Analog IIR Filter Design

Commonly used analog filters:

- **Lowpass Butterworth filters**
  all-pole filters characterized by magnitude response. (N=filter order)

\[
G(j\omega) = |H(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2N}}
\]

\[
G(s) = H(s)H(-s) = \frac{1}{1 + (\frac{s}{\omega_c})^{2N}}
\]

Poles of \(H(s)H(-s)\) are equally spaced points on a circle of radius \(\omega_c\) in s-plane

\[\omega_c\]

poles of \(H(s)\)

N=4
Butterworth Filters

- Lowpass Butterworth filters
  monotonic in pass-band & stop-band

`maximum flat response`: (2N-1) derivatives are zero at
\[ \omega = 0 \quad \text{and} \quad \omega = \infty \]
Analog IIR Filter Design

Commonly used analog filters:

- **Lowpass Chebyshev filters (type-I)**
  
  All-pole filters characterized by magnitude response

  \[
  G(j\omega) = \left|H(j\omega)\right|^2 = \frac{1}{1 + \varepsilon^2 T_N^2\left(\frac{\omega}{\omega_c}\right)}
  \]

  \[
  G(s) = H(s)H(-s)
  \]

  \(\varepsilon\) is related to passband ripple
  
  \(T_N(x)\) are Chebyshev polynomials:

  \[
  T_0(x) = 1
  \]

  \[
  T_1(x) = x
  \]

  \[
  T_2(x) = 2x^2 - 1
  \]

  ... 

  \[
  T_N(x) = 2xT_{N-1}(x) - T_{N-2}(x)
  \]
Chebyshev & Elliptic Filters

• Lowpass Chebyshev filters (type-I)
  – All-pole filters, poles of $H(s)H(-s)$ are on ellipse in $s$-plane
  – Equiripple in the pass-band
  – Monotone in the stop-band

• Lowpass Chebyshev filters (type-II)
  – Pole-zero filters based on Chebyshev polynomials
  – Monotone in the pass-band
  – Equiripple in the stop-band

• Lowpass Elliptic (Cauer) filters
  – Pole-zero filters based on Jacobian elliptic functions
  – Equiripple in the pass-band and stop-band
  – (hence) yield smallest-order for given set of specs

\[ |H(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 U_N\left(\frac{\omega}{\omega_c}\right)} \]
Analog IIR Filter Design

Frequency Transformations:

- **Principle**: prototype low-pass filter (e.g. cut-off frequency $= 1$ rad/sec) is transformed to properly scaled low-pass, high-pass, band-pass, band-stop,... filter

- **Example**: replacing $s$ by $\frac{s}{\omega_c}$ moves cut-off frequency to $\omega_c$

- **Example**: replacing $s$ by $\frac{\omega_c}{s}$ turns LP into HP, with cut-off frequency $\omega_c$

- **Example**: replacing $s$ by $\frac{s^2 + \omega_1 \omega_2}{s(\omega_2 - \omega_1)}$ turns LP into BP
Analog -> Digital

• Principle:
  – design analog filter (LP/HP/BP/…), and then convert it to a digital filter.

• Conversion methods:
  – convert differential equation into difference equation
  – convert continuous-time impulse response into discrete-time impulse response
  – convert transfer function $H(s)$ into transfer function $H(z)$

• Requirement:
  – the left-half plane of the $s$-plane should map into the inside of the unit circle in the $z$-plane, so that a stable analog filter is converted into a stable digital filter.
Analog -> Digital

(I) convert differential equation into difference equation:

- in a difference equation, a derivative dy/dt is replaced by a ‘backward difference’ (y(kT)-y(kT-T))/T=(y[k]-y[k-1])/T, where T=sampling interval.
- similarly, a second derivative, and so on.
- eventually (details omitted), this corresponds to replacing s by (1-1/z)/T in $H_a(s)$ (=analog transfer function):

$$H(z) = H_a(s) \bigg|_{s = \frac{1-z^{-1}}{T}}$$

- stable analog filters are mapped into stable digital filters, but pole location for digital filter confined to only a small region (o.k. only for LP or BP)
Analog -> Digital

(II) convert continuous-time impulse response into discrete-time impulse response:

- given continuous-time impulse response $h_c(t)$, discrete-time impulse response is $h[k] = h_c(kT_d)$ where $T_d$ = sampling interval.
- eventually (details omitted) this corresponds to a (many-to-one) mapping

![Diagram showing the relationship between the s-plane and the z-plane.](image)

$$z = e^{sT_d}$$

$s = 0 \Rightarrow z = 1$

$s = \pm j\pi / T_d \Rightarrow z = -1$

- aliasing (!) if continuous-time response has significant frequency content above the Nyquist frequency (i.e. not bandlimited)
Example: Filter Design by Impulse Invariance

If $h_c(t)$ is the impulse response of continuous-time filter, and $h_c(nT_d)$ is equally spaced samples of it.

Let

$$h[n] = T_d h_c(nT_d) \quad (7.4)$$

then the corresponding frequency responses meet following equation:

$$H(e^{j\omega}) = T_d \cdot \frac{1}{T_d} \sum_{k=-\infty}^{\infty} H_c \left( j \frac{\omega}{T_d} + j \frac{2\pi}{T_d} k \right) \quad (7.5)$$

If the continuous-time filter is band-limited, so that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T_d \quad (7.6)$$

then

$$H(e^{j\omega}) = H_c \left( j \frac{\omega}{T_d} \right), \quad |\omega| \leq \pi \quad (7.7)$$
If the continuous-time filter is not band-limited, then the interference (aliasing) between successive terms exists. 

Many-to-one mapping
Example: A Low-Pass Filter

Specifications:

\[ 0.89125 \leq |H(e^{j\omega})| \leq 1, \quad 0 \leq |\omega| \leq 0.2\pi \]  

\[ |H(e^{j\omega})| \leq 0.17783, \quad 0.3\pi \leq |\omega| \leq \pi \]  

Choose \( T_d = 1 \) (i.e. \( \omega = \Omega \)), from Eq.(7.7) we obtain

\[ H(e^{j\omega}) = H_c(j \frac{\omega}{T}), \quad |\omega| \leq \pi \]

Then

\[ 0.89125 \leq |H_c(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq 0.2\pi \]

\[ |H_c(j\Omega)| \leq 0.17783, \quad 0.3\pi \leq |\Omega| \leq \pi \]

Let \( \Omega_p = 0.2\pi \), and \( \Omega_s = 0.3\pi \), then

\[ |H_c(j0.2\pi)| \geq 0.89125 \]

\[ |H_c(j0.3\pi)| \leq 0.17783, \]
By using Butterworth filter, then

\[ |H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega / \Omega_c)^{2N}} \]

Consequently,

\[ 1 + \left( \frac{0.2\pi}{\Omega_c} \right)^{2N} = \left( \frac{1}{0.89125} \right)^2 \]

\[ 1 + \left( \frac{0.3\pi}{\Omega_c} \right)^{2N} = \left( \frac{1}{0.17783} \right)^2 \]

Then

\[ \left( \frac{0.2\pi}{0.3\pi} \right)^{2N} = \frac{\left( \frac{1}{0.89125} \right)^2 - 1}{\left( \frac{1}{0.17783} \right)^2 - 1} \]

\[ \implies N = 5.8858 \]

Since \( N \) must be integer. So, \( N = 6 \). And we obtain \( \Omega_c = 0.7032 \)

We can obtain 12 poles of \( |H_c(s)|^2 \). They are uniformly distributed in angle on a circle of radius \( \Omega_c = 0.7032 \)
\[ H_c(s) = \frac{K_0}{\prod_{k=1}^{N} (s - s_k)} \]

\[ K_0 = 0.12093 \]

so that

\[ H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)} \]

**Discrete Filter**

\[
H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}.
\]

\( (7.19) \)

Figure 7.6 (b) Frequency response of sixth-order Butterworth filter
Analog -> Digital

- (III) convert continuous-time system transfer function into discrete-time system transfer function: **Bilinear Transform**
  - mapping that transforms (whole!) $jw$-axis of the $s$-plane into unit circle in the $z$-plane only once, i.e. that avoids aliasing of the frequency components.

\[
H(z) = H_a(s)
\bigg|_{s = \frac{2(1-z^{-1})}{T(1+z^{-1})}}
\]

- for low-frequencies, this is an approximation of $z = e^{sT}$
- for high frequencies: significant frequency compression ('warping')
- sometimes pre-compensated by 'pre-warping'

\[
s = 0 \Rightarrow z = 1
\]

\[
s = j\infty \Rightarrow z = -1
\]
The bilinear transformation avoids the problem of aliasing problem because it maps the entire imaginary axis of the s-plane onto the unit circle in the z-plane. The price paid for this, however, is the nonlinear compression the frequency axis (warping).

\[
s = \frac{2}{T_d} \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}
\]

\[
s = \sigma + j\Omega = \frac{2}{T_d} \left[ \frac{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})} \right]
\]

\[
= \frac{2}{T_d} \left[ \frac{2 j \sin(\omega/2)}{2 \cos(\omega/2)} \right] = \frac{2 j}{T_d} \tan(\omega/2)
\]

If \( \sigma = 0 \), then

\[
\Omega = \frac{2}{T_d} \tan(\omega/2)
\]
Conclusions/Software

- IIR filter design considerably more complicated than FIR design (stability, phase response, etc.)
- (Fortunately) IIR Filter design abundantly available in commercial software
- Matlab:
  
  \[
  [b,a]=\text{butter/cheby1/cheby2/ellip}(n,\ldots,Wn),
  \]
  
  IIR LP/HP/BP/BS design based on analog prototypes, pre-warping, bilinear transform, ...