The Discrete Fourier Transform

- What is Discrete Fourier Transform (DFT)?
  (*Note:* It's not DTFT – discrete-time Fourier transform)
  - A linear transformation (matrix)
  - Samples of the Fourier transform (DTFT) of an aperiodic (with finite duration) sequence
  - Extension of Discrete Fourier Series (DFS)

- Review: FT, DTFT, FS, DFS

Time signal	*Transform*	*Coeffs. (periodic/aperiodic)*	*Coeffs. (conti./discrete)*
Analog aperiodic	**FT**	Aperiodic	Continuous
Analog periodic	**FT** **FS**	Aperiodic Aperiodic	Continuous (impulse) Discrete
Discrete aperiodic	**DTFT**	Periodic	Continuous
Discrete periodic	**DFS**	Periodic	Discrete
Discrete finite-duration	**DFT**		

✧ The Discrete Fourier Series

- Properties of W_N

$W_N = e^{-j2\pi/N}$, thus $W_N^k = e^{-j\frac{2\pi}{N}k}$

-- W_N is periodic with period *N*. (It is essentially cos and sin) : $W_N^k = W_N^{k\pm N} = W_N^{k\pm 2N} = \cdots$

-- $\sum_{k=0}^{N-1} W_N{}^{lk} = \begin{cases} N, & \text{if } l = mN \\ 0, & \text{if } l \neq mN \end{cases}$

(Pf) (i) If $l = m \cdot N$, $W_N^{lk} = W_N^{mk \cdot N} = W_N^0 = 1$

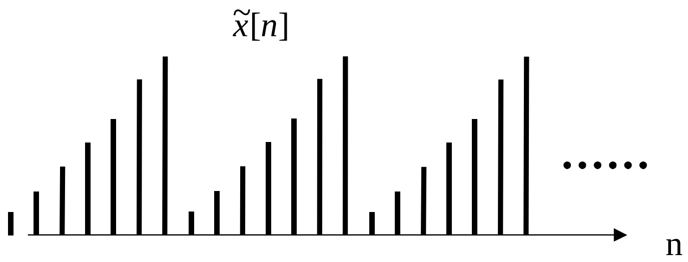
$$\sum_{k=0}^{N-1} W_N^{lk} = \sum_{k=0}^{N-1} 1 = N$$

(ii) If $l \neq m \cdot N$, $W_N^l \neq 1$

$$\sum_{k=0}^{N-1} W_N^{lk} = \frac{1 - W_N^{l \cdot N}}{1 - W_N^l} = \frac{1-1}{1 - W_N^l} = 0$$

-- $Y[l] = \frac{1}{N} \sum_{k=0}^{N-1} W_N{}^{lk} = \sum_{m=-\infty}^{\infty} \delta[l - mN]$

- **DFS** for periodic sequences



$$\tilde{x}[n] = \tilde{x}[n + rN], \qquad period \;\; N$$

Its DFS representation is defined as follows:

Synthesis equation: $\tilde{x}[n] = \frac{1}{N} \sum_{k=o}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N{}^{-kn}$

Analysis equation: $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_n{}^{kn}$

Note: The tilde in $\tilde{x}$ indicates a periodic signal.

$\tilde{X}[k]$ is periodic of period *N*.

Pf) $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$

Pick an r ($0 \le r < N$)

$\times W_N^{rn}$ → $$\tilde{x}[n]W_N^{rn} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}\cdot W_N^{rn}$$

$\sum_{n=0}^{N-1}$ → $$\begin{aligned}\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{rn} &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}\cdot W_N^{rn}\\ &= \frac{1}{N}\sum_{k=0}^{N-1}(\tilde{X}[k]\sum_{n=0}^{N-1}W_N^{(r-k)n})\\ &= \tilde{X}[0]\cdot 0 + \tilde{X}[1]\cdot 0 + \cdots + \tilde{X}[k=r]\cdot 1 + \cdots\\ &= \tilde{X}[r]\end{aligned}$$

That is, $\tilde{X}[r] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{rn}$. QED

Example: Periodic Rectangular Pulse Train

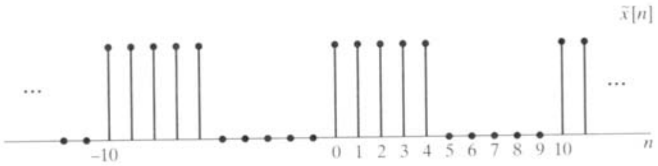


Figure 8.1 Periodic sequence with period $N = 10$ for which the Fourier series representation is to be computed.

$$\tilde{X}[k] = \sum_{n=0}^{4}W_{10}^{Kn} = \frac{1-W_{10}^{5k}}{1-W_{10}^{k}} = e^{-j\frac{4\pi k}{10}}\frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{10}\right)}$$

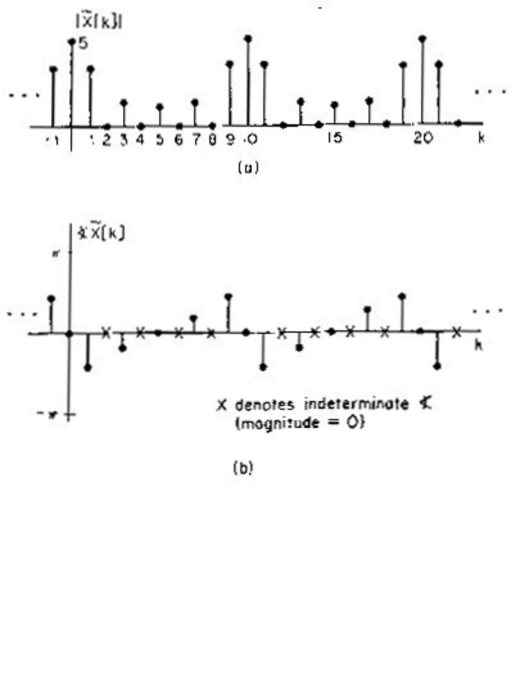


Figure 8.2 Magnitude and phase of the Fourier series coefficients of the sequence of Fig. 8.1.

✧ Sampling the Fourier Transform

Compare two cases:

(1) Periodic sequence $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$

(2) Finite duration sequence $x[n]$ = one period of $\tilde{x}[n]$

An aperiodic sequence:

$$\begin{array}{lcl} x[n] & \rightarrow FT \rightarrow & X\left(e^{j\omega}\right) \\ \updownarrow ? & & \downarrow \text{samples} \\ \tilde{x}[n] \leftarrow IDFS \leftarrow & \tilde{X}[k] = & X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{N}k} \end{array}$$

Compare:

$$\begin{array}{lcl} x(t) & \rightarrow FT \rightarrow & X(j\Omega) \\ \downarrow \text{samples} & & \updownarrow ? \\ x[n] \rightarrow DTFT & \rightarrow & X\left(e^{j\omega}\right) \end{array}$$

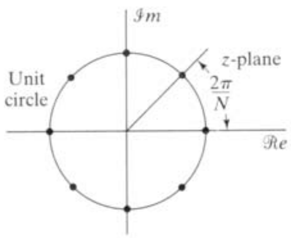


Figure 8.7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Example: $x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & otherwiase \end{cases}$ $\quad \tilde{x}[n] = \begin{cases} 1, & r10 \le n \le 4 + r10 \\ 0, & 5 + r10 \le n \le 9 + r10 \end{cases}$

r = integer

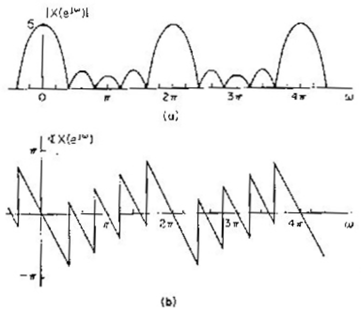


Figure 8.4 Magnitude and phase of the Fourier transform of one period of the sequence in Fig. 8.1.

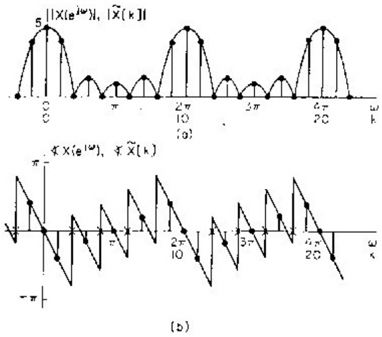
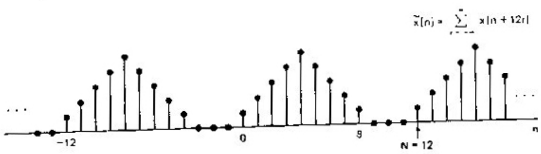


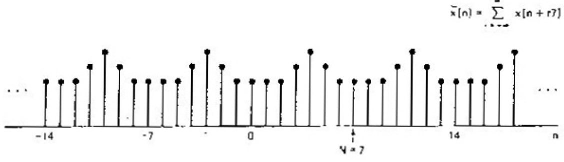
Figure 8.5 Overlay of Figs. 8.2 and 8.4 illustrating the DFS coefficients of a periodic sequence as samples of the Fourier transform of one period.

$$
\begin{aligned}
\tilde{x}[n] &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-Kn} && \text{(IDFS)} \\
&= \frac{1}{N}\sum_{k}\left(X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{N}k}\right)W_N^{-kn} && \text{(Sampling)} \\
&= \frac{1}{N}\sum_{k}\left(\sum_{m=-\infty}^{\infty}x[m]e^{-j\omega m}\right)\Bigg|_{\omega=\frac{2\pi}{k}} W_N^{-kn} && \text{(FT)} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=-\infty}^{\infty}x[m]e^{-j\frac{2\pi}{N}km}\right)W_N^{-kn} \\
&= \frac{1}{N}\sum_{m=-\infty}^{\infty}x[m]\left\{\sum_{k=0}^{N-1}W_N^{km}W_N^{-kn}\right\} && \text{(Interchange } \textstyle\sum \text{)} \\
&= x[n]*\sum_{r=-\infty}^{\infty}\delta[n+rN] = \sum_{r=-\infty}^{\infty}x[n+rN]
\end{aligned}
$$

N = 12



N = 7



If x[n] has finite length and we take a sufficient number of equally spaced samples of its Fourier Transform (a number greater than or equal to the length of x[n]), then x[n] is recoverable from $\tilde{x}[n]$.

- Two ways (equivalently) to define DFT:

  (1) N samples of the DTFT of a finite duration sequence $x[n]$

  (2) Make the periodic replica of $x[n] \rightarrow \tilde{x}[n]$

  Take the DFS of $\tilde{x}[n]$

  Pick up one segment of $\tilde{X}[k]$

$$\begin{array}{ccc} x[n] & \rightarrow DFT \rightarrow & X[k] \\ \downarrow \text{ periodic} & & \uparrow \text{ one segment} \\ \tilde{x}[n] & \rightarrow DFS \rightarrow & \tilde{X}[k] \end{array}$$

✧ Properties of the Discrete Fourier Series

-- Similar to those of FT and z-transform

- **Linearity**

$$\left.\begin{array}{l} \tilde{x}_1[n] \leftrightarrow \tilde{X}_1[k] \\ \tilde{x}_2[n] \leftrightarrow \tilde{X}_2[k] \end{array}\right\} \Rightarrow a\tilde{x}_1[n] + b\tilde{x}_2[n] \leftrightarrow a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

- **Shift**

$$\tilde{x}[n] \leftrightarrow \tilde{X}[k] \implies \begin{array}{l} \tilde{x}[n-m] \leftrightarrow W_N^{km}\tilde{X}[k] \\ W_N^{-nl}\tilde{x}[n] \leftrightarrow \tilde{X}[k-l] \end{array}$$

- **Duality**

Def: $$\begin{cases} \tilde{x}[n] = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} & (*) \\ \tilde{X}[k] = \displaystyle\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{nk} & (\#) \end{cases}$$

$$\begin{cases} \tilde{x}[n] \leftrightarrow \tilde{X}[k] \\ \tilde{X}[k] \leftrightarrow N\tilde{x}[-k] \end{cases}$$

- **Symmetry** $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$

$$\mathrm{Re}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_e[k]\left(=\frac{1}{2}\left(\tilde{X}[k]+\tilde{X}^*[-k]\right)\right)$$

$$j\,\mathrm{Im}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_o[k]\left(=\frac{1}{2}\left(\tilde{X}[k]-\tilde{X}^*[-k]\right)\right)$$

$$\tilde{x}_e[n]=\frac{1}{2}\left(\tilde{x}[n]+\tilde{x}^*[-n]\right) \leftrightarrow \mathrm{Re}\{\tilde{X}[k]\}$$

$$\tilde{x}_o[n]=\frac{1}{2}\left(\tilde{x}[n]-\tilde{x}^*[-n]\right) \leftrightarrow j\,\mathrm{Im}\{\tilde{X}[k]\}$$

If $\tilde{x}[n]$ is real, $\tilde{X}[k]=\tilde{X}^*[-k]$.
$$\Rightarrow \begin{cases} \left|\tilde{X}[k]\right| = \left|\tilde{X}[-k]\right| \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$$
$$\Rightarrow \begin{cases} \mathrm{Re}\{\tilde{X}[k]\} = \mathrm{Re}\{\tilde{X}[-k]\} \\ \mathrm{Im}\{\tilde{X}[k]\} = -\mathrm{Im}\{\tilde{X}[-k]\} \end{cases}$$

- **Periodic Convolution**

$\tilde{x}_1[n], \tilde{x}_2[n]$ are periodic sequences with period N

$$\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m] \leftrightarrow \tilde{X}_1[k]\tilde{X}_2[k]$$

$$\tilde{x}_3[n]=\tilde{x}_1[n]\tilde{x}_2[n] \leftrightarrow \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}_1[l]\tilde{X}_2[k-l]$$

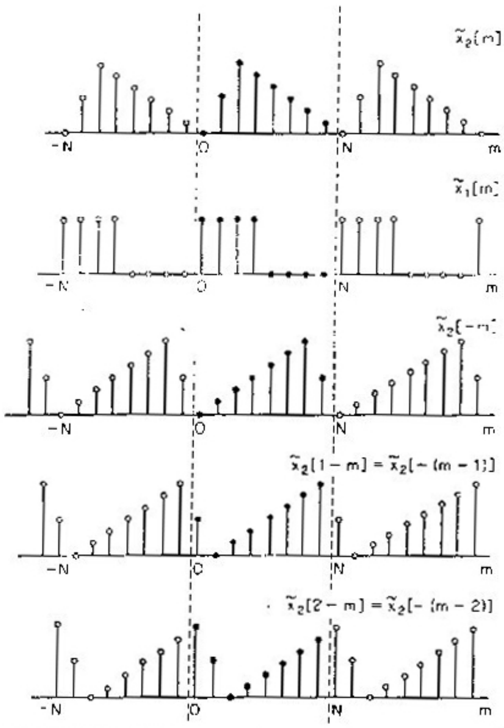


Figure 8.6 Procedure for forming the periodic convolution of two periodic sequences.

✧ Discrete Fourier Transform

- *Definition*

$x[n]$: length N, $0 \le n \le N-1$

Making the periodic replica:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN]$$
$$\equiv x[(n \text{ modulo } N)]$$
$$\equiv x[((n))_N]$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Keep one segment (finite duration)

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases} \qquad \text{That is, } \tilde{X}[k] = X[((k))_N]$$

This finite duration sequence $X[k]$ is the **discrete Fourier transform** (DFT) of $x[n]$

$$\textit{Analysis eqn}: \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$
$$\textit{Synthesis eqn}: \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

Remark: DFT formula is the same as DFS formula. Indeed, many properties of DFT are derived from those of DFS.

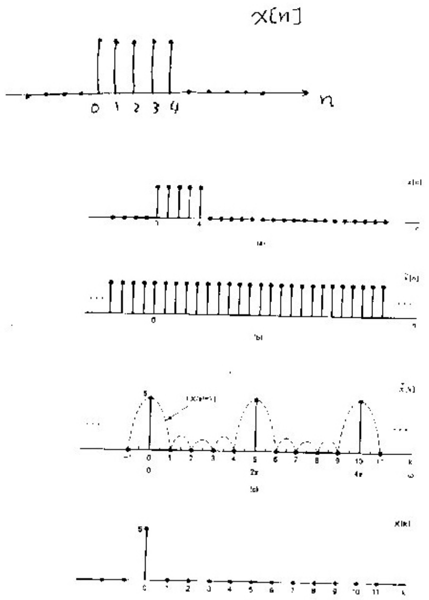


Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

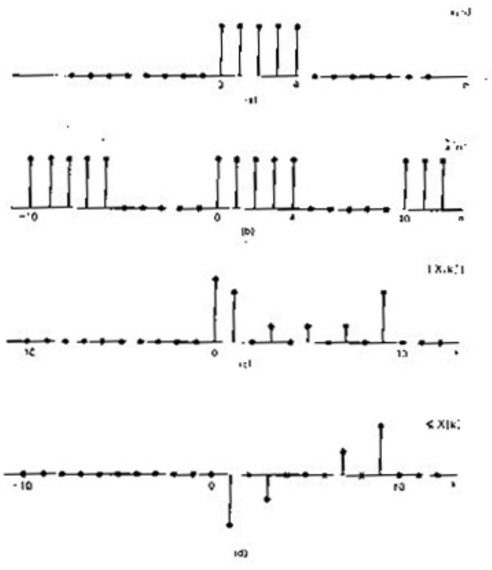


Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)

✧ Properties of Discrete Fourier Transform

- **Linearity**

$$\left.\begin{matrix} x_1[n] \leftrightarrow X_1[k] \\ x_2[n] \leftrightarrow X_2[k] \end{matrix}\right\} \Rightarrow ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k]$$

$$length = \max[N_1, N_2]$$

- **Circular Shift**

$$x[n] \leftrightarrow X[k] \quad ==> \quad \begin{matrix} x[((n-m))_N] \leftrightarrow W_N^{km} X[k] \\ W_N^{-\ln} x[n] \leftrightarrow X[((k-l))_N] \end{matrix}$$

(Pf) From the right side of the 2nd eqn.

$$\begin{matrix} W_N^{Km} X[k] = e^{j\frac{2\pi}{N}km} X[k] \rightarrow e^{j\frac{2\pi}{N}km} \tilde{X}[k] \\ \updownarrow \text{DFT} \qquad\qquad\qquad \downarrow \text{IDFS} \\ x[((n-m))_N] \leftarrow x[((n-m))_N] = \tilde{x}[n-m] \end{matrix} \qquad \text{QED}$$

Remark: This is *circular* shift not *linear* shift. (Linear shift of a periodic sequence = circular shift of a finite sequence.)

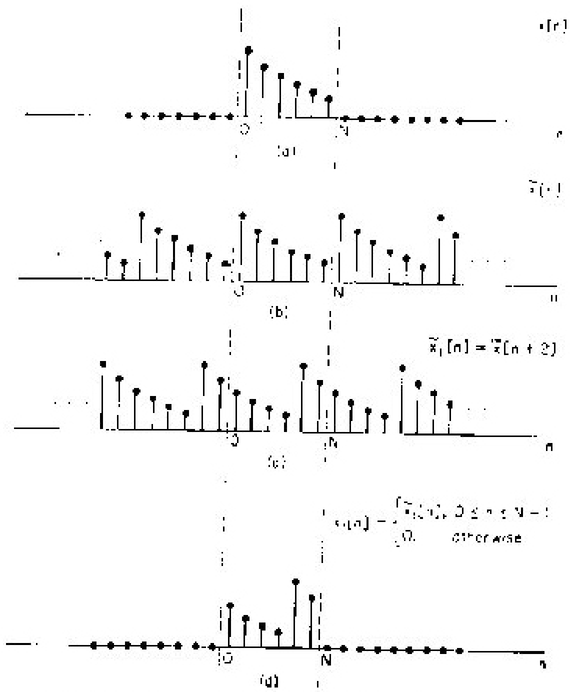


Figure 8.12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear phase factor.

- **Duality**

$$x[n] \leftrightarrow X[k]$$
$$X[n] \leftrightarrow Nx[((-k))_N], \quad 0 \le \mathrm{k} \le \mathrm{N-1}$$

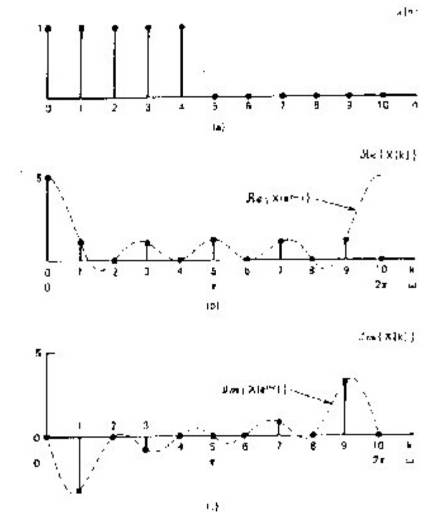


Figure 8.13 Illustration of duality (a) Real finite-length sequence $x[n]$. (b) and (c) Real and imaginary parts of corresponding DFT $X[k]$.

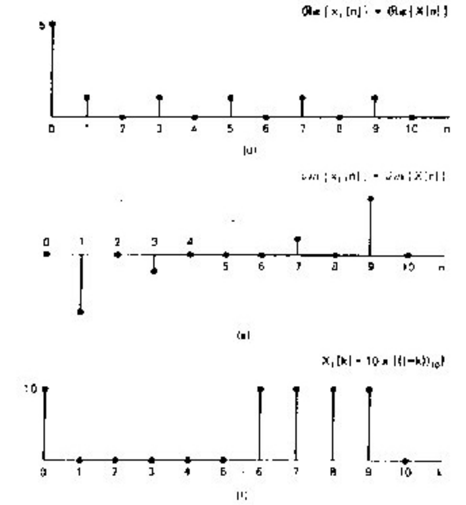


Figure 8.13 (continued) (d) and (e) The real and imaginary parts of the dual sequence $x_1[n] = X[n]$. (f) The DFT of $x_1[n]$.

- **Symmetry Properties**

$$x_{ep}[n] = \text{periodic conjugate-symmetric}$$
$$\equiv \tilde{x}_e[n]$$
$$= \frac{1}{2}\{x[((n))_N] + x^*[((n))_N]\}, \quad 0 \le n \le N\text{-}1$$
$$= \begin{cases} \frac{1}{2}\{x[n] + x^*[N-n]\}, & 1 \le n \le N\text{-}1 \\ \mathrm{Re}\{x[0]\}, & n = 0 \end{cases}$$

$$x_{op}[n] = \text{periodic conjugate-antisymmetric}$$
$$= \begin{cases} \frac{1}{2}\{x[n] - x^*[N-n]\}, & 1 \le n \le N\text{-}1 \\ \mathrm{Im}\{x[0]\}, & n = 0 \end{cases}$$

$$x_{ep}[n] \leftrightarrow \mathrm{Re}\{X[k]\} \qquad x_{op}[n] \leftrightarrow j\,\mathrm{Im}\{X[k]\}$$

If $x[n]$ real, $X[k] = X^*[((-k))_N], \quad 0 \le k \le N-1$

$$\Rightarrow \begin{cases} |X[k]| = |X[((-k))_N]| \\ \angle\{X[k]\} = -\angle X[((-k))_N] \end{cases} \qquad \Rightarrow \begin{cases} \mathrm{Re}\{X[k]\} = \mathrm{Re}\{X[((-k))_N]\} \\ \mathrm{Im}\{X[k]\} = -\mathrm{Im}\{X[((-k))_N]\} \end{cases}$$

$$\begin{cases} \mathrm{Re}\{x[n]\} \leftrightarrow X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\} \\ \mathrm{Im}\{x[n]\} \leftrightarrow X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\} \end{cases}$$

- **Circular Convolution**

$$x_3[n] = x_1[n] \Theta x_2[n]$$

$$\equiv \sum_{m=0}^{N-1} x[m]x[((n-m))_N] \quad N\text{-point circular convolution}$$

$$x_1[n] \Theta x_2[n] \leftrightarrow X_1[k]X_2[k]$$

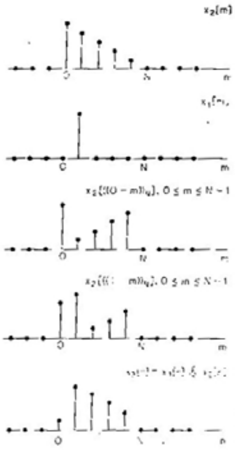


Figure 8.14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse.

Example: N-point circular convolution of two constant sequences of length N

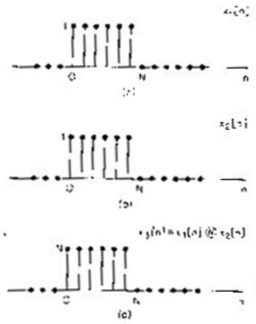


Figure 8.15 N-point circular convolution of two constant sequences of length N.

2L-point circular convolution of two constant sequences of length L

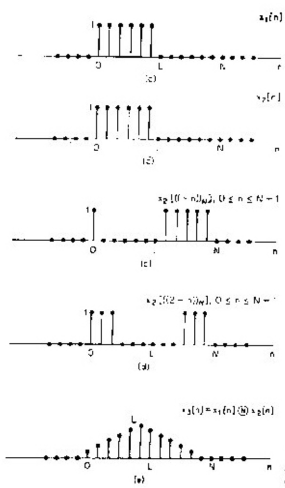


Figure 8.16 2L-point circular convolution of two constant sequences of length L.

✧ Linear Convolution Using DFT

- Why using DFT? There are fast DFT algorithms (FFT)

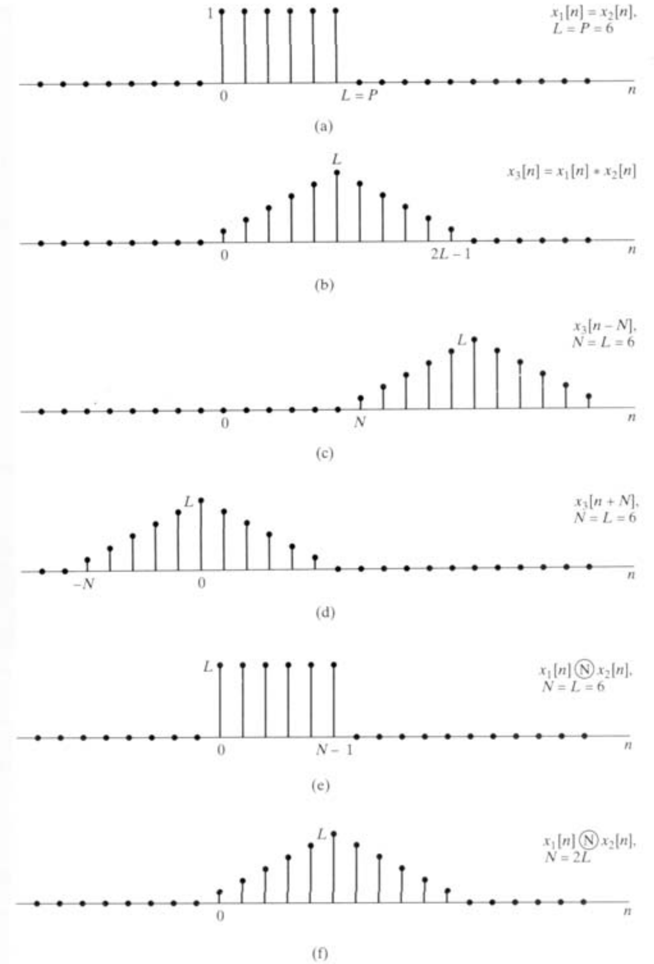


Figure 8.18 Illustration that circular convolution is equivalent to linear convolution followed by aliasing. (a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved. (b) The linear convolution of $x_1[n]$ and $x_2[n]$. (c) $x_3[n-N]$ for $N=6$. (d) $x_3[n+N]$ for $N=6$. (e) $x_1[n]$ ⑥ $x_2[n]$, which is equal to the sum of (b), (c), and (d) in the interval $0 \le n \le 5$. (f) $x_1[n]$ ⑫ $x_2[n]$.

- How to do it?

  (1) Compute the *N*-point DFT of $x_1[n]$ and $x_2[n]$ separately

  → $X_1[k]$ and $X_2[k]$

  (2) Compute the product $X_3[k] = X_1[k]X_2[k]$

  (3) Compute the *N*-point IDFT of $X_3[k]$ → $x_3[n]$

- Problems: (a) Aliasing

  (b) *Very* long sequence

- **Aliasing**

  $x_1[n]$, length *L* (nonzero values)

  $x_2[n]$, length *P*

In order to avoid aliasing, $N \geq L + P - 1$

(What do we mean avoid aliasing? The preceding procedure is *circular* convolution but we want *linear* convolution. That is, $x_3[n]$ equals to the linear convolution of $x_1[n]$ and $x_2[n]$)

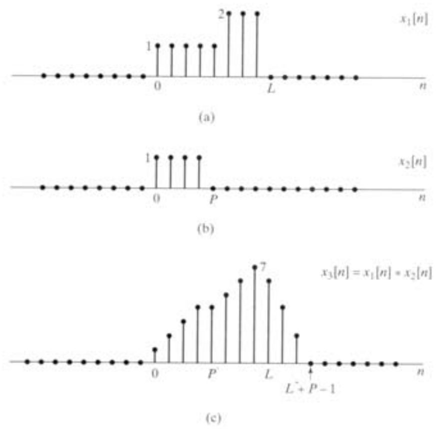


Figure 8.19 An example of linear convolution of two finite-length sequences.

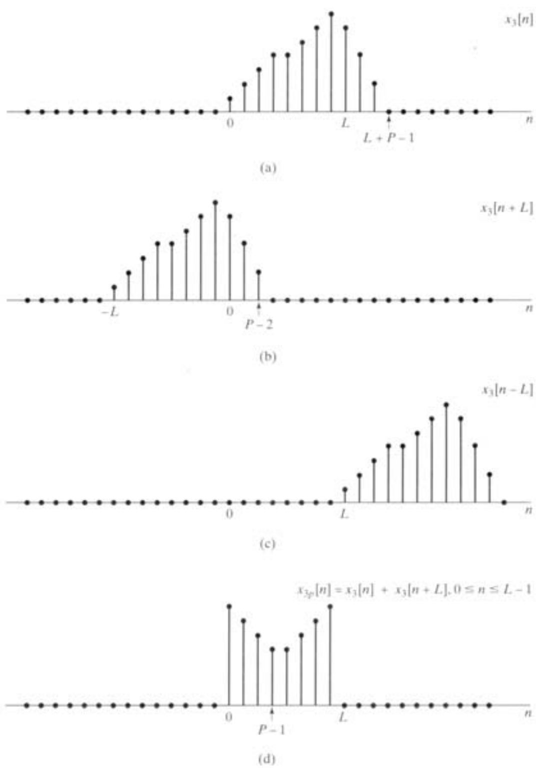


Figure 8.20 Interpretation of circular convolution as linear convolution followed by aliasing for the circular convolution of the two sequences $x_1[n]$ and $x_2[n]$ in Figure 8.19.

$x_1[n]$ pad with zeros → length N
$x_2[n]$ pad with zeros → length N

Interpretation: (Why call it aliasing?)

$X_3[k]$ has a (time domain) bandwidth of size $L+P-1$

(That is, the nonzero values of $x_3[n]$ can be at most $L+P-1$)

Therefore, $X_3[k]$ should have at least $L+P-1$ samples. If the sampling rate is insufficient, aliasing occurs on $x_3[n]$.

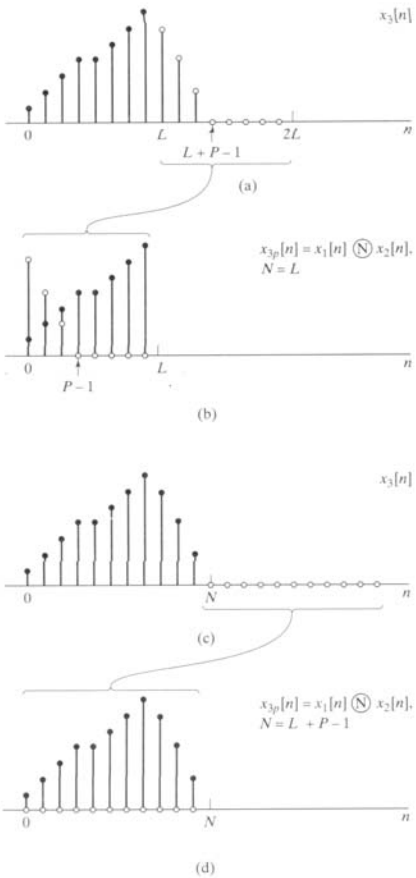


Figure 8.21 Illustration of how the result of a circular convolution "wraps around." (a) and (b) $N = L$, so the aliased "tail" overlaps the first $(P-1)$ points. (c) and (d) $N = (L+P-1)$, so no overlap occurs.

- **Very long sequence** (FIR filtering)
  - Block convolution
- Method 1 – **overlap and add**

Partition the long sequence into sections of shorter length.

For example, the filter impulse response $h[n]$ has finite length P and the input data $x[n]$ is nearly "infinite".

Let $x[n] = \sum_{r=0}^{\infty} x_r[n - rL]$ where $x_r[n] = \begin{cases} x[n + rL], & 0 \le n \le L\text{-}1 \\ 0, & \text{otherwise} \end{cases}$

The system (filter) output is a linear convolution:

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL] \text{ where } y_r[n] = x_r[n] * h[n]$$

Remark: The convolution length is $L + P - 1$. That is, the $L + P - 1$ point DFT is used. $y_r[n]$ has $L + P - 1$ data points; among them, (*P*-1) points should be added to the next section.

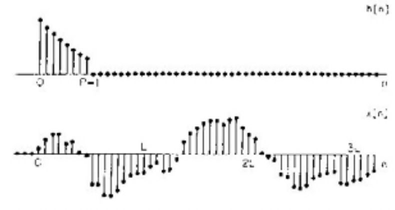


Figure 8.22 Finite-length impulse response $h[n]$ and indefinite-length signal $x[n]$ to be filtered.

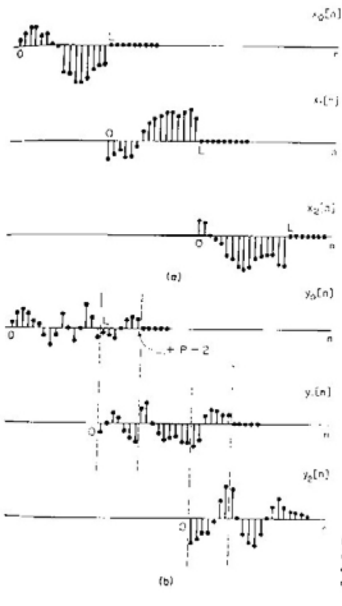


Figure 8.23 (a) Decomposition of $x[n]$ in Fig. 8.22 into nonoverlapping sections of length L. (b) Result of convolving each section with $h[n]$.

This is called **overlap-add method**.

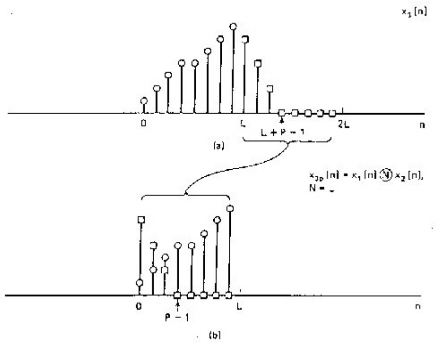
(*Key:* The input data are partitioned into *nonoverlapping* sections → the section outputs are overlapped and added together.)

- Method 2 – **overlap and save**

Partition the long sequence into overlapping sections.

After computing DFT and IDFT, throw away some (incorrect) outputs.

For each section (length *L*, which is also the DFT size), we want to retain the correct data of length $(L-(P-1))$ points



Let $h[n]$, length *P*

$x_r[n]$, length *L* (*L*>*P*)

Then, $y_r[n]$ contains (*P*-1) incorrect points at the beginning.

Therefore, we divide into sections of length *L* but each section overlaps the preceding section by (*P*-1) points.

$$x_r[n] = x[n + r(L - P + 1) - (P - 1)],\ 0 \le \text{n} \le \text{L - 1}$$

This is called **overlap-save method**.

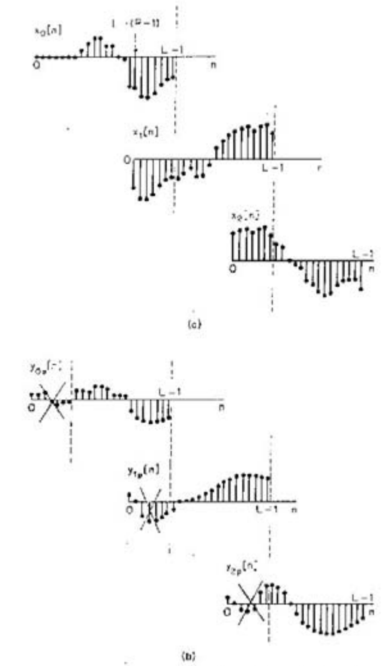


Figure 8.24 (a) Decomposition of x[n] in Fig. 8.22 into overlapping sections of length L. (b) Result of convolving each section with h[n]. The portions of each filtered section to be discarded in forming the linear convolution are indicated.