}=z^{-2}Y(z)+z^{-1}y[-1]+y[-2]$$
$$Z\{y[n-3]\}=z^{-3}Y(z)+z^{-2}y[-1]+z^{-1}y[-2]+y[-3]$$

For causal signals, their single-sided *z*-transforms are identical to their two-sided *z*-transforms.

Ex., Find $y[n]$ of the difference eqn.

$y[n]-0.5y[n-1]=x[n]$ with $x[n]=1, n\ge 0$, and $y[-1]=1$

(Sol) Take the single-sided *z*-transform of the above eqn.

→ $Y(z)-0.5\{z^{-1}Y(z)+y[-1]\}=X(z)=\dfrac{1}{1-z^{-1}}$

→ $Y(z)=\left\{\dfrac{1}{1-0.5z^{-1}}\right\}\left\{0.5+\dfrac{1}{1-z^{-1}}\right\}$

$=\dfrac{0.5}{1-0.5z^{-1}}+\dfrac{1}{(1-0.5z^{-1})(1-z^{-1})}$

→ $Y(z)=\dfrac{2}{1-z^{-1}}-\dfrac{0.5}{1-0.5z^{-1}}$

Take the inverse *z*-transform

→ $y[n]=2-0.5(0.5)^n$, $n\ge 0$