Filter Design

✧ Introduction

- Filter – An important class of LTI systems
- We discuss frequency-selective filters mostly: LP, HP, …
- We concentrate on the design of ***causal*** filters.
- Three stages in filter design:
  - Specification: application dependent
  - "Design": approximate the given spec using a causal discrete-time system
  - Realization: architectures and circuits (IC) implementation
- IIR filter design techniques
- FIR filter design techniques

Frequency domain specifications

Magnitude: $|H(e^{j\omega})|$, *Phase*: $\angle H(e^{j\omega})$

Ex., Low-pass filter: Passband , Transition, Stopband

Frequencies: Passband cutoff ω_p

Stopband cutoff ω_s

Transition bandwidth $\omega_s - \omega_p$

Error tolerance δ_1, δ_2

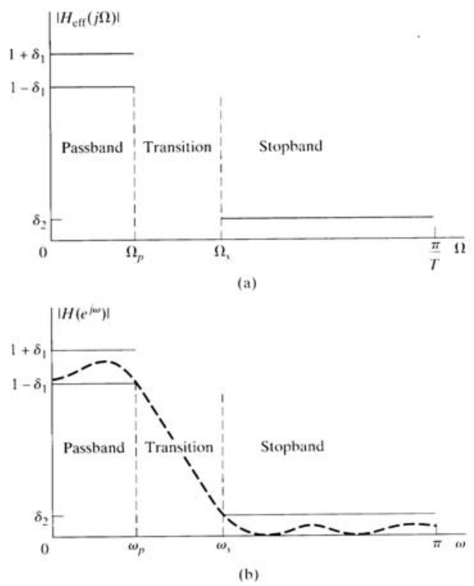


Figure 7.2 (a) Specifications for effective frequency response of overall system in Figure 7.1 for the case of a lowpass filter. (b) Corresponding specifications for the discrete-time system in Figure 7.1.

✧ Analog Filters

● Butterworth Lowpass Filters

- Monotonic magnitude response in the passband and stopband
- The magnitude response is maximally flat in the passband.

  For an Nth-order lowpass filter

  $\Rightarrow$ The first (2N-1) derivatives of $|H_c(j\Omega)|^2$ are zero at $\Omega = 0$.

  $$|H_c(j\Omega)|^2 = \frac{1}{1+(\frac{j\Omega}{j\Omega_c})^{2N}}$$

  N: filter order

  Ω_c: 3-dB cutoff frequency (magnitude = 0.707)

- Properties

  (a) $|H_c(j\Omega)|_{\Omega=0} = 1$

  (b) $|H_c(j\Omega)|^2{}_{\Omega=\Omega_c} = 1/2$ or $|H_c(j\Omega)|_{\Omega=\Omega_c} = 0.707$

  (c) $|H_c(j\Omega)|^2$ is monotonically decreasing (of Ω)

  (d) $N \to \infty$ ➔ $|H_c(j\Omega)| \to$ ideal lowpass

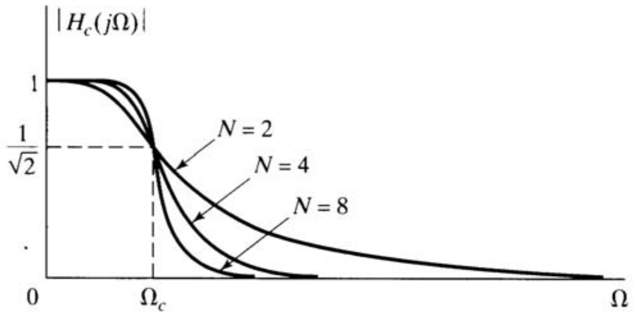


Figure B.2 Dependence of Butterworth magnitude characteristics on the order *N*.

- Poles

$$H_c(s)H_c(-s) = \frac{1}{1+(\frac{s}{j\Omega_c})^{2N}}$$

Roots: $s_k = (-1)^{\frac{1}{2N}}(j\Omega_c) = \Omega_c e^{j\frac{\pi}{2N}(2k+N-1)}, \quad k = 0,1,\ldots,2N-1$

(a) $2N$ poles in pairs: $s_k, -s_k$ symmetric w.r.t. the imaginary axis; never on the imaginary axis. If N odd, poles on the real axis.

(b) Equally spaced on a circle of radius Ω_c

(c) $H_c(s)$ causal, stable ← all poles on the left half plane

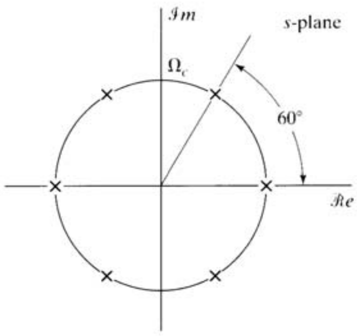


Figure B.3 s-plane pole locations for a third-order Butterworth filter.

- Usage (There are only two parameters N, Ω_c)

Given specifications $\varepsilon, \Omega_p, \delta_2, \Omega_s$ → N, Ω_c

$$|H(j\Omega)|^2 = \frac{1}{1+(\frac{\Omega}{\Omega_c})^{2N}} = \frac{1}{1+\varepsilon^2(\frac{\Omega}{\Omega_p})^{2N}}$$

Thus, $|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2}$ at $\Omega = \Omega_p$ => $\Omega_c = \Omega_p \big/ \varepsilon^{\frac{1}{N}}$

At $\Omega = \Omega_s$, $|H(j\Omega)|^2_{\Omega_x} = \delta_2^2 = \frac{1}{1+\varepsilon^2(\Omega_s/\Omega_p)^{2N}}$ $\qquad N = \frac{\log[(\frac{1}{\delta_2})^2 - 1]}{2\log(\Omega_s/\Omega_c)}$

Chebyshev Filters

- **Type I:** Equiripple in the passband; monotonic in the stopband

  Type II: Equiripple in the stopband; monotonic in the passband

- Same *N* as the Butterworth filter, it would have a sharper transition band. (A smaller *N* would satisfy the spec.)

- **Type I:**

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)}$$

where $V_N(x)$ is the *N*th-order Chebyshev polynominal

$$V_N(x) = \cos(N\cos^{-1}(x)),\ 0 < V_N(x) < 1 \ \textit{for}\ 0 < x < 1$$

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x)$$

$V_N(x)|_{x=1} = 1$ for all N

<The first several Chebyshev polynominals>

N	$V_N(x)$
0	1
1	x
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$

- Properties (Type I)

(a) $$|H_c(j\Omega)|^2_{\Omega=0} = \begin{cases} 1, & \text{if N odd} \\ \dfrac{1}{1+\varepsilon^2}, & \text{if N even} \end{cases}$$

(b) The magnitude squared frequency response oscillates between 1 and $\dfrac{1}{1+\varepsilon^2}$ within the passband:

$$|H_c(j\Omega)|^2_{\Omega=\Omega_c} = \frac{1}{1+\varepsilon^2} \quad \text{at} \quad \Omega = \Omega_c$$

(c) $|H_c(j\Omega)|^2$ is monotonic outside the passband.

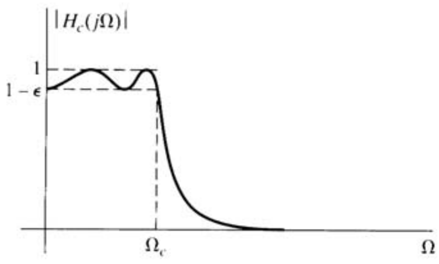


Figure B.4 Type I Chebyshev lowpass filter approximation.

- Poles (Type I)

  On the ellipse specified by the following:

  Length of minor axis = $2a\Omega_c$, $a = \frac{1}{2}\left(\alpha^{\frac{1}{N}} - \alpha^{-\frac{1}{N}}\right)$

  Length of major axis = $2b\Omega_c$, $b = \frac{1}{2}\left(\alpha^{\frac{1}{N}} + \alpha^{-\frac{1}{N}}\right)$

  and $\alpha = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$

  (a) Locate equal-spaced points on the major circle and minor circle with angle

  $$\Phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{N},\ k = 0,1,\cdots,N-1$$

  (b) The poles are (x_k, y_k): $x_k = a\Omega_c \cos\phi_k$, $y_k = b\Omega_c \sin\phi_k$

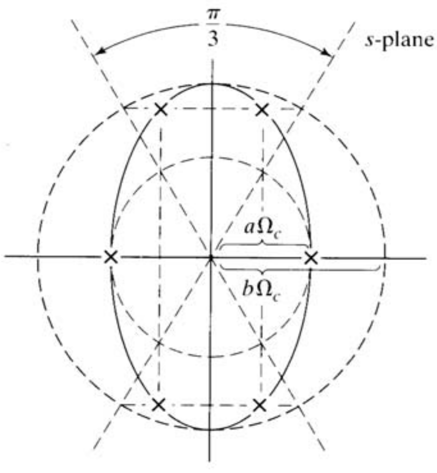


Figure B.5 Location of poles for a third-order type I lowpass Chebyshev filter.

- **Type II:**

$$|H_a(j\Omega)|^2 = \frac{1}{1+[\varepsilon^2 V_N^2(\Omega_c/\Omega)]^{-1}}$$ has both poles and zeros.

- Usage (There are only two parameters N, Ω_c)

  Given specifications $\varepsilon, \Omega_p, \delta_2, \Omega_s$ ➔ N, Ω_c

$\Omega_c = \Omega_p$

$$N = \frac{\log[(\sqrt{1-\delta_2^2} + \sqrt{1-\delta_2^2(1+\varepsilon^2)})/\varepsilon\delta_2]}{\log[(\Omega_s/\Omega_p) + \sqrt{(\Omega_s/\Omega_p)^2 - 1}]}$$

$$= \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} \qquad \left(\delta_2 = \frac{1}{\sqrt{1+\delta^2}}\right)$$

Elliptic Filters

- Equiripple at both the passband and the stopband
- Optimum: smallest $(\Omega_s - \Omega_p)$ at the same N

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N^2(\Omega/\Omega_p)}$$

where $U_N(x)$: Jacobian elliptic function (Very complicated! Skip!)

- Usage (There are only two parameters N, Ω_c)

  Given specifications $\varepsilon, \Omega_p, \delta_2, \Omega_s$ ➔ N, Ω_c

$$N = \frac{K(\Omega_p/\Omega_s)K(\sqrt{1-(\varepsilon^2/\delta^2)})}{K(\varepsilon/\delta)K(\sqrt{1-(\Omega_p/\Omega_s)^2})} \qquad \left(\delta_2 = \frac{1}{\sqrt{1+\delta^2}}\right)$$

where $K(x)$ is the complete elliptic integral of the first kind

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}$$

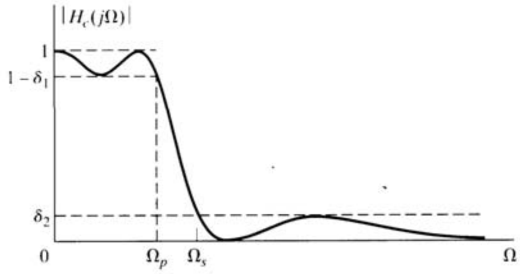


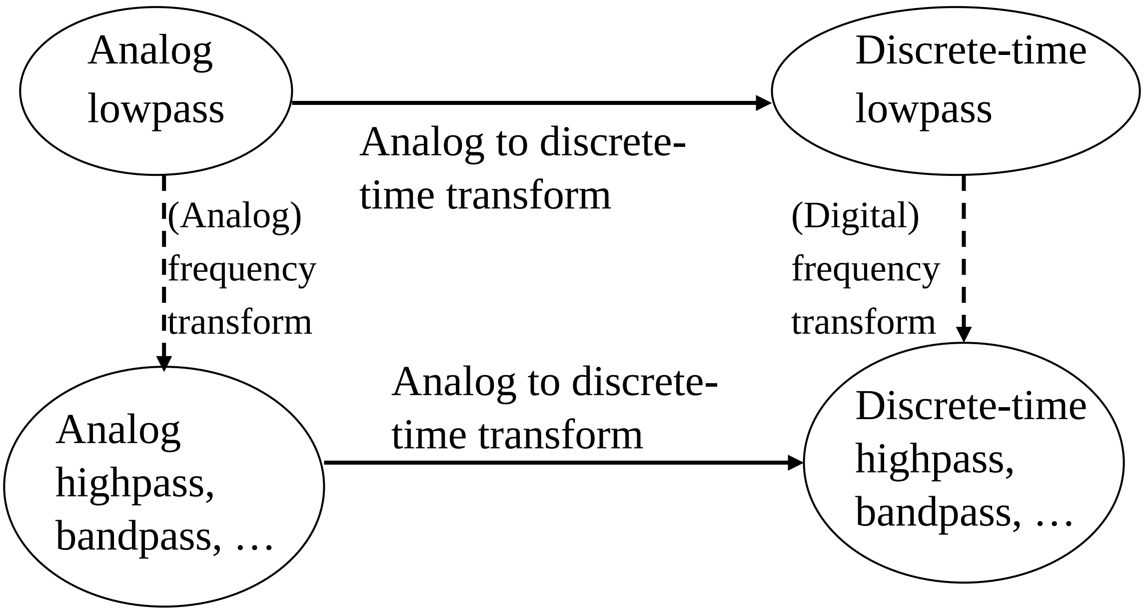
Figure B.6 Equiripple approximation in both passband and stopband.

Remark: The drawback of the elliptic filters: They have more nonlinear phase response in the passband than a comparable Butterworth filter or a Chebyshev filter, particularly, near the passband edge.

✧ Design Digital IIR Filters from Analog Filters

- Why based on analog filters?
  - Analog filter design methods have been well developed.
  - Analog filters often have simple *closed-from* design formulas.

    ← Direct digital filter design methods often don't have *closed-form* formulas.
- There are two types of transformations
  - Transformation from analog to discrete-time
  - Transformation from one type filter to another type (so called *frequency transformation*)



- Methods in analog to discrete-time transformation
  - Impulse invariance
  - Bilinear transformation
  - Matched-z transformation
- Desired properties of the transformations
  - Imaginary axis of the s-plane → The unit circle of the z-plane
  - Stable analog system → Stable discrete-time system

    (Poles in the left s-plane → Poles inside the unit circle)

- Steps in the design

  (1) Digital specifications → Analog specifications

  (2) Design the desired analog filter

  (3) Analog filter → Discrete-time filter

Impulse Invariance

-- Sampling the impulse of a continuous-time system

$$h[n] = T_d h_c(nT_d)$$
$$= T_d h_c(t)\,|_{t=nT_d}$$

T_d : Sampling period

← *Important:* to avoid aliasing

← Does not show up in the final discrete formula if we start from the digital specifications, ...

- Frequency response

Sampling in time ➔ Sifted duplication in frequency

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$

If $H_c(j\Omega)$ is band-limited and $f_d = 1/T_d$ is higher than the Nyquist sampling frequency (no aliasing)

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \qquad |\omega| \le \pi$$

Remark: This is not possible because the IIR analog filter is typically not bandlimited.

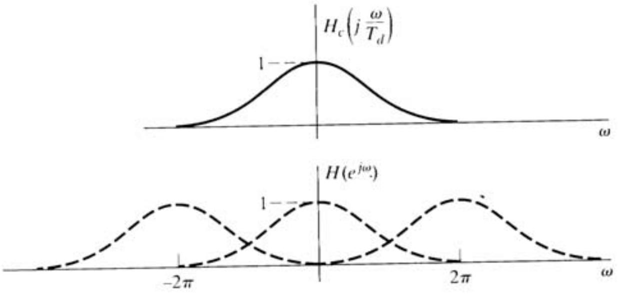


Figure 7.3 Illustration of aliasing in the impulse invariance design technique.

Approach 1: Sampling $h[n]$

Approach 2: Map $H_c(s)$ to $H(z)$ because we need $H(z)$ to implement a digital filter anyway.

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\begin{aligned} h[n] &= T_d h_c(nT_d) \\ &= T_d \sum_{K=1}^{N} A_k e^{s_k n T_d} u[n] \\ &= \sum_{K=1}^{N} (T_d A_k)(e^{s_k T_d})^n u[n] \end{aligned}$$

$$H(z) = \sum_{K=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Essentially, **factorize and map:**

Analog pole

⇓

Discrete-time pole

Remarks: (1) Stability is preserved:

LHS poles → poles inside the unit circle

(2) No simple correspondence for zeros

***Design Example:* Low-pass filter**

Using Butterworth continuous-time filter

Given specifications in the digital domain

"-1 dB in passband" and "-15 dB in stopband"

$$0.89125 \leq |H(e^{j\omega})| \leq 1, \qquad 0 \leq |\omega| \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.17783, \qquad 0.3\pi \leq |\omega| \leq \pi$$

Step 1: Convert the above specifications to the analog domain

(Assume "negligible aliasing")

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T_d}) \qquad |\omega| \le \pi$$

$$0.89125 \le |H(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi/T_d$$

$$|H(j\Omega)| \le 0.17783, \qquad 0.3\pi/T_d \le |\Omega| \le \pi/T_d$$

Step 2: Design a Butterworth filter that satisfies the above specifications. That is, select proper N, Ω_c.

$$\begin{cases} |H_c(j\frac{0.2\pi}{T_d})| \ge 0.89125 \\ |H_c(j\frac{0.3\pi}{T_d})| \le 0.17783 \end{cases}$$

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

Thus, $$\begin{cases} 1+\left(\frac{0.2\pi}{T_d\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \\ 1+\left(\frac{0.3\pi}{T_d\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2 \end{cases}$$

➔ $N = 5.8858, \quad T_d\Omega_c = 0.70474$

➔ (Taking integer) $N = 6, \quad T_d\Omega_c = 0.7032$

(Meet passband spec. exactly; overdesign at stopband)

<Case 1: Assume $T_d = 1 \Rightarrow \; s_k = \Omega_c e^{j\frac{\pi}{2N}(2k+N-1)}$

<Case 2: Assume $T_d \ne 1 \Rightarrow \; s_k = \left(\frac{0.7032}{T_d}\right) e^{j\frac{\pi}{2N}(2k+N-1)}$

$$H_c(s) = \frac{0.12093}{(s^2+0.365s+0.495)(s^2+0.995s+0.495)(s^2+1.359s+0.495)}$$

Step 3: Convert analog filter to discrete-time

Analog pole s_k

$\Downarrow$

Discrete-time pole e^{s_k}

<Case 1: Assume $T_d = 1 \Rightarrow$ $z_k = \exp\left[0.7032\, e^{j\frac{\pi}{2N}(2k+N-1)}\right]$

<Case 2: Assume $T_d \neq 1 \Rightarrow$ $z_k = \exp\left[T_d\left(\frac{0.7032}{T_d}\right)e^{j\frac{\pi}{2N}(2k+N-1)}\right]$

They are identical! (In general, this is true.)

$$H(z) = \frac{0.287 - 0.447z^{-1}}{1 - 1.297z^{-1} + 0.695z^{-2}} + \frac{-2.143 + 1.145z^{-1}}{1 - 1.069z^{-1} + 0.370z^{-2}} + \frac{1.856 - 0.630z^{-1}}{1 - 0.997z^{-1} + 0.257z^{-2}}$$

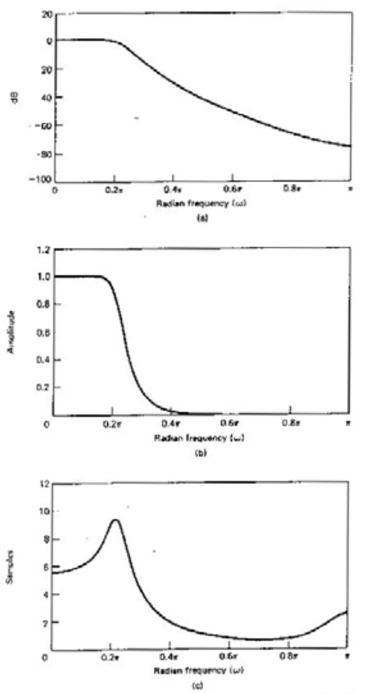


Figure 7.6 Frequency response of sixth-order Butterworth filter transformed by impulse invariance. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

Remark: (1) Impulse invariance method has a precise control on the shape of the time signal.

(2) Except for aliasing, the shape of the frequency response is preserved.

- **Bilinear Transform**

  - Avoid aliasing but distort the frequency response – uneven stretch of the frequency axis.

  - $$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \text{ or } z = \frac{1+ {sT_d}/{2}}{1- {sT_d}/{2}}$$

    $$H_c(s) \to H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

    Note: $j\Omega$ axis on the s-plane → unit circle on the z-plane

    LHS of the s-plane → Interior of the unit circle on the z-plane

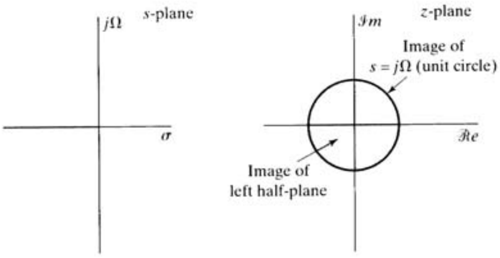


Figure 7.6 Mapping of the *s*-plane onto the *z*-plane using the bilinear transformation.

  - How the $j\Omega$ axis is mapped to the unit circle?

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\Bigg|_{Z=e^{j\omega}} = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right)$$

$$= \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}\left(j\sin\frac{\omega}{2}\right)}{2e^{-j\omega/w}\left(\cos\frac{\omega}{2}\right)}\right]$$

$$= \frac{2j}{T_d}\tan\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \text{ or } \omega = 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right)$$

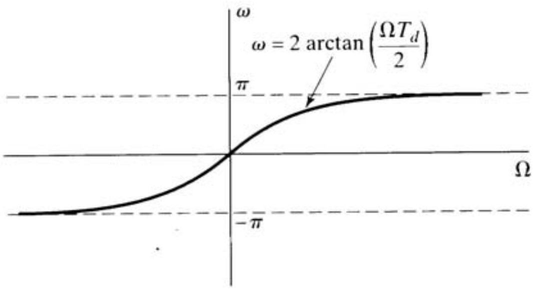


Figure 7.7 Mapping of the continuous-time frequency axis onto the discrete-time frequency axis by bilinear transformation.

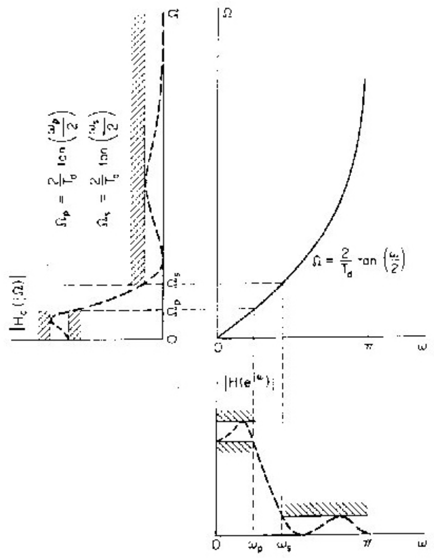


Figure 7.10 Frequency warping inherent in the bilinear transformation of a continuous-time lowpass filter into a discrete-time lowpass filter. To achieve the desired discrete-time cutoff frequencies, the continuous-time cutoff frequencies must be prewarped as indicated.

Problem in design – nonlinear distortion in magnitude and phase

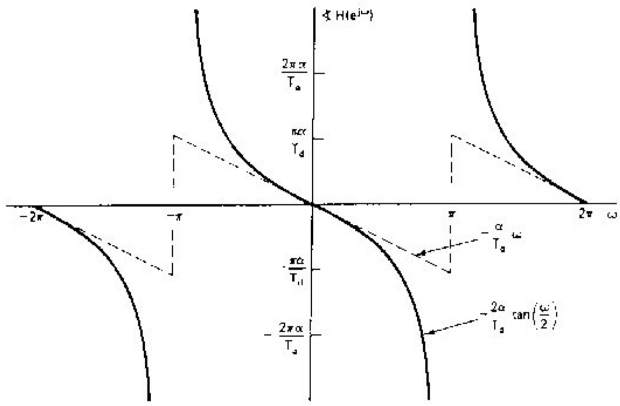


Figure 7.11 Illustration of the effect of the bilinear transformation on a linear phase characteristic. (Dashed line is linear phase and solid line is phase resulting from bilinear transformation.)

- Steps in the design

  (1) Digital specifications to analog specifications: prewarp

  (2) Design the desired analog filter

  (3) Analog filter to discrete-time filter: bilinear transform

Design Example: **Lowpass filter**

Using Butterworth continuous-time filter

Given specifications in the digital domain (same as the previous ex.)

$$0.89125 \le |H(e^{j\omega})| \le 1, \qquad 0 \le |\omega| \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.17783, \qquad 0.3\pi \le |\omega| \le \pi$$

Step 1: Prewarp $\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right)$

Passband freq. $\Omega_p = \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right)$

Stopband freq. $\Omega_s = \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right)$

Let $T_d = 1$ since T_d will disappear after "analog to discrete".

Step 2: Design a Butterworth filter -- select proper N, Ω_c.

$$\begin{cases} |H_c(j2\tan(0.1\pi))| \ge 0.89125 \\ |H_c(j2\tan(0.15\pi))| \le 0.17783 \end{cases}$$

Because $|H_c(j\Omega)|^2 = \dfrac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$

➔
$$\begin{cases} 1+\left(\dfrac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.89125}\right)^2 \\ 1+\left(\dfrac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.17783}\right)^2 \end{cases}$$

➔ $N = 5.30466,$

➔ $N = 6, \quad T_d\Omega_c = 0.76622$

(Meet stopband spec. exactly; exceed passband spec.)

$$H_c(s) = \frac{0.20238}{(s^2+0.3996s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)}$$

Step 3: Convert analog filter to discrete-time

$$H_c(s) \rightarrow H(z) = H_c\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$$

$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})}$$

$$\times \frac{1}{(1-0.9044z^{-1}+0.2155z^{-2})}$$

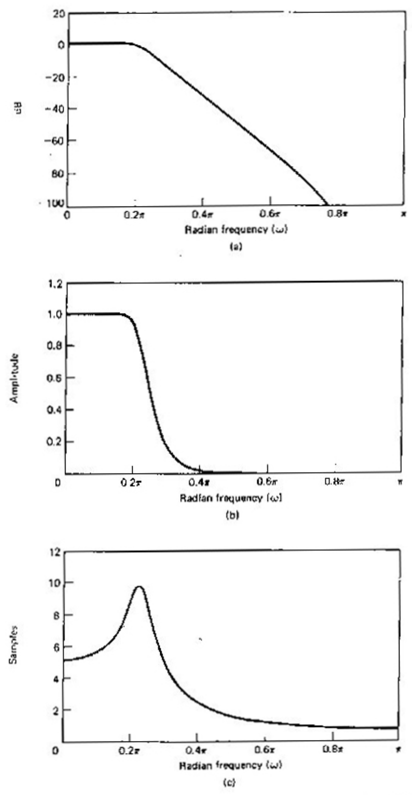


Figure 7.13 Frequency response of sixth-order Butterworth filter transformed by bilinear transform. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

Remarks: (1) Bilinear transforms warps frequency values but preserves the magnitude. Therefore, the discrete-time Butterworth filter still has the maximal flat property; Chebyshev and Ellipic filters have equal ripple property.

(2) Although we may obtain $H_c(s)$ in closed form, it is often difficult to find the locations of poles and zeros of $H(z)$ from $H_c(s)$ directly.

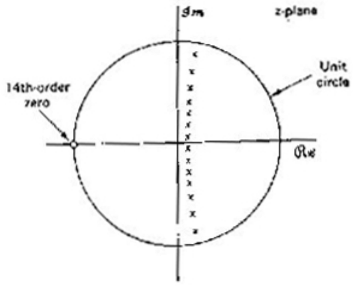
Bilinear Transform Design Example using 4 analog filters:

$$\begin{cases} 0.99 \le \left|H\left(e^{j\omega}\right)\right| \le 1.01, & |\omega| \le 0.4\pi \\ \left|H\left(e^{j\omega}\right)\right| \le 0.001, & 0.6\pi \le |\omega| \le \pi \end{cases}$$

Butterworth: 14th order

Chebyshev I and II: 8th order

Elliptic: 6th order



Butterworth

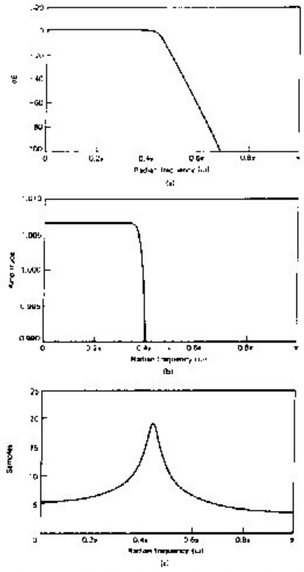
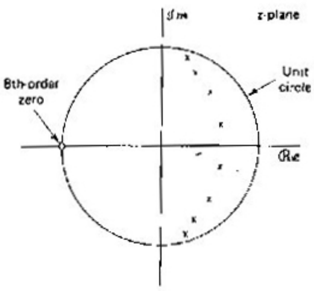


Figure 7.14 Frequency response of 14th-order Butterworth filter in Example [illegible]. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

Butterworth



Chebyshev type-I

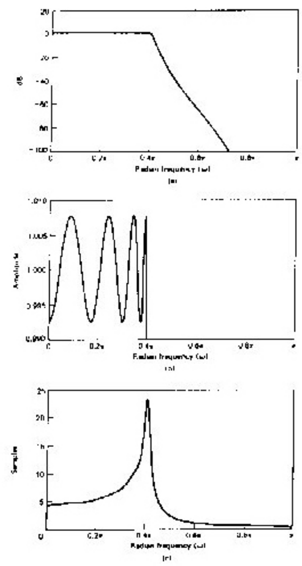
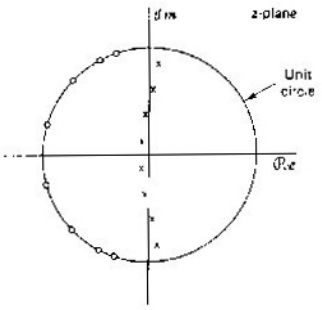


Figure 7.16 Frequency response of eighth-order Chebyshev type I filter in Example 7.6. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

Chebyshev type-I



Chebyshev type-II

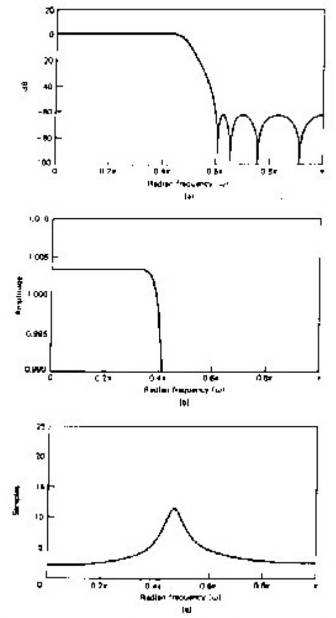
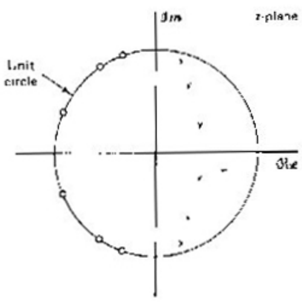


Figure 7.17 Frequency response of eighth-order Chebyshev type II filter in Example 7.6. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

Chebyshev type-II



Elliptic

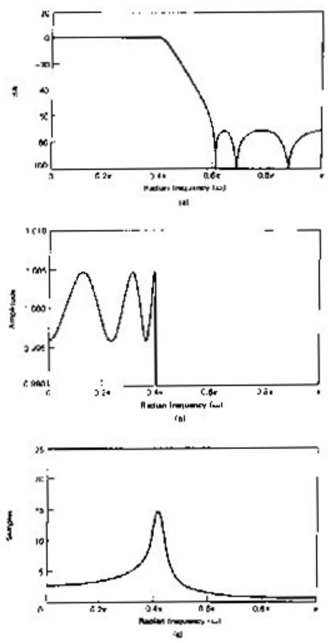


Figure 7.19 Frequency response of [illegible]-order elliptic filter in Example 7.7. (a) Log magnitude in dB. (b) Detailed plot of magnitude in passband. (c) Group delay.

Elliptic

● Frequency Transformation

-- Transform one-type (often lowpass) filter to another type.

Typically, we first design a *frequency-normalized prototype lowpass* filter. Then, use an algebraic transformation to derive the desired lowpass, high pass , …, filters from the prototype lowpass filter.

<Prototype filter> → <Desired filter>

Z → z

$$Z^{-1} = G(z^{-1})$$

$$H_{lp}(Z)\Big|_{Z^{-1}=G(z^{-1})} \rightarrow H(z)$$

Typically, this transform is made of all-pass like factors

$$G(z^{-1}) = \pm\prod_{K=1}^{N}\left(\frac{z^{-1}-\alpha_k}{1-\alpha_k z^{-1}}\right)$$

Remarks: The desired properties of *G*(.) are

(1) transforms the unit circle in *Z* to the unit circle in *z*,

(2) transforms the interior of the unit circle in *Z* to the interior of the unit circle in *z*,

(3) *G*(.) is rational.

Example: **Lowpass to lowpass** (with different passband and stopband frequency, but magnitude is not changed)

$$Z^{-1} = \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$$

Check the relationship between θ (the Z filter) and ω (the z filter). α is a parameter. Different α offers different "shapes" of the transformed filters in ω.

$$e^{-j\theta} = \frac{e^{-j\omega}-\alpha}{1-\alpha e^{-j\omega}}$$

$$\omega = \tan^{-1}\left[\frac{(1-\alpha^2)\sin\theta}{2\alpha+(1+\alpha^2)\cos\theta}\right]$$

If θ_p is to be mapped to ω_p, then

$$\alpha = \frac{\sin[(\theta_p-\omega_p)/2]}{\sin[(\theta_p+\omega_p)/2]}$$

■ Various Digital to Digital Transformations

Filter Type	Transformation	Associated Design Formulas
Lowpass	$Z^{-1}=\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$	$\alpha=\frac{\sin\left(\frac{\theta_p-\omega_p}{2}\right)}{\sin\left(\frac{\theta_p+\omega_p}{2}\right)}$ $\omega_p = desired\ cutoff\ freq.$
Highpass	$Z^{-1}=-\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha=-\frac{\cos\left(\frac{\theta_p-\omega_p}{2}\right)}{\cos\left(\frac{\theta_p+\omega_p}{2}\right)}$ $\omega_p = desired\ cutoff\ freq.$
Bandpass	$Z^{-1}=-\frac{z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+\frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+1}$	$\alpha=\frac{\cos\left(\frac{\omega_{p2}+\omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)}$ $k=\cot\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} = desired\ lower\ cutoff\ freq.$ $\omega_{p2} = desired\ upper\ cutoff\ freq.$
Bandstop	$Z^{-1}=\frac{z^{-2}-\frac{2\alpha}{1+k}z^{-1}+\frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2}-\frac{2\alpha}{1+k}z^{-1}+1}$	$\alpha=\frac{\cos\left(\frac{\omega_{p2}+\omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)}$ $k=\tan\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} = desired\ lower\ cutoff\ freq.$ $\omega_{p2} = desired\ upper\ cutoff\ freq.$

✧ Design of FIR Filters by Windowing

- Why FIR filters?

  -- Always stable

  -- *Exact* linear phase

  -- Less sensitive to inaccurate coefficients

  <Disadvantage> Higher complexity (storage, multiplication) due to higher orders

- Design Methods

  (1) Windowing

  (2) Frequency sampling

  (3) Computer-aided design

  Remark: No meaningful analog FIR filters

- Windowing technique advantages

  -- Simple

  -- Pick up a “segment” (window) of the ideal (infinite) $h_d[n]$

  -- Filter order = window length = (M+1)

General form: $h[n] = h_{\mathrm{d}}[n]w[n]$

Filter impulse response = Desired response x Window

Example: **Rectangular window**

Window shape: $w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$

➔ $h[n] = \begin{cases} h_{\mathrm{d}}[n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$

- Because the filter specifications are (often) given in the frequency domain $H_d(e^{j\omega})$.

  We take the inverse DTFT to obtain $h_d[n]$.

$$h_{\mathrm{d}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{d}}\left(e^{j\omega}\right) \cdot e^{j\omega n} d\omega$$

or, $$H_{\mathrm{d}}\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_{\mathrm{d}}[n] e^{-j\omega n}$$

Now, because of the inclusion of $w[n]$,

$$H\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_{\mathrm{d}}\left(e^{j\theta}\right)\cdot W\left(e^{j(\omega-\theta)}\right)d\theta \quad \text{(A periodic convolution)}$$

That is, $H(e^{j\omega})$ is "smeared" version of $H_d(e^{j\omega})$.

Why $W(e^{j\omega})$ cannot be $\delta(e^{j\omega})$? (If so, $H(e^{j\omega}) = H_d(e^{j\omega})$!)

Parameters (to choose): (1) Window size (order of filter)

(2) Window shape

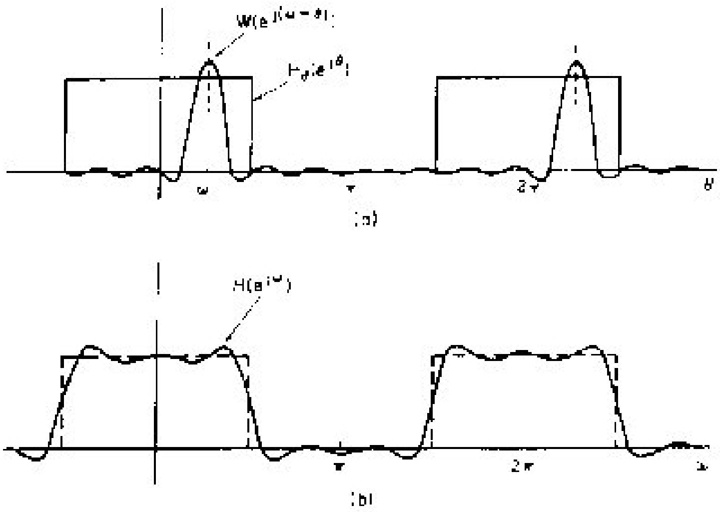


Figure 7.27 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

- **Rectangular Window**: $w[n]=\begin{cases}1, & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$

-- Narrow mainlobe

-- High sidelobe (Gibbs phenomenon)

-- Frequency response

$$\begin{aligned}W\left(e^{j\omega}\right)&=\sum_{n=0}^{M}1\cdot e^{-j\omega n}\\&=\frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}\\&=e^{-j\omega\frac{M}{2}}\frac{\sin\left[\omega\frac{(M+1)}{2}\right]}{\sin\left(\frac{\omega}{2}\right)}\end{aligned}$$

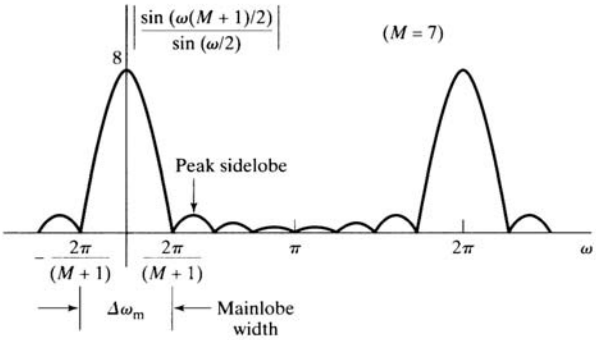


Figure 7.20 Magnitude of the Fourier transform of a rectangular window ($M = 7$).

-- Mainlobe ~ $\dfrac{4\pi}{M+1}$, $M \uparrow$, $W(e^{j\omega}) \to \delta(e^{j\omega})$

-- Peak sidelobe ~ -13 dB (lower than the mainlobe)

Area under each lobe remains constant with increasing *M*

→ Increasing M does not lower the (relative) amplitude of the sidelobe.

(Gibbs phenomemnon)

Remarks: For frequency selective filters (ideal lowpass, highpass, …),

narrow mainlobe → sharp transition

lower sidelobe → oscillation reduction

- ## Commonly Used Windows

  -- Sidelobe amplitude (area) vs. mainlobe width

  -- Closed form, easy to compute

Bartlett (triangular) Window:

$$w[n] = \begin{cases} \frac{2n}{M}, & 0 \le n \le \frac{M}{2} \\ 2 - \frac{2n}{M}, & \frac{M}{2} < n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hanning Window:

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\mathrm{n}}{\mathrm{M}}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hamming Window:

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\mathrm{n}}{\mathrm{M}}\right),, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Blackman Window:

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\mathrm{n}}{\mathrm{M}}\right) + 0.08\cos\left(\frac{2\mathrm{n}}{\mathrm{M}}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

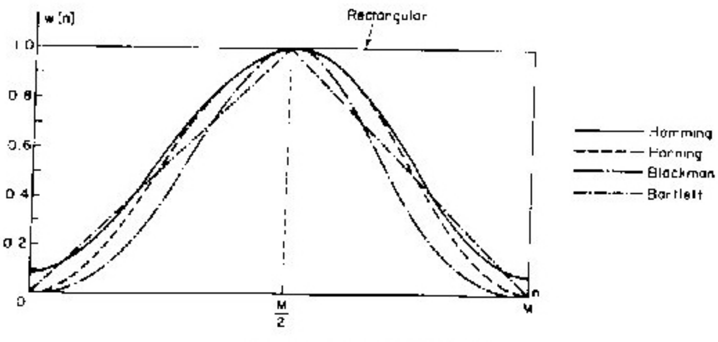


Figure 7.29 Commonly used windows.

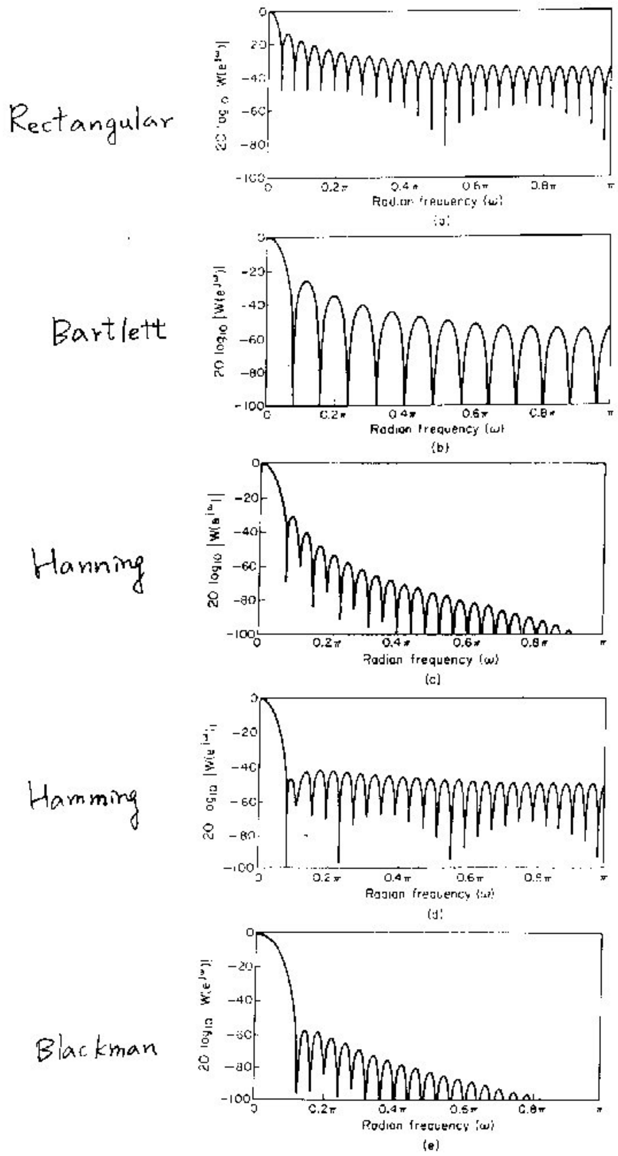


Figure 7.30 (continued) (c) Hanning. (d) Hamming. (e) Blackman.

- *Comparison of Commonly Used Windows*

Window Type	Peak Sidelobe Amplitude (Relative)	Approximate Width of Mainlobe	Equivalent Kaise Window β	Transition Width of Equivalent Kaiser Window
Rectangular	-13	$4\pi/(M+1)$	0	$1.81\pi/M$
Bartlett	-25	$8\pi/M$	1.33	$2.37\pi/M$
Hanning	-31	$8\pi/M$	3.86	$5.01\pi/M$
Hamming	-41	$8\pi/M$	4.86	$6.27\pi/M$
Blackman	-57	$12\pi/M$	7.04	$9.19\pi/M$

● Generalized Linear Phase Filters

-- We wish $H(e^{j\omega})$ be (general) linear phase.

<Window> Choose windows such that

$$w[n] = w[M-n], \quad 0 \le n \le M$$

That is, symmetric about *M*/2 (samples)

$$W\left(e^{j\omega}\right) = W_e\left(e^{j\omega}\right) \cdot e^{-j\omega\frac{M}{2}}$$, where $W_e\left(e^{j\omega}\right)$ is real.

<Desired filter> Suppose the desired filter is also generalized linear phase

$$H_d\left(e^{j\omega}\right) = H_e\left(e^{j\omega}\right) \cdot e^{-j\omega\frac{M}{2}}$$

<Filter> $H(e^{j\omega})$ is a periodic convolution of $H_d(e^{j\omega})$ and $W(e^{j\omega})$

$$\begin{aligned} H\left(e^{j\omega}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right) \cdot W_e\left(e^{j(\omega-\theta)}\right) \cdot e^{-j\theta\frac{M}{2}} e^{-j\frac{(\omega-\theta)M}{2}} d\theta \\ &= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right) \cdot W_e\left(e^{j(\omega-\theta)}\right) d\theta}_{A_e\left(e^{j\omega}\right)} \cdot e^{-j\omega\frac{M}{2}} \end{aligned}$$

$A_e\left(e^{j\omega}\right)$ is real.

Thus, $H(e^{j\omega})$ is also generalized linear phase.

Example: **Linear phase lowpass filter**

Ideal lowpass: $$H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega\frac{M}{2}}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Impulse response: $$h_{lp}[n] = \frac{\sin\left[\omega_c\left(n-\frac{M}{2}\right)\right]}{\pi\left(n-\frac{M}{2}\right)}$$

Designed filter: $$h[n] = \frac{\sin\left[\omega_c\left(n-\frac{M}{2}\right)\right]}{\pi\left(n-\frac{M}{2}\right)} \cdot w[n]$$

ω_c : 1/2 amplitude of $H\left(e^{j\omega}\right)$ = cutoff frequency of the dieal lowpass filter

Peak to the left of ω_c occurs at ~ 1/2 mainlobe width

-Peak to the right of ω_c occurs at ~ 1/2 mainlobe width

Transition bandwidth $\Delta\omega$ ~ mainlobe width- (smaller)

Peak approximation error: proportional to sidelobe area

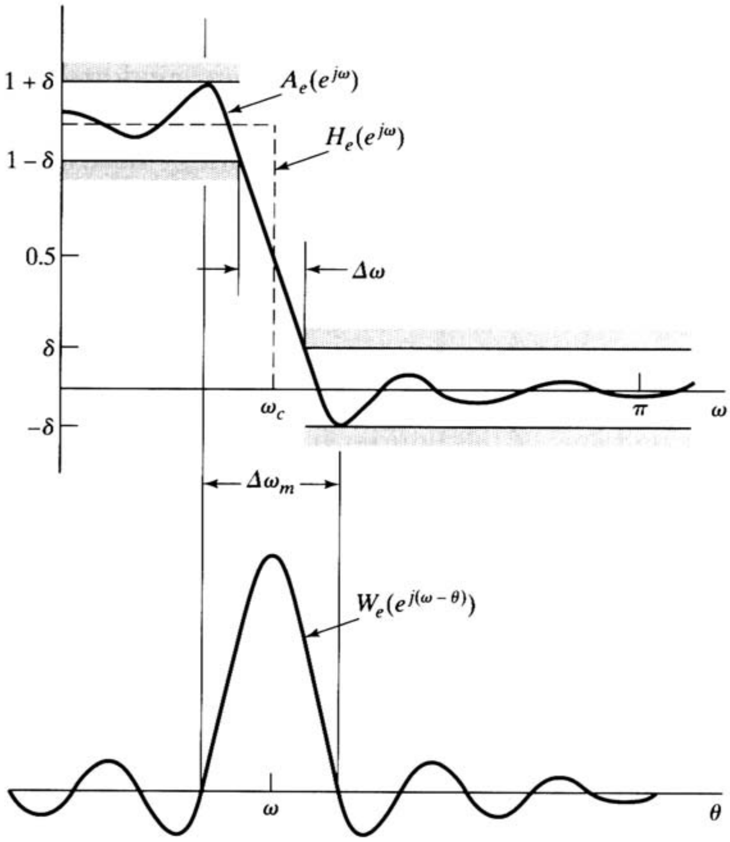


Figure 7.23 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

Kaiser Window

-- Nearly optimal trade-off between mainlobe width and sidelobe area

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\left(1-\left[{(n-\alpha)}/{\alpha}\right]^2\right)^{1/2}\right]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

where $I_0(\cdot)$: zeroth-order modified Bessel function of the first kind

α : $M/2$

β : shape parameter; $\beta = 0$, rectangular window

$\beta \uparrow$, mainlobe width $\uparrow$, sidelobe area $\downarrow$

-- $A \equiv -20 \cdot \log_{10} \delta$

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.0 & A < 21 \end{cases}$$

-- $\Delta\omega = \omega_s - \omega_p$ (stopband – passband)

$M = \dfrac{A-8}{2.285 \cdot \Delta\omega}$ (within +-2 over a wide range of $\Delta\omega$ and A)

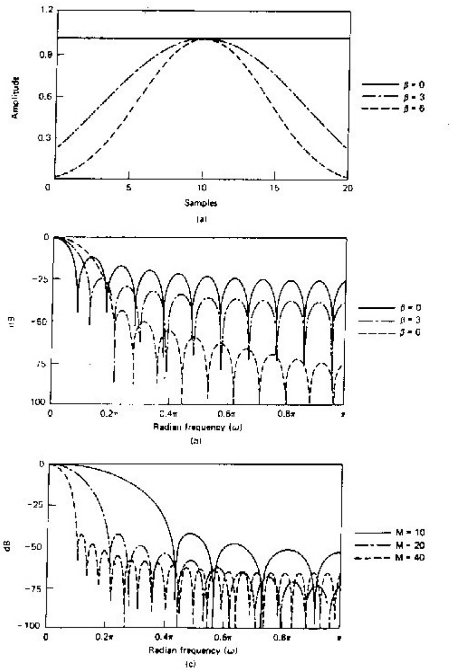


Figure 7.32 (a) Kaiser windows for $\beta = 0$, 3, and 6 and $M = 20$. (b) Fourier transforms corresponding to windows in (a). (c) Fourier transforms of Kaiser windows with $\beta = 6$ and $M = 10$, 20, and 40.

***Kaiser window example* – lowpass**

Specifications: $\delta_1 = \delta_2 = 0.001$

Ideal lowpass cutoff: $\omega_c = \frac{\omega_s + \omega_p}{2} = 0.5\pi$

Select parameters: $\begin{cases} \Delta\omega = \omega_s - \omega_p = 0.2\pi \\ A = -20\log_{10}\delta = 60 \end{cases} \rightarrow \begin{cases} \beta = 5.653 \\ M = 37 \end{cases}$

$$\alpha = M/2 = 18.5$$

This is a type II, linear phase (odd M, even symmetry) filter.

Approximation error: $|H_d(e^{j\omega})| - |H(e^{j\omega})|$

$$E_A(e^{j\omega}) = \begin{cases} 1 - A_e(e^{j\omega}), & 0 \le \omega < \omega_p \\ 0 - A_e(e^{j\omega}), & \omega_s < \omega \le \pi \end{cases}$$

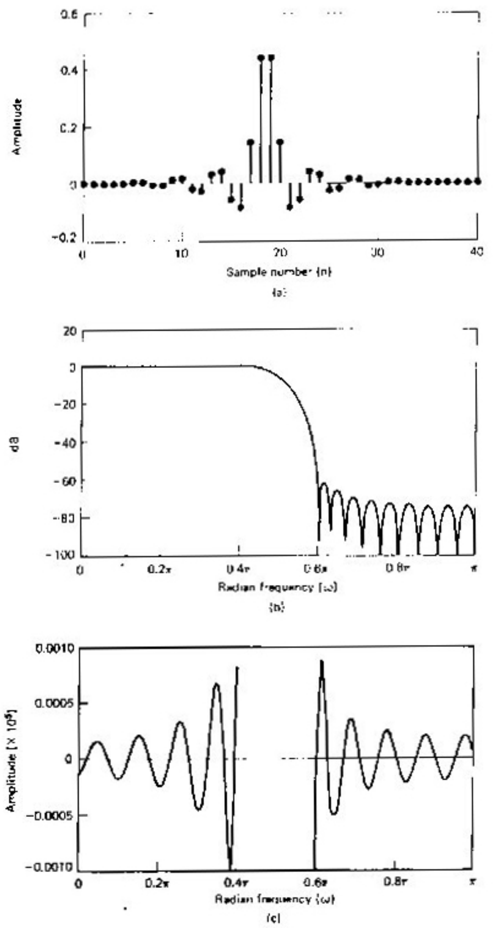


Figure 7.33 Response functions for Example 7.11. (a) Impulse response ($M = 37$). (b) Log magnitude. (c) Approximation error.

***Kaiser window example* – highpass**

Ideal highpass: $H_{\mathrm{hp}}\left(e^{j\omega}\right)=\begin{cases}0, & 0\le|\omega|<\omega_{\mathrm{c}}\\ e^{-j\omega\frac{M}{2}}, & \omega_{\mathrm{c}}<|\omega|\le\pi\end{cases}$

$$h_{\mathrm{hp}}[n]=\frac{\sin\pi\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)}-\frac{\sin\omega_{\mathrm{c}}\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)}$$

Specifications: $\delta_1=\delta_2=0.021$

Highpass cutoff: $\omega_{\mathrm{c}}=\frac{\omega_{\mathrm{s}}+\omega_{\mathrm{p}}}{2}=\frac{0.35\pi+0.5\pi}{2}$

Select parameters: $\begin{cases}\Delta\omega\\ A\end{cases}\rightarrow\begin{cases}\beta=2.6\\ M=24\end{cases}$

This is a Type I filter.

Check! Approximation error = 0.0213 > 0.021!!

Increase M to 25 → Not good! This is a Type II filter: a zero at –1.→ $H_d\left(e^{j\pi}\right)=0$

But we want it to be 1 because this is a highpass filter.

Increase M to 26. Okay!

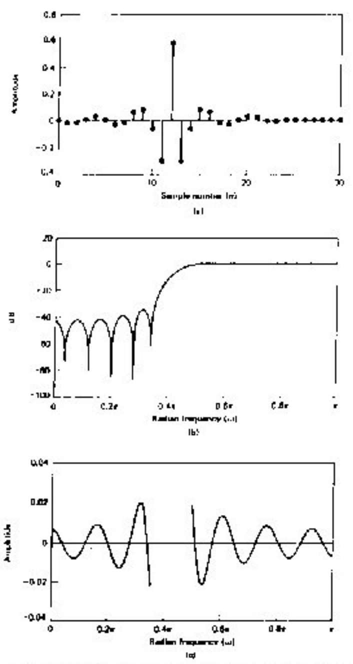


Figure 7.34 Response functions for type I FIR highpass filter. (a) Impulse response (M = 24). (b) Log magnitude. (c) Approximation error.

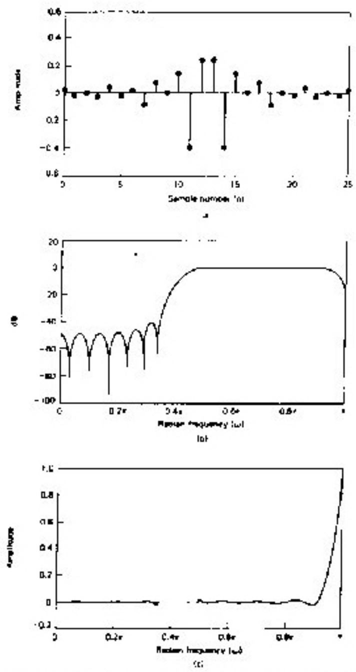


Figure 7.35 Response functions for type II FIR highpass filter. (a) Impulse response (M = 25). (b) Log magnitude. (c) Approximation error.

M = 24 M = 25

***Kaiser window example* – differentiator**

Ideal differentiator: $\sim \dfrac{d}{dt}$

$$H_{\text{diff}}\left(e^{j\omega}\right) = (j\omega)\cdot e^{-j\omega\frac{M}{2}}, \quad -\pi < \omega < \pi$$

$$h_{\text{diff}}[n] = \frac{\cos\pi\left(n-\frac{M}{2}\right)}{\left(n-\frac{M}{2}\right)} - \frac{\sin\pi\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)^2}$$

Note that both terms in $h_{diff}[n]$ are odd symmetric.

Hence, $h[n] = -h[M-n]$.

This must be a Type III or Type IV system.

<Comparison>

Case 1: M=10, $\beta = 2.4$ → Type III

Zeros at 0 and –1. Approximation is not good at $\omega = \pi$.

Case 2: M=5, $\beta = 2.4$ → Type III

Zeros at 0. Approximation error is smaller.

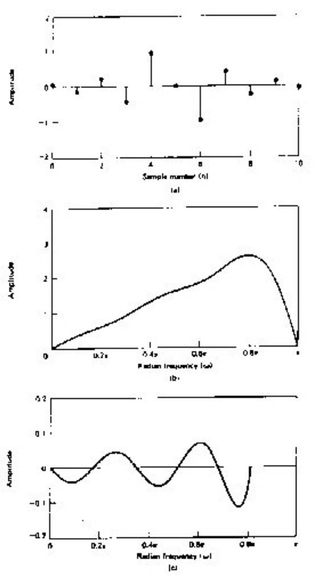


Figure 7.37 Response functions for type III FIR discrete-time differentiator. (a) Impulse response (M = 10). (b) Magnitude. (c) Approximation error.

M = 10

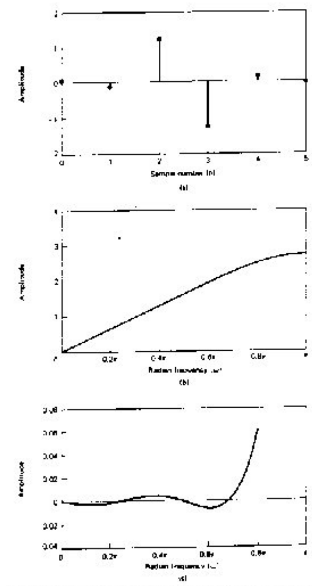


Figure 7.38 Response functions for type IV FIR discrete-time differentiator. (a) Impulse response (M = 5). (b) Magnitude. (c) Approximation error.

M = 5

- **Frequency Sampling Method**
  - In frequency domain, matches (M+1) samples of the ideal frequency response.
  - Observe that $H\left(e^{j\omega}\right)=\sum_{n=0}^{M} h[n]\cdot e^{-j\omega n}$

    If take samples, then $H\left(e^{j\omega}\right)\Big|_{\omega=\left(\frac{2\pi}{M+1}\right)\cdot k} = \sum_{n=0}^{M} h[n]\cdot e^{-j\left(\frac{2\pi}{M+1}\right)\cdot kn}$

    Now, take samples of the desired model as the target,

$$\tilde{H}(k) = H_{\mathrm{d}}\left(e^{j(2\pi/(M+1))k}\right), \quad k = 0,1,\cdots,M$$

    The final filter impulse response is

$$h[n] = \begin{cases} \frac{1}{M+1}\sum_{n=0}^{M}\tilde{H}(k)\cdot e^{j\left(\frac{2\pi}{M+1}\right)k\cdot n}, & k = 0,1,\cdots,M \\ 0, & \text{otherwise} \end{cases}$$

  - Take (M+1) samples of the desired $H_d\left(e^{j\omega}\right)$ and then take the inverse Fourier transform of these (M+1) samples to form $h[n]$. Consequently, the FT of $h[n]$ ($H\left(e^{j\omega}\right)$) would match the desired $H_d\left(e^{j\omega}\right)$ at these (M+1) sample points, BUT $H\left(e^{j\omega}\right)$ may not be adequate at the other points $\omega \neq \frac{2\pi}{M+1}k$.

  ← Modify some of the $H_d\left(e^{j\omega}\right)$ sampled values to change $H\left(e^{j\omega}\right)$. Introduce the *transition samples*.

  Remark: Adjust the number and the values of transition samples → an optimization problem. (Rabiner et al., "An Approach to the Appoximation Problem for Nonrecursive Digital Filters," IEEE Trans. Audio Electroacoust., Vol.AU-18, June 1970, pp.83-106)

✧ Optimum Approximation of FIR Filters

- Why computer-aided design?

  -- Optimum: minimize an error criterion

  -- More freedom in selecting constraints.

  (In windowing method: must $\delta_1 = \delta_2 = \delta$)

- Several algorithms – ***Parks-McClellan algorithm*** (1972)

  Type I linear phase FIR filter

  Its symmetry property: $h_e[n] = h_e[-n]$ (omit delay)

  Check its frequency response:

$$
\begin{aligned}
A_e\left(e^{j\omega}\right) &= \sum_{n=-L}^{L} h_e[n] \cdot e^{-j\omega n} \\
&= h_e[0] + \sum_{n=1}^{L} 2h_e[n] \cdot \cos(\omega n) \\
&= a_0 + \sum_{n=1}^{L} a_k \cdot (\cos(\omega))^k \\
&= \sum_{n=0}^{L} a_k \cdot (\cos(\omega))^k \\
&= P(x)\Big|_{x=\cos\omega}
\end{aligned}
$$

Note that $P(x) = \sum a_k x^k$ is an Lth-order polynominal. In the above process, we use a polynominal expression of cos(.), $\cos(\omega n) = T_n(\cos\omega)$, where $T_n(\cdot)$ is the nth-order Chebyshev polynominal. Thus, we shift our goal from finding (L+1) values of $\{h_e[n]\}$ to finding (L+1) values of $\{a_k\}$.

(want to use the polynominal approximation algorithms.)

<*Our Problem* now>

Adjustable parameters: $\{a_k\}$, (L+1) values

Specifications: ω_p, ω_p, $\delta_1/\delta_2 = K$, and L (L is often preselected)

Error criterion: $E(\omega) = W(\omega) \cdot \left[H_d\left(e^{j\omega}\right) - A_e\left(e^{j\omega}\right)\right]$

Goal: minimize the maximum error

$$\min_{\{h_e[n]\}^L} \left(\max_{\omega\in F} |E(\omega)|\right),$$ F: passband and stopband

(*Note:* Often, no constraint on the transition band)

(Why choose this minimization target? Even error values!

Recall: In the rectangular windowing method, we actually minimize $\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right) - H\left(e^{j\omega}\right)\right|^2 d\omega$. Although the total squared error can be small but errors at some frequencies may be large.)

<Alternation Theorem>

F_P:	closed subset consists of (the union) of disjoint closed subsets of the real axis x	*Example*, lowpass: $[0,\omega_p],[\omega_s,\pi]$ → $x=\cos\omega$ → $[1,\cos\omega_p],[\cos\omega_s,1]$
$P(x)$:	rth-order polynominal $P(x)=\sum_{k=0}^{r} a_k x^k$	$P(\cos\omega)=\sum_{k=0}^{L} a_k(\cos\omega)^k$
$D_P(x)$:	desired function of x continuous on F_P	$D_P(x)=\begin{cases}1, & x_p \le x \le 1\\ 0, & -1 \le x \le x_s\end{cases}$ $x=\cos\omega$
$W_P(x)$:	weighting: positive, continuous on F_P	$W_P(x)=\begin{cases}1/K, & x_p \le x \le 1\\ 1, & -1 \le x \le x_s\end{cases}$
$E_P(x)$:	weighted error $E_P(x)=W_P(x)[D_P(x)-P(x)]$	$E_P(x)=W_P(x)[D_P(x)-P(x)]$
$\|E\|$:	maximum error $\|E\| = \max_{x\in F_P} E_P(x)$	$\|E\|=\delta_2$

$P(x)$ is the *unique* rth-order polynominal that minimizes $\|E\|$

if and only if $E_P(x)$ exhibits *at least* (r+2) ***alternations***

Alternation: There exist (r+2) values x_i in F_P such that

$E_P(x_i) = -E_P(x_{i+1}) = \pm\|E\|,\ i = 1,2,\cdots,(r+1)$, where $x_1 < x_2 < \cdots < x_{r+2}$.

Remark: Two conditions here for alternation: value and sign.

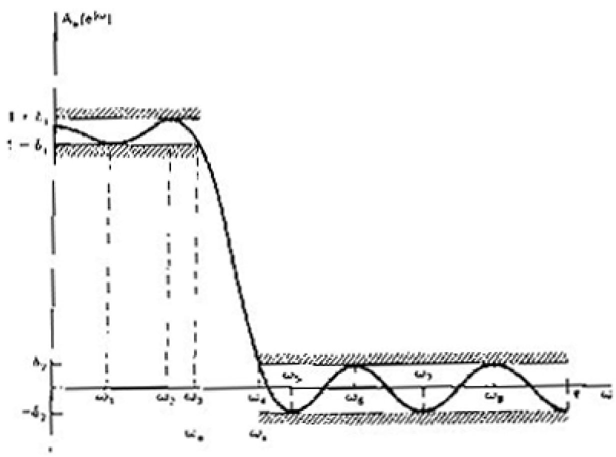


Figure 7.42 Typical example of a lowpass filter approximation that is optimal according to the alternation theorem for $L = 7$.

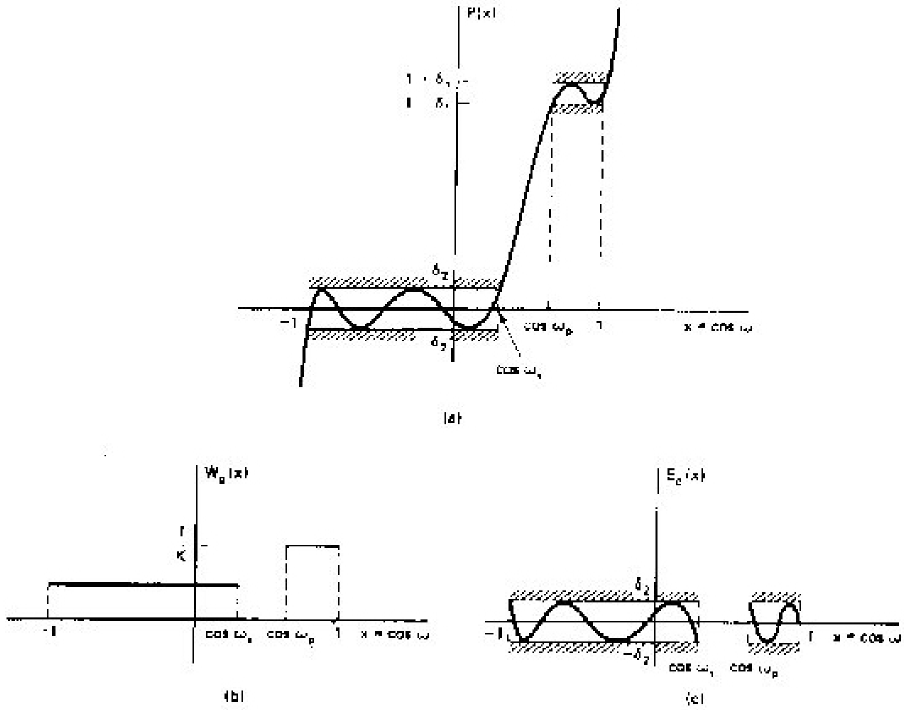


Figure 7.43 Equivalent polynomial approximation functions as a function of $x = \cos\omega$. (a) Approximating polynomial. (b) Weighting function. (c) Approximation error.

Type I linear phase FIR filter

(1) Maximum number of alternations of errors = (L+3)

(2) Alternations always occur at ω_p and ω_s

(3) Equiripple except possibly at $\omega = 0$ and $\omega = \pi$

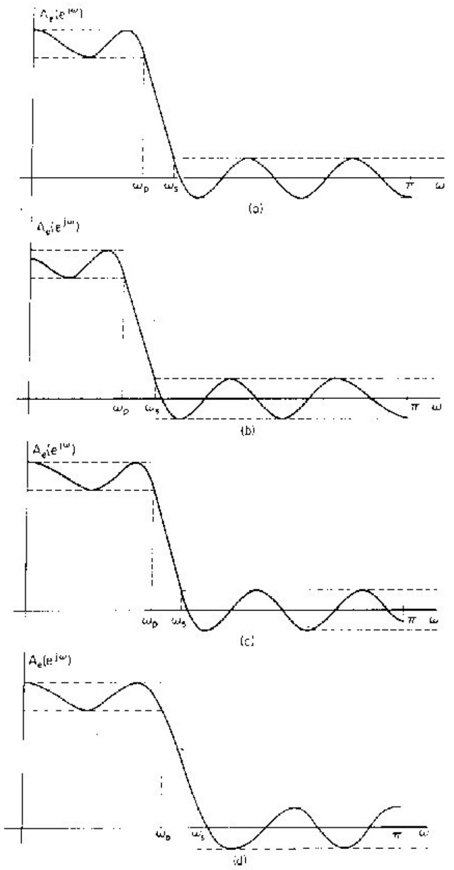


Figure 7.44 Possible optimum lowpass filter approximations for $L = 7$. (a) $L + 3$ alternations (extraripple case). (b) $L + 2$ alternations (extremum at $\omega = \pi$). (c) $L + 2$ alternations (extremum at $\omega = 0$). (d) $L + 2$ alternations (extremum at both $\omega = 0$ and $\omega = \pi$).

(Reasons)

(a) Locations of extrema: Lth-order polynominal has at most L-1 extrema. Now, in addition, the local extrema may locate at band edges $\omega = 0, \pi, \omega_p, \omega_s$. Hence, at most, there are (L+3) extrema or alternations.

(Note: Because $x = \cos\omega$, $\frac{dP(\cos\omega)}{d\omega} = 0$, at $\omega = 0$ and $\omega = \pi$.)

(b) If ω_p is not an alternation, for example, then because of the +- sign sequence, we loose two alternations → (*L*+1) alternations ← violates the (*L*+2) alternation theorem.

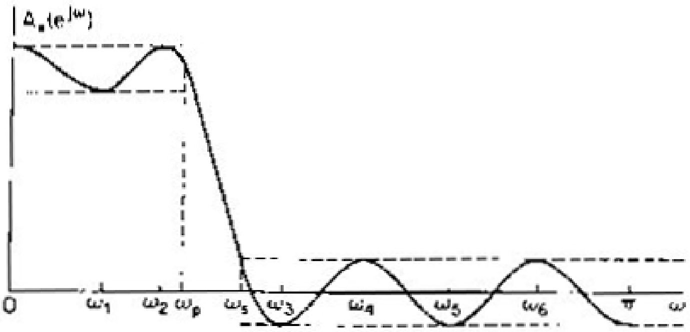


Figure 7.45 Illustration that the passband edge ω_p must be an alternation frequency.

(c) The only possibility that the extrema can be a non-alternation is that it locates at $\omega = 0$ or $\omega = \pi$. In either case, we have (*L*+2) alternations – minimum requirement.

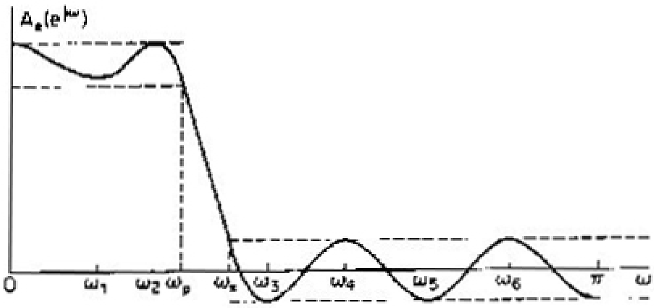


Figure 7.46 Illustration that the frequency response must be equiripple in the approximation bands.

Type II linear phase FIR filter

Its symmetry property: $h_e[n] = h_e[M-n]$, M odd

Frequency response:

$$H\left(e^{j\omega}\right) = e^{-j\omega\frac{M}{2}}\left\{\sum_{n=1}^{(M+1)/2} b[n]\cdot\cos(\omega(n-1/2))\right\}$$

$$= e^{-j\omega\frac{M}{2}}\cos\left(\frac{\omega}{2}\right)\left\{\sum_{n=1}^{(M+1)/2}\tilde{b}[n]\cdot\cos(\omega n)\right\}$$

→ $H\left(e^{j\omega}\right) = e^{-j\omega\frac{M}{2}}\cos\left(\frac{\omega}{2}\right)P(\cos\omega)$,

where $P(\cos\omega) = \sum_{k=0}^{L} a_k(\cos\omega)^k$

Problem: How to handle $\cos\left(\frac{\omega}{2}\right)$?

Transfer specifications!

Let $H_d\left(e^{j\omega}\right) = D_p(\cos\omega) = \begin{cases}\dfrac{1}{\cos\left(\frac{\omega}{2}\right)}, & 0 \le \omega \le \omega_p \\ 0, & \omega_s \le \omega \le \pi\end{cases}$

Original	New
Ideal: $D(\cos\omega) \Leftarrow \cos\left(\frac{\omega}{2}\right)P(\cos\omega)$	Ideal: $\dfrac{D(\cos\omega)}{\cos\left(\frac{\omega}{2}\right)} \Leftarrow P(\cos\omega)$

Thus,

$$W(\omega) = W_p(\cos\omega) = \begin{cases}\dfrac{\cos\left(\frac{\omega}{2}\right)}{K}, & 0 \le \omega \le \omega_p \\ \cos\left(\frac{\omega}{2}\right), & \omega_s \le \omega \le \pi\end{cases}$$

Parks-McClellan Algorithm

<Type I Lowpass>

According to the preceding theorems, errors

$E(\omega) = W(\omega)\cdot\left[H_d\left(e^{j\omega}\right) - A_e\left(e^{j\omega}\right)\right]$ has alternations at $\omega_i, i = 1,\ldots, L+2$, if $A_e\left(e^{j\omega}\right)$ is the *optimum* solution.

That is, let $\delta = \|E\|$, the maximum error,

$$W(\omega_i)\cdot\left[H_d\left(e^{j\omega_i}\right) - A_e\left(e^{j\omega_i}\right)\right] = (-1)^{i+1}\delta, \quad i = 1,2,.., L+2.$$

Because $A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k (\cos\omega)^k = a_0 1 + a_1\cos\omega + a_2(\cos\omega)^2 + \cdots$,

at ω_1: $a_0 1 + a_1\cos\omega_1 + a_2(\cos\omega_1)^2 + \cdots \quad \leftrightarrow \quad a_0 1 + a_1 x_1 + a_2 (x_1)^2 + \cdots$

at ω_2: $a_0 1 + a_1\cos\omega_2 + a_2(\cos\omega_2)^2 + \cdots \quad \leftrightarrow \quad a_0 1 + a_1 x_2 + a_2 (x_2)^2 + \cdots$

…

Hence,

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & \dfrac{1}{W(\omega_1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & \dfrac{-1}{W(\omega_2)} \\ \vdots & \ddots & & \ddots & & \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & \dfrac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \delta \end{bmatrix} = \begin{bmatrix} H_d\left(e^{j\omega_1}\right) \\ H_d\left(e^{j\omega_2}\right) \\ \vdots \\ H_d\left(e^{j\omega_{L+2}}\right) \end{bmatrix}$$

Remark: For Type I lowpass filter, ω_p and ω_s must be two of the alternation frequencies $\{\omega_i\}$.

Now, we have *L*+2 simultaneous equations and *L*+2 unknowns, $\{a_i\}$ and δ.

The solutions are

$$\delta = \frac{\sum_{k=1}^{L+2} b_k H_d\left(e^{j\omega_k}\right)}{\sum_{k=1}^{L+2} \dfrac{b_k (-1)^{k+1}}{W(\omega_k)}}, \quad b_k = \prod_{\substack{i=1 \\ i\neq k}}^{L+2} \frac{1}{(x_k - x_i)}.$$

Once we know $\{a_i\}$, we can calculate $A_e\left(e^{j\omega}\right)$ for all ω.

However, there is short cut. We can calculate $A_e\left(e^{j\omega}\right)$ for all ω directly based on $W(\omega_k), H_d\left(e^{j\omega_k}\right)$ and ω_k without solving for $\{a_i\}$.

$$A_e\left(e^{j\omega}\right) = P(\cos\omega) = \frac{\sum_{k=1}^{L+1}\left[\frac{d_k}{(x-x_k)}\right]c_k}{\sum_{k=1}^{L+1}\left[\frac{d_k}{(x-x_k)}\right]},$$

where $c_k = H_d\left(e^{j\omega_k}\right) - \dfrac{(-1)^{k+1}\delta}{W(\omega_k)}$,

$$d_k = \prod_{i=1, i\neq k}^{L+1} \frac{1}{(x_k - x_i)}$$

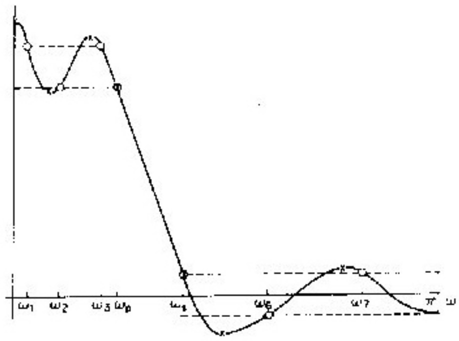


Figure 7.47 Illustration of the Parks McClellan algorithm for equiripple approximation.

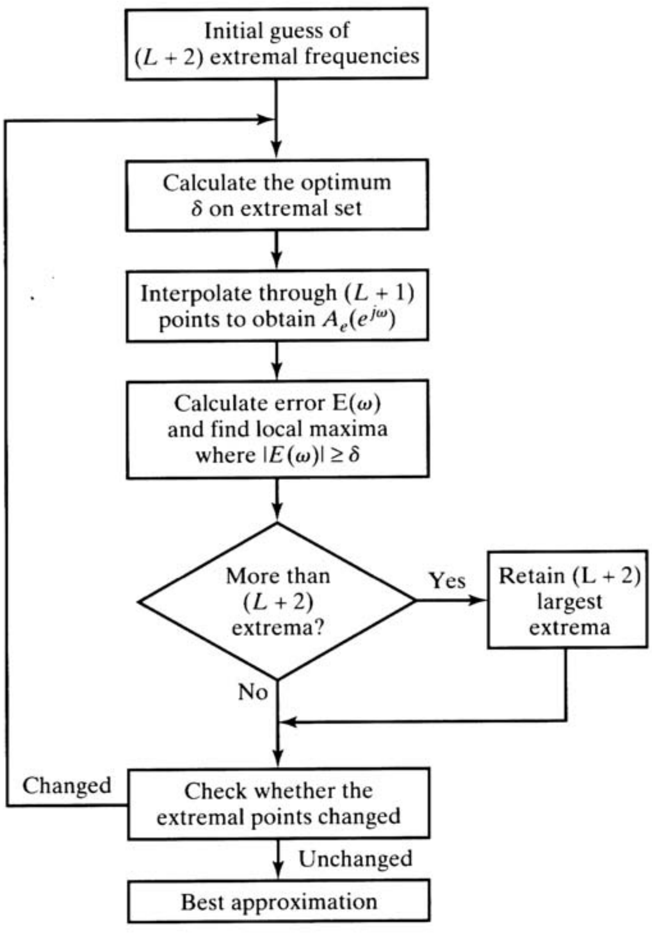


Figure 7.41 Flowchart of Parks–McClellan algorithm.

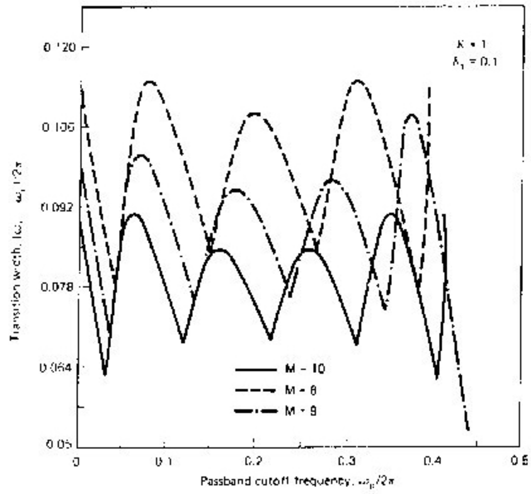


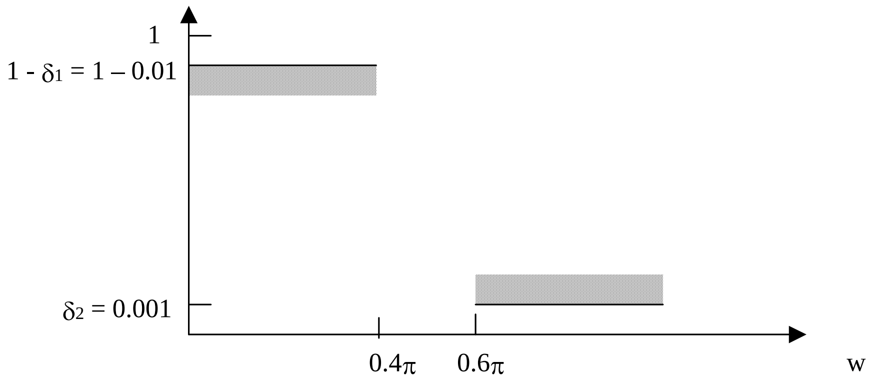
Figure 7.49 Dependence of transition width on cutoff frequency for optimum approximations of a lowpass filter

-- How to decide *M* (for lowpass)? (Experimental formula)

$$M = \frac{-10\log_{10}(\delta_1\delta_2) - 13}{2.324 \cdot \Delta\omega}$$

$$\Delta\omega = \omega_s - \omega_p$$

Example: Lowpass Filter



$$K = \frac{\delta_1}{\delta_2} = 10$$

$$M = \frac{-10\log_{10}(\delta_1\delta_2) - 13}{2.324 \cdot \Delta\omega} \Rightarrow \text{M = 26}$$

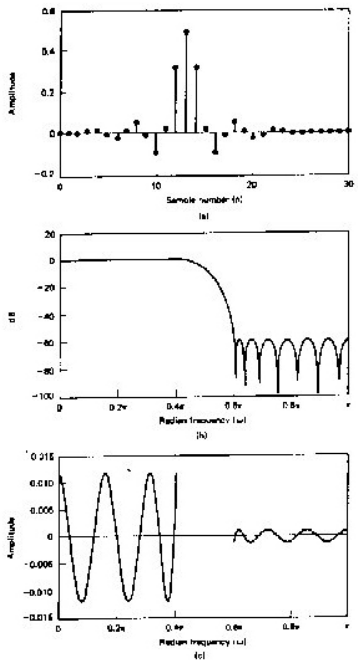


Figure 7.50 Optimum type I FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$, and $M = 26$. (a) Impulse response. (b) Log magnitude. (c) Approximation error (unweighted).

But the maximum errors in the passband and stopband are 0.0116 and 0.00116, respectively.
$\Rightarrow$ M = 27

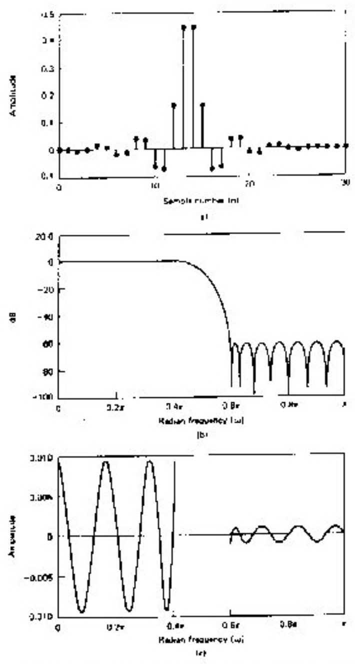


Figure 7.51 Optimum type II FIR lowpass filter for $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $K = 10$ and $M = 27$. (a) Impulse response. (b) Log magnitude. (c) Approximation error (unweighted).

Remark: The Kaiser window method requires a value M = 38 to meet or exceed the same specifications.

Example: Bandpass filter

Note: (1) From the alternation theorem

$\Rightarrow$ the minimum number of alternations for the optimum approximation is L + 2.

(2) Multiband filters can have more than L+3 alternations.

(3) Local extrema can occur in the transition regions.

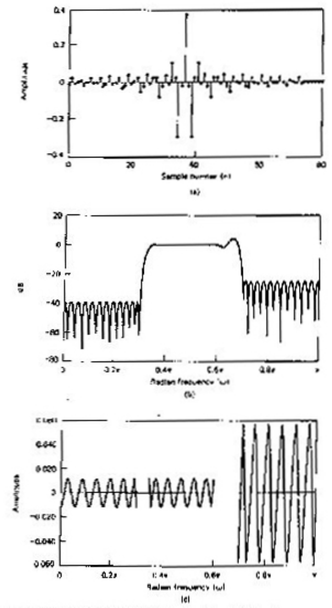


Figure 7.54 Optimum FIR bandpass filter for M = 34 (a) Impulse response. (b) Log magnitude. (c) Approximation error (unweighted).

- **IIR vs. FIR Filters**

Property	FIR	IIR
Stability	Always stable	Incorporate stability constraint in design
Analog design	No meaningful analog equivalent	Simple transformation from analog filters
Phase linearity	Can be exact linear	Nonlinear typically
Computation	More multiplications and additions	Fewer
Storage	More coefficients	Fewer
Sensitivity to coefficient inaccuracy	Low sensitivity	Higher
Adaptivity	Easy	Difficult