Principles of Communications Lecture 2: Signals and Systems(2)

Chih-Wei Liu 劉志尉
National Chiao Tung University
cwliu@twins.ee.nctu.edu.tw

Outlines

- Signal Models & Classifications
- Signal Space & Orthogonal Basis
- Fourier Series &Transform
- Signals & Linear Systems
- Correlation & Power Spectral Density
- Sampling Theory
- DFT & FFT

Signals & Linear Systems

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) \qquad y(t) = H\{x(t)\}$$

- **Linear:** Satisfies superposition principle

$$y(t) = H\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\}$$
$$= H\{\alpha_1 x_1(t)\} + H\{\alpha_2 x_2(t)\} = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

- **Time-invariant:** Delayed input produces an output with the same delay. $H\{x(t - t_0)\} = y(t - t_0)$

- **LTI** – Linear Time-invariant system. Our focus!
- How to completely describe an LTI system?

Impulse Response

- $h(t)$: response due to an impulse input applied at t=0.

$$h(t) \equiv H\{\delta(t)\}.$$

- Then, for an arbitrary input x(t),

Recall that $x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda$, then

$$y(t) = H\{x(t)\} = H\{\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda\}$$

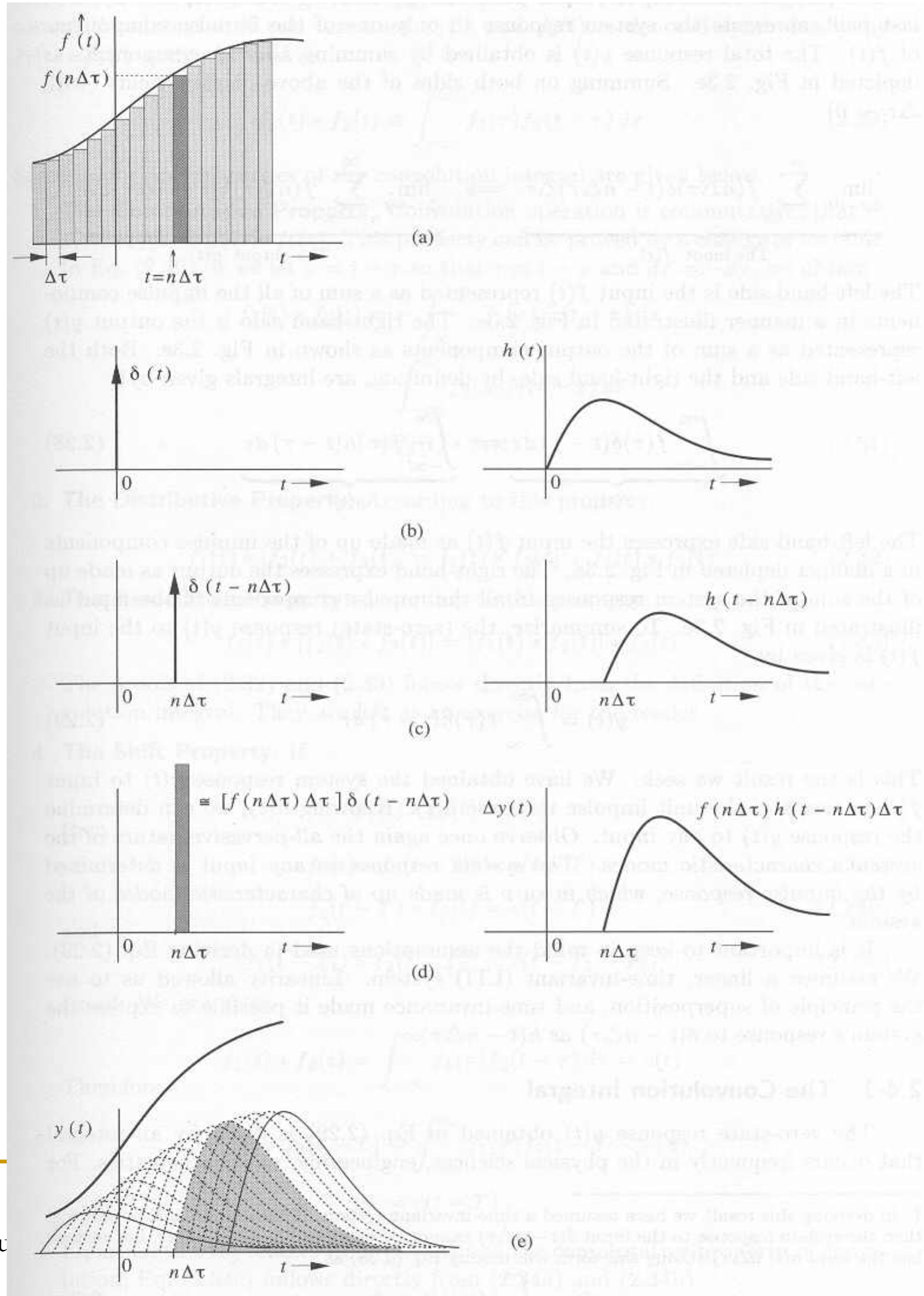
$$= \int_{-\infty}^{\infty} x(\lambda)H\{\delta(t-\lambda)\}d\lambda$$

Convolution integral
Superposition integral

$$y(t) = H\{x(t)\} = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \equiv x(t) * h(t)$$

Convolution Illustration

(Lathi, 1998, p.119)



Transfer Function

- The convolution form holds for linear system.
- Duality of signal $x(t)$ & system $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

- **Transfer Function**: $H(f) \equiv \Im\{h(t)\}$

$$\Im\{y(t)\} = Y(f) = \Im\{\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda\} = H(f)X(f)$$

Remark: Generally, $H(f)X(f)$ is easier in computation than $x(t)*h(t)$.

Stability & Causality

- A system is **BIBO** (bounded-input, bounded-output) **stable** if its output is bounded for <u>any</u> bounded input.

$$\left|y(t)\right| = \left|\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda\right| \le \max\{|x(t)|\}\int_{-\infty}^{\infty} |h(\lambda)|d\lambda < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} |h(\lambda)|d\lambda < \infty \Rightarrow \text{main element of Dirichlet condition}$$

- A system is **causal** if the current output does not depend on future input.

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

$$\Rightarrow h(t) = 0, \text{ for } t < 0$$

Paley-Wiener Condition

- Causality leads to constraints on $H(f)$.
- <u>*Necessary condition*</u>: (If violated, cannot find causal H.)

$$\text{If } \int_{-\infty}^{\infty} |H(f)|^2 df < \infty, \text{ and } h(t) = 0 \text{ for } t < \infty,$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{|\ln|H(f)||}{1+f^2} df < \infty.$$

(Papoulis, *Fourier Integral and Its Applications*, pp.215-217)

- *Remarks*: (1) $|H(f)|$ cannot grow too fast.

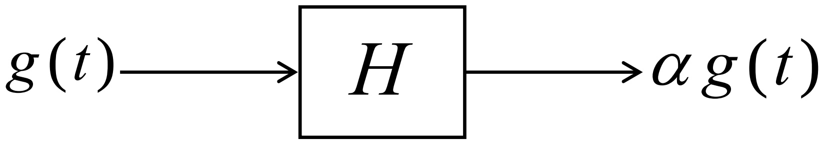
  (2) $|H(f)|$ cannot be <u>*exactly zero*</u> over a finite band of frequency.

  ($ln0 \rightarrow \infty$)
- <u>*Sufficient cond.*</u>: If $\int_{-\infty}^{\infty} |H(f)|^2 df < \infty$ and $\int_{-\infty}^{\infty} \frac{|\ln|H(f)||}{1+f^2} df < \infty,$

$$\Rightarrow \exists \angle H \text{ such that } H(f) \text{ is causal } (h(t) = 0 \text{ for } t < 0).$$

Eigenfunction & Eigenvalue

- The "waveform" (function) does not change shape after passing through an LTI system.



- If $H\{g(t)\} = \alpha g(t)$, where α is a constant, then α is the **eigenvalue** for the **eigenfunction** $g(t)$.

- Recall that, any arbitrary periodic input can be represented by a summation of complex exponential …

Eigenfunctions of LTI Systems

- For an arbitrary LTI system, $g(t) = Ae^{st}$, is an eigenfucntion, where s is any complex number, and $H(s)$ is the corresponding eigenvalue.

  Note: Special case: $s=2\pi f$

  (Pf)

$$y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} h(\lambda) A e^{s(t-\lambda)} d\lambda$$

$$= [\int_{-\infty}^{\infty} h(\lambda) e^{-s\lambda} d\lambda] A e^{st} = \alpha g(t),$$

$$\text{where } \alpha = \int_{-\infty}^{\infty} h(\lambda) A e^{-s\lambda} d\lambda = H(s).$$

Remarks

- Since any arbitrary periodic input can be represented by a summation of complex exponential, consequently, its output will be the summation of complex exponential Fourier series, i.e.

$$y(t) = \sum_{n=-\infty}^{\infty} X_n H(nf_0) e^{j2\pi n f_0 t}$$

- If the system is distortionless, then we only to characterize the system $h(t)$ (or its transfer function), and the eigenvalue $H(f)$.
- For distortionless system $y(t)=Ax(t-t_0)$:
  - the amplitude response is constant and the phase shift is linear with frequency
- How about the system with transmission distortion?

Distortion in Transmission

- <u>**Distortionless**: The system output is identical to its input except for</u> a possible change in *amplitude* and a *constant delay*.

$$y(t) = H\{x(t)\} = H_0 x(t - t_0) \rightarrow H(f) = H_0 e^{-j2\pi f t_0}$$

- Three major types of **distortion** (channel)

1. Amplitude distortion: linear system but the *amplitude* response is not constant.
2. Phase (delay) distortion: linear system but the *phase shift* is not a linear function of frequency.
3. Nonlinear distortion: nonlinear system

Group Delay

- Definition:

$$T_g(f) = -\frac{1}{2\pi}\frac{d\theta(f)}{df}, \quad \theta(f) = \angle H(f).$$

For a distortionless system, $\theta(f) = \angle H_0 - 2\pi f t_0$

$\Rightarrow T_g(f) = t_0$, which is a constant.

- If $T_g(f)$ is not a constant, sinusoidal inputs of different frequencies have different delays. ← Nonlinear phase

Phase Delay

- **Phase delay**: At a single freq component

$$T_p(f) = -\frac{\theta(f)}{2\pi f}$$

Ex.: Input: $x(t) = A\cos 2\pi f_1 t$

Output: $y(t) = A\,|H(f_1)|\cos(2\pi f_1 t + \theta(f_1))$

$$= A\,|H(f_1)|\cos\left(2\pi f_1\left(t + \frac{\theta(f_1)}{2\pi f_1}\right)\right)$$

Example of Delays

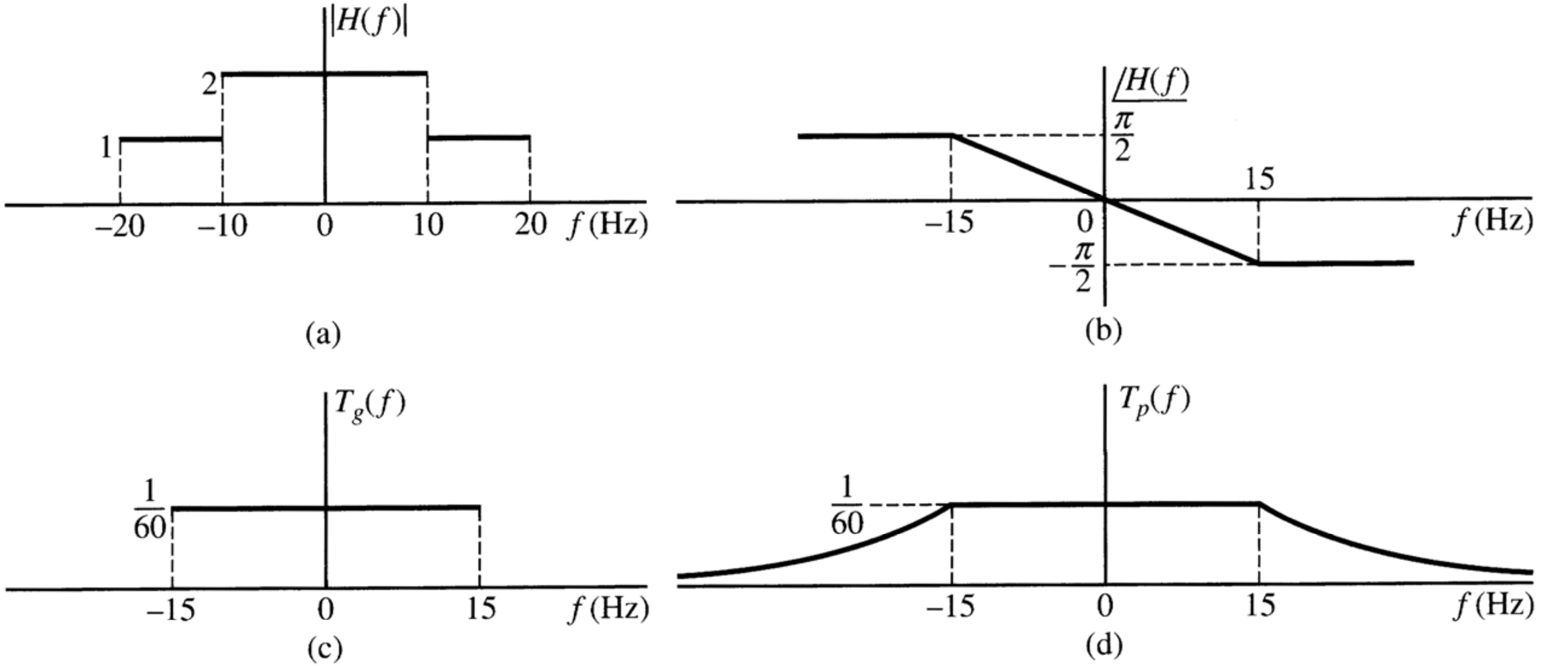


Figure 2.18
Amplitude and phase response and group and phase delays of the filter for Example 2.23. (a) Amplitude response. (b) Phase response. (c) Group delay. (d) Phase delay.

Nonlinear Distortion

Ex.: System: $y(t) = x^2(t)$

Input: $x(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$

Output:

$$y(t) = A_1^2 \cos^2(\omega_1 t) + 2A_1 A_2 \cos(\omega_1 t)\cos(\omega_2 t) + A_2^2 \cos^2(\omega_2 t)$$

$$= \tfrac{1}{2}(A_1^2 + A_2^2) + \tfrac{1}{2}[A_1^2 \cos(2\omega_1 t) + A_2^2 \cos(2\omega_2 t)]$$

$$+ A_1 A_2 [\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t]$$

Intermodulation: $(\omega_1 + \omega_2)$, $(\omega_1 - \omega_2)$

Harmonic distortion: $2\omega_1$, $2\omega_2$

Ideal Filters

Constant amplitude response and linear phase response

Not a causal system

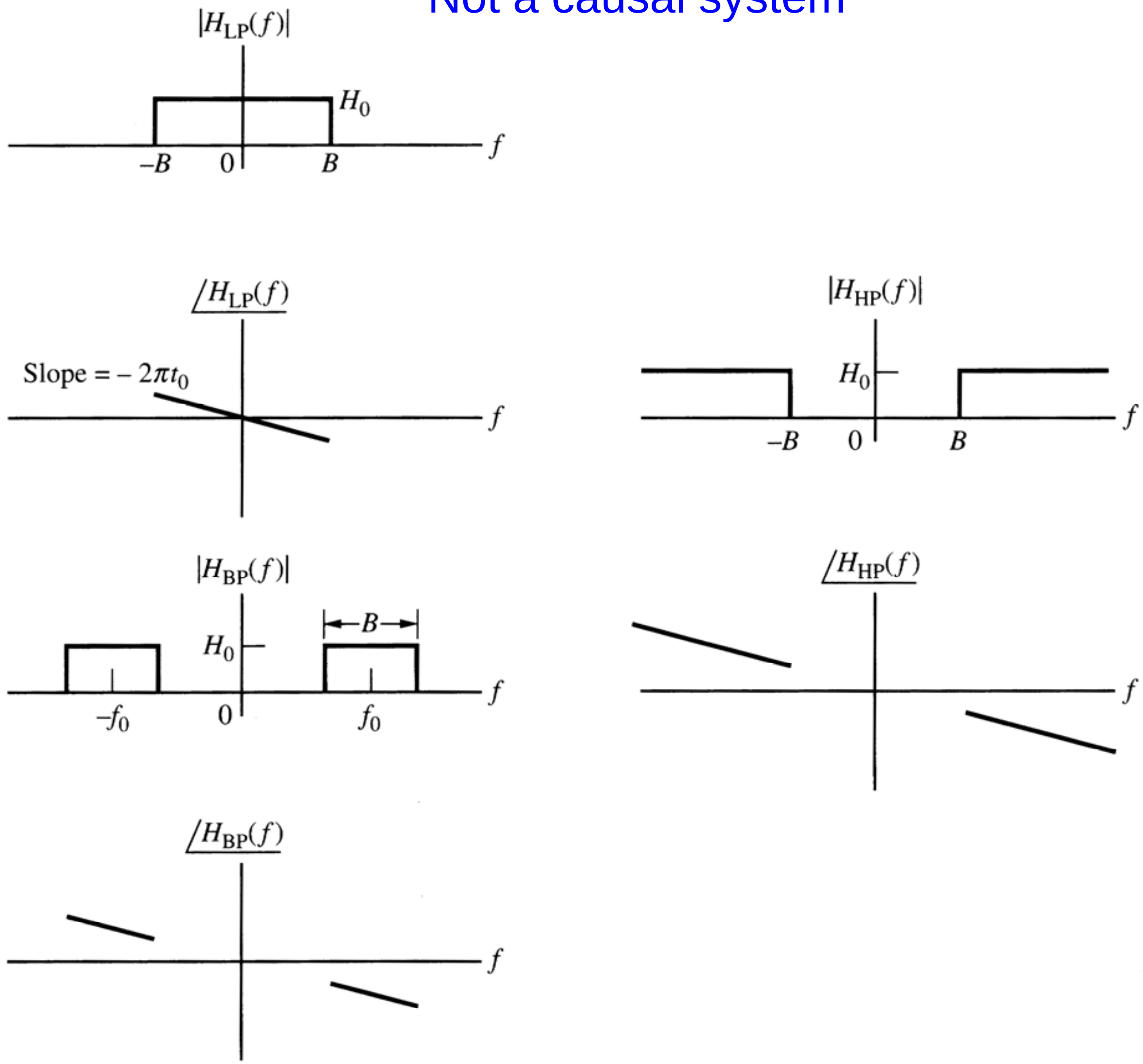


Figure 2.21
Amplitude-response and phase-response functions for ideal filters.

Realizable Filters

- For lowpass filters
  - Butterworth filter : simple
  - Chebyshev filter: smaller maximum deviation
  - Bessel filter : approximately linear phase
- For bandpass and highpass filters
  - Start from lowpass filters
  - Followed by suitable frequency transformation
- The details, in DSP or ADSP course…

Realizable Filters (2)

- Approximations to the ideal filters

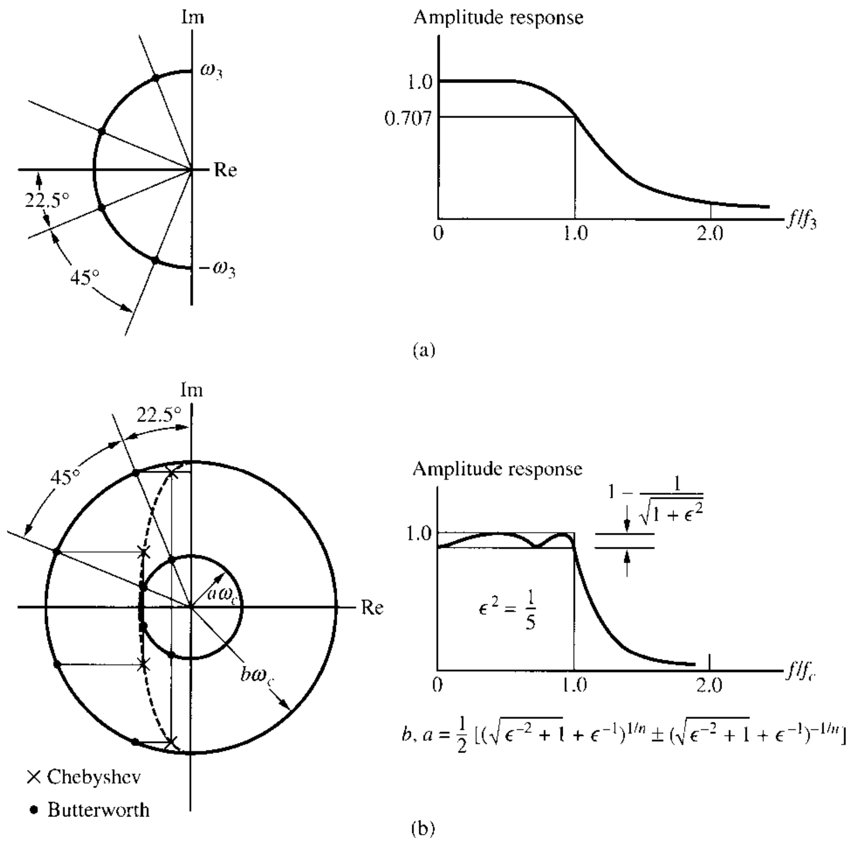


Figure 2.23
Pole locations and amplitude responses for fourth-order Butterworth and Chebyshev filters. (a) Butterworth filter. (b) Chebyshev filter.

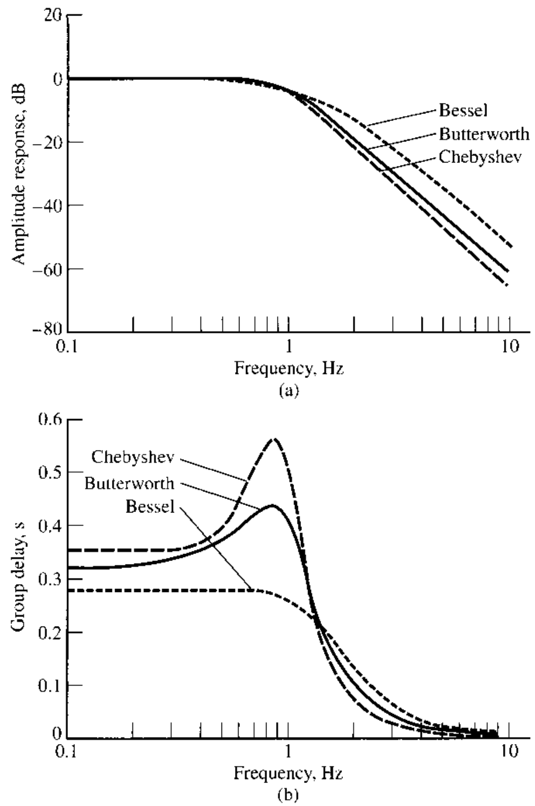


Figure 2.24
Comparison of third-order Butterworth, Chebyshev (0.1-dB ripple), and Bessel filters. (a) Ampli response. (b) Group delay. All filters are designed to have a 1-Hz, 3-dB bandwidth.

Time-Bandwidth Product

- A narrow (duration) time signal has a wide (frequency) bandwidth, and vice versa.

$$(duration) \times (bandwidth) \geq \text{constant}$$

Remark: In a way, it is similar to the uncertain principle in physics.

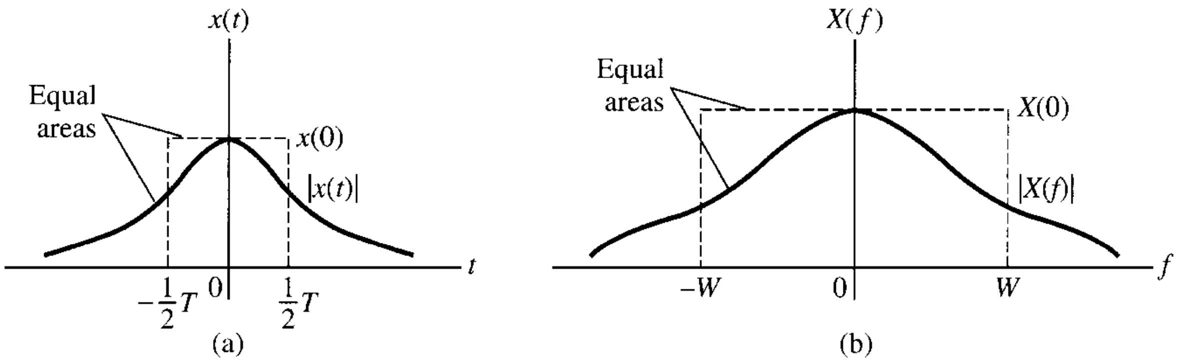


Figure 2.25
Arbitrary pulse signal and spectrum. (a) Pulse and rectangular approximation. (b) Amplitude spectrum and rectangular approximation.

Time-Bandwidth Product (conti)

- Definition of ***T***: (equal areas)

From (a) : $Tx(0) = \int_{-\infty}^{\infty} |x(t)|\,dt \geq \int_{-\infty}^{\infty} x(t)dt = X(f)|_{f=0} = X(0)$

- Definition of ***W***: (equal areas)

From (b) : $2WX(0) = \int_{-\infty}^{\infty} |X(f)|\,df \geq \int_{-\infty}^{\infty} X(f)df = x(0)$

➔ (not a rigorous proof)

$$2W \geq \frac{x(0)}{X(0)} \geq \frac{1}{T} \Rightarrow TW \geq \frac{1}{2}$$

Time Average

- Time average: $<v(t)>$

$$v(t) \text{ aperiodic}, \quad < v(t) > \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t)dt$$

$$v(t) \text{ periodic}, \quad < v(t) > \equiv \frac{1}{T_0} \int_{T_0} v(t)dt$$

Remark: $<v(t)> = X_0$ Fourier Series DC component

Time Average Autocorrelation

- **Autocorrelation** function of *energy* signals

$$\phi(\tau) \equiv \lim_{T\to\infty} \int_{-T}^{T} x(\lambda)x^*(\lambda+\tau)d\lambda = \int_{-\infty}^{\infty} x(\lambda)x^*(\lambda+\tau)d\lambda = x(\tau) * x^*(-\tau)$$

- **Autocorrelation** function of *power* signals

$$R(\tau) \equiv \left\langle x(t)x^*(t+\tau) \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x^*(t+\tau)dt = \frac{1}{T_0} \oint_{T_0} x(t)x^*(t+\tau)dt$$

<*Interpretation*> $\phi(\tau)$, $R(\tau)$ measure the *similarity* between the signal at time t and at time $t+\tau$.

Note: $R(\tau)$ can be defined on random signals.

Power/Energy Spectral Density Function

- **Energy spectral density**: (energy signal)

$$G(f) = \Im[\phi(\tau)] = |X(f)|^2$$

$$\phi(\tau) \equiv \Im^{-1}\{G(f)\} = \Im^{-1}[X(f)X^*(f)] = \Im^{-1}[X(f)] * \Im^{-1}[X^*(f)]$$

$$= x(\tau) * x^*(-\tau) = \int_{-\infty}^{\infty} x(\lambda)x^*(\lambda+\tau)d\lambda = \lim_{T\to\infty}\int_{-T}^{T} x(\lambda)x^*(\lambda+\tau)d\lambda$$

- **Power spectral density (psd):** (power signal)

$$S(f) \equiv \Im\{R(\tau)\} = \sum_{-\infty}^{\infty} |X_n|^2 \,\delta(f - nf_0)$$

Check it !!

<*Interpretation*> $G(f)$ and $S(f)$ represents the signal energy or power per unit frequency at freq f. ← squared magnitude of the FT .

Properties of R(τ)

- $|R(\tau)| \le R(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} S(f)df$ = power

  $\max\{R(\tau)\} = R(0)$ (Similar result for $\varphi(\tau)$; skip)

  $\varphi(0) = \int_{-\infty}^{\infty} G(f)df \ge \phi(\tau)$
- $R(\tau)$ are even for real *x*(*t*):

  $$R(-\tau) \equiv \langle x(t)x^*(t-\tau)\rangle = R(\tau).$$
- If *x*(*t*) does not contain a periodic component:

  $$\lim_{|\tau|\to\infty} R(\tau) = \langle x(t)\rangle^2.$$ (DC square)
- If *x*(*t*) is periodic with period T_0, then $R(\tau)$ is periodic in τ with the same period.
- *S*(*f*) is non-negative. $S(f) = \Im\{R(\tau)\} \ge 0, \ \forall f$

Crosscorrelation

- **Cross-correlation** of two *power* signals:

$$R_{xy}(\tau) \equiv \left\langle x(t)y^*(t+\tau)\right\rangle = \left\langle x(t-\tau)y^*(t)\right\rangle$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)y^*(t+\tau)dt$$

(cf: Z&T, p.318; Carlson, p.142~;- τ)

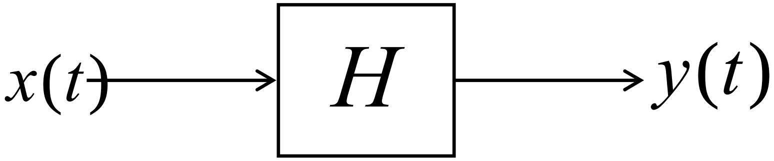
- **Cross-correlation** of two *energy* signals:

$$\varphi_{xy}(\tau) \equiv \int_{-\infty}^{\infty} x(t)y^*(t+\tau)dt$$

Remarks: $R_{xy}(\tau) = R_{yx}^*(-\tau)$, $\phi_{xy}(\tau) = \phi_{yx}^*(-\tau)$

Crosscorrelation of LTI I/O

- Input-Output correlation



$$1.\ R_{yx}(\tau) = \int_{-\infty}^{\infty} h(\lambda) R_x(\tau+\lambda)\, d\lambda = h(\tau) * R_x(-\tau)$$

$$2.\ R_y(\tau) = h(\tau) * R_{xy}(-\tau) = h(\tau) * h^*(-\tau) * R_x^*(\tau)$$

$$3.\ S_{yx}(f) = H(f) S_x(f)$$

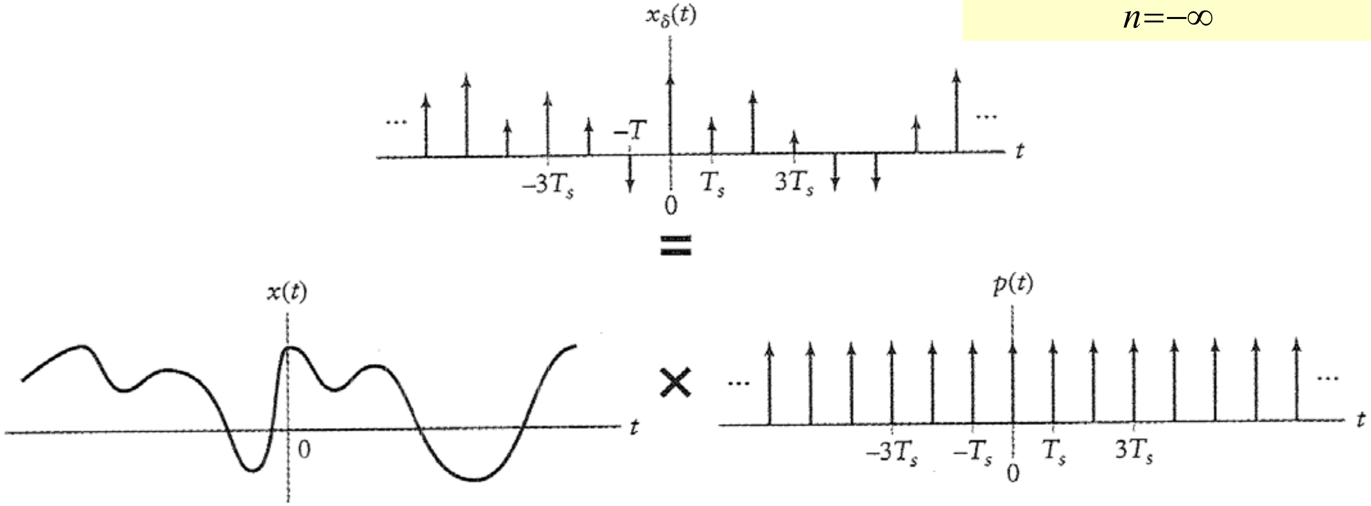
$$4.\ S_y(f) = H(f) H^*(f) S_x(f) = \left| H(f) \right|^2 S_x(f)$$

- *Note*: In proving them, we use:

$$\Im\{h(-\tau)\} = H(-f) \qquad \Im\{h^*(\tau)\} = H^*(-f)$$

Sampling Theory

- Ideal sampling using impulse train $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$



- **Impulse train** (an analog signal)
  T_s: the sampling period
- Analog (continuous-time) signal: $x(t)$
- **Sampled** (continuous-time) signal: $x_\delta(t)$

Mathematical Model

- Time domain:

$$x_\delta(t) = x(t)s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

- Freq domain:

$$X_\delta(f) = X(f) * S(f) = X(f) * [f_s \sum_{n=-\infty}^{\infty}\delta(f-nf_s)]$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s)$$

- Sampling → Duplicate shifted spectrum at nf_s
- Perfect Recovering → Retain only one copy of spectra
- **Aliasing**: If $f_s<2W$. the replicas of $X(f)$ overlap in frequency domain. That is, the higher frequency components of $X(f)$ overlap with the lower frequency components of $X(f-f_s)$.

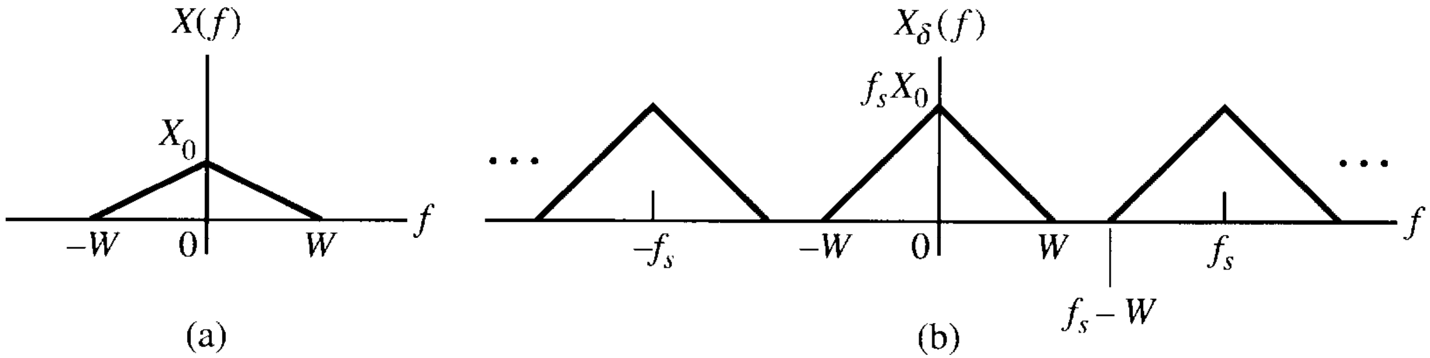
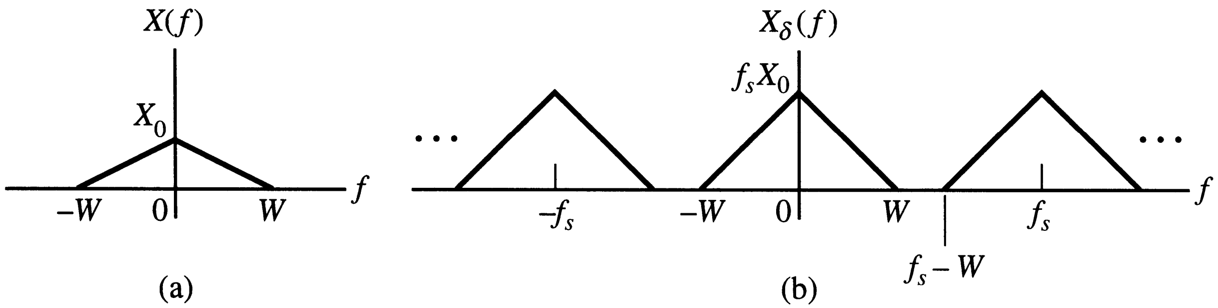


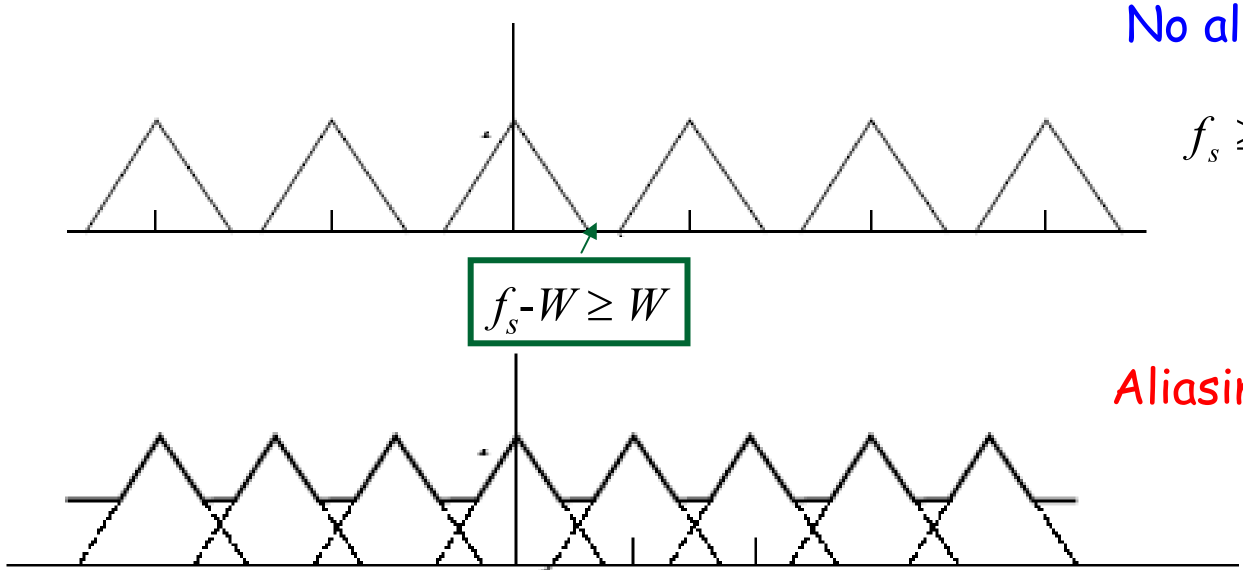
Figure 2.27
Signal spectra for lowpass sampling. (a) Assumed spectrum for $x(t)$. (b) Spectrum of the sampled signal.

Band-limited Sampling



No aliasing

$f_s \geq 2W$



Aliasing !!

distortion

Nyquist Sampling Theorem

- Let *x*(*t*) be a bandlimited signal with *X*(*f*) = 0 for $|f| \geq W$. (i.e., no components at frequencies greater than *W*.) Then, *x*(*t*) is uniquely determined by its samples

$$x[n] = x(nT_s),\ n = 0, \pm 1, \pm 2, \ldots$$

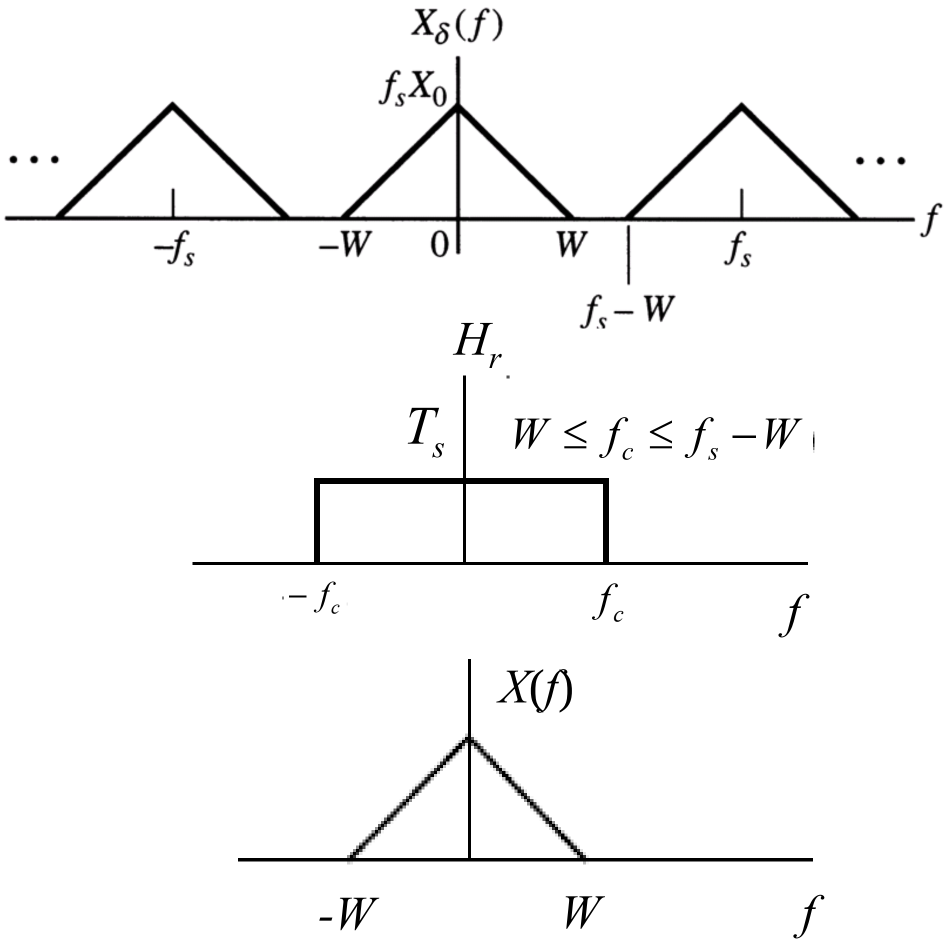
if $f_s = \dfrac{1}{T_s} \geq 2W.$ f_s is the sampling rate.

← Uniform sampling thm for **lowpass** signals (p.78)

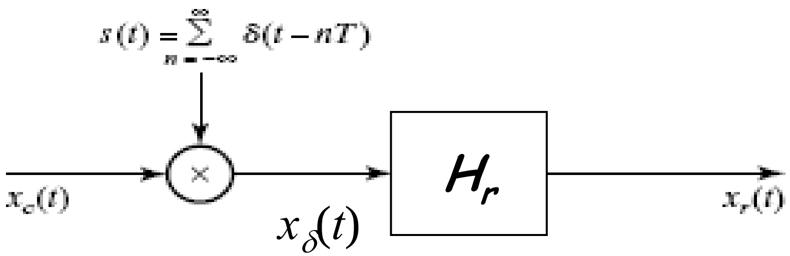
Undersampling: $f_s < 2W$

Oversampling: $f_s > 2W$

Recovery of Band-limited Sampling



By passing the equivalent impulse train x_δ through an ideal lowpass filter H_r with a cutoff at f_c and a gain of T_s



Reconstruction

- Ideal **reconstruction** filter (interpretation in time)

$$H(f) = H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}, \qquad W \le B \le f_s - W$$

$$\Rightarrow Y(f) = f_s H_0 X(f) e^{-j2\pi f t_0}$$

$$\Rightarrow y(t) = f_s H_0 x(t - t_0)$$

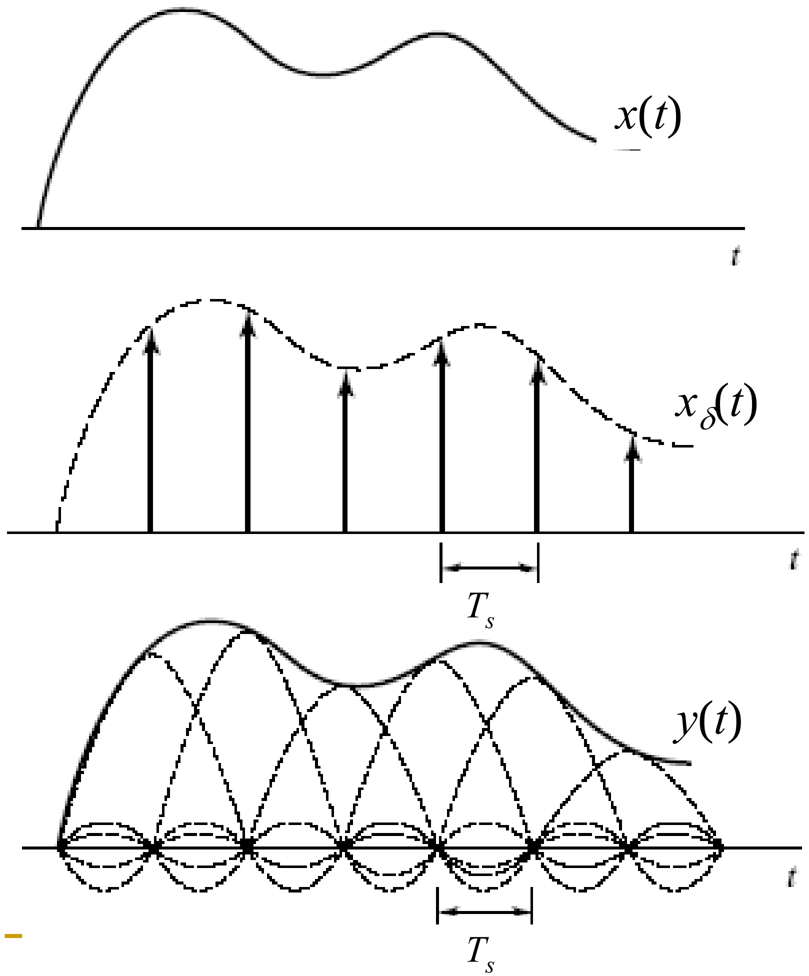
- In time domain:

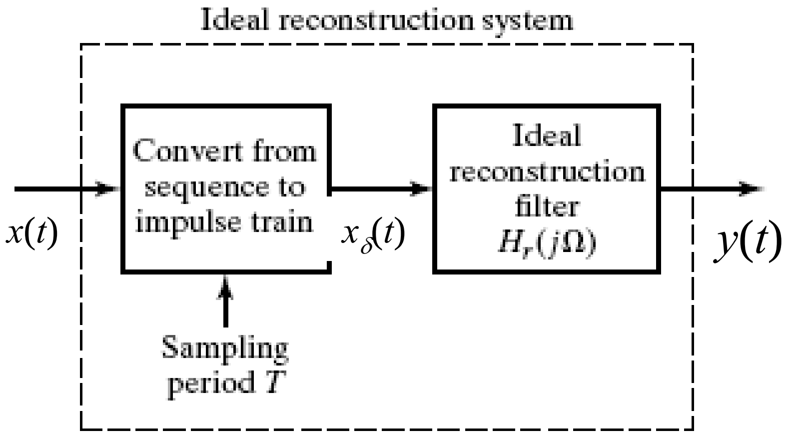
$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h(t - nT_s)$$

$$= 2BH_0 \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}[2B(t - t_0 - nT_s)]$$

interpolation

Ideal Band-limited Interpolation

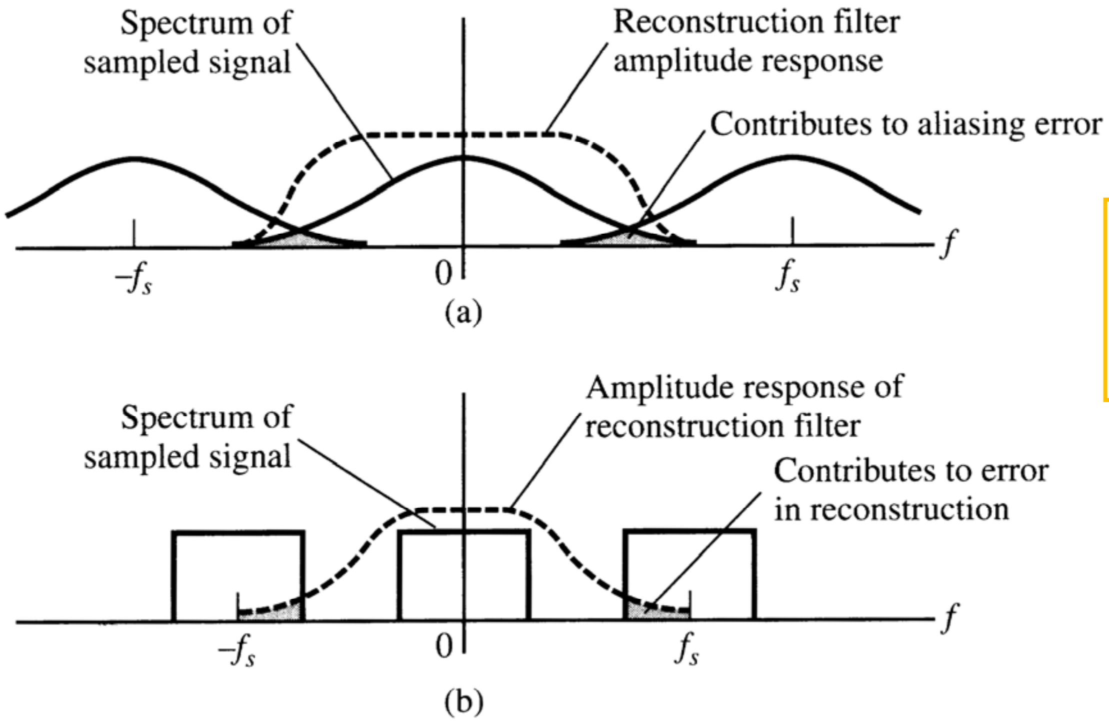




$B = \frac{1}{2} f_s$, $H_0 = T_s$, and $t_0 = 0$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}[(f_s t - n)]$$

Reconstruction Errors

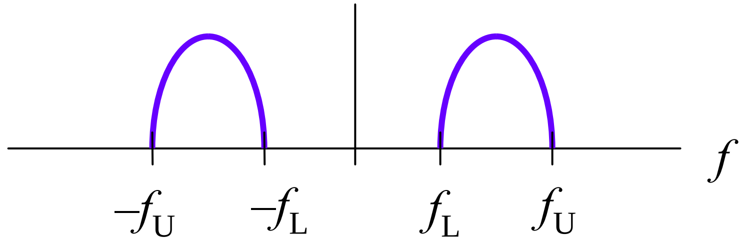


Two types of errors

Figure 2.28
Spectra illustrating two types of errors encountered in reconstruction of sampled signals. (a) Illustration of aliasing error in the reconstruction of sampled signals. (b) Illustration of error due to nonideal reconstruction filter.

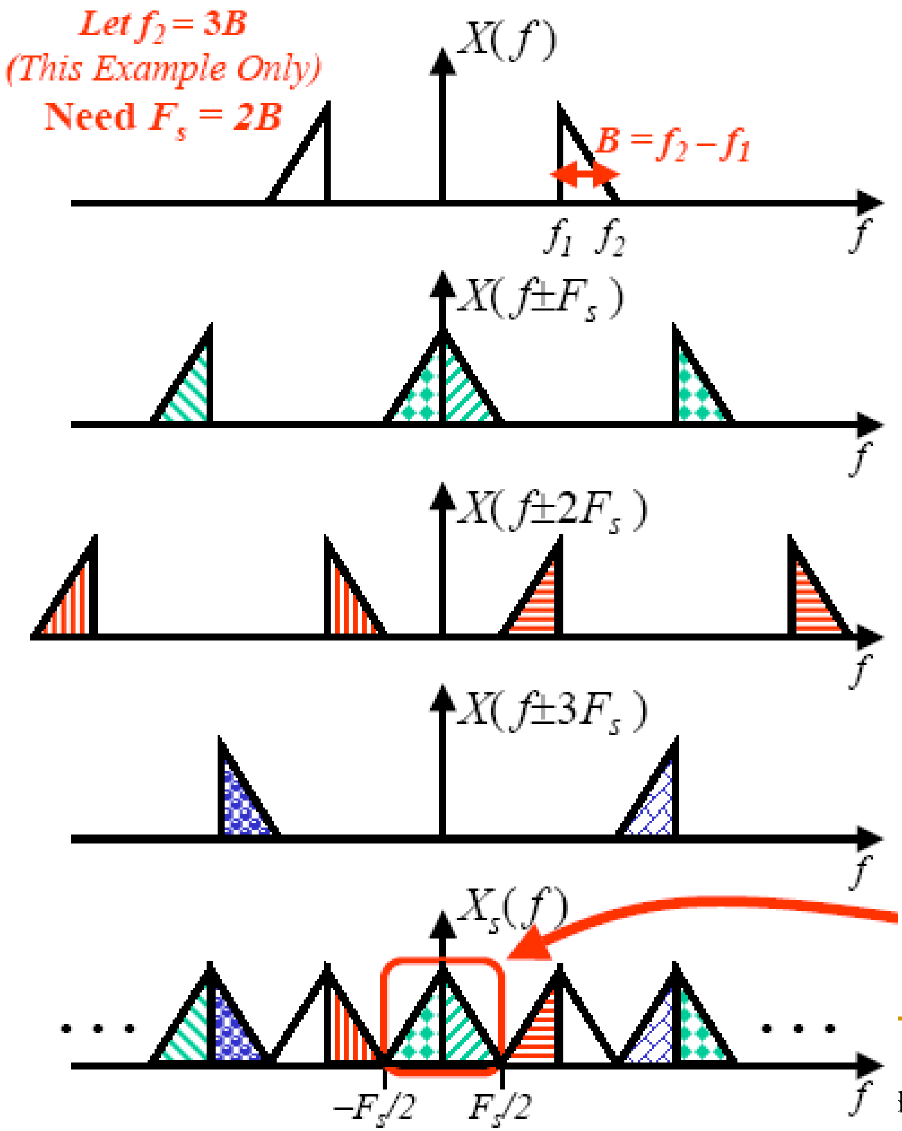
Bandpass Sampling?

- Bandpass signal can be obtained by modulating a lowpass signal



- Bandwidth BW $= f_U - f_L$
- Usually, $\boldsymbol{f_U \gg BW}$
- One can of course sample the bandpass signal with $f_s \geq 2f_U$ to prevent aliasing (by sampling theorem), but …

BP-Sampling

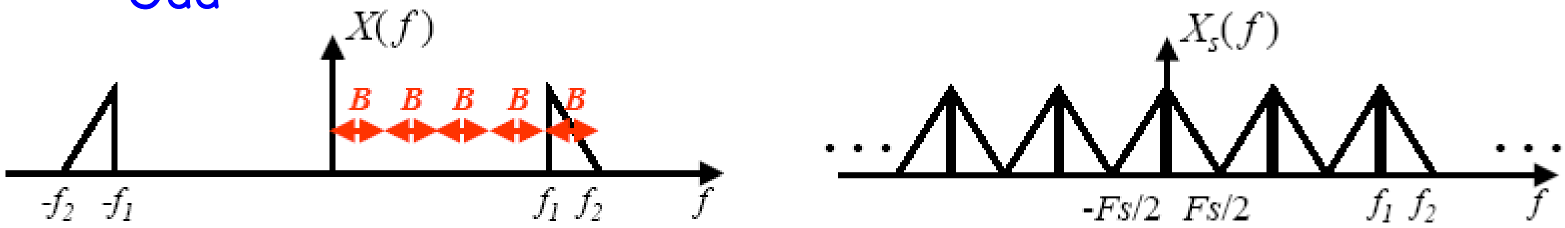


BP sampled signal is just a down-shifted version of the bandpass signal

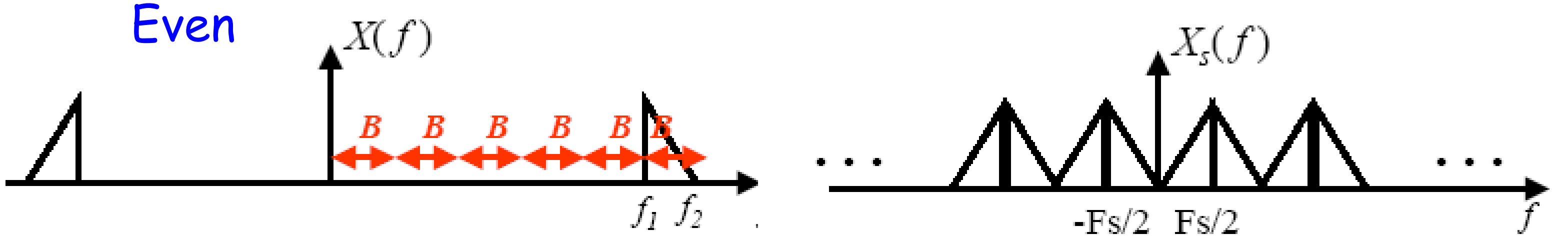
BP-Sampling: Simple Case

- Consider the simple case $f_u = L \times B$ BW=B
- L is an integer: choose $f_s = 2B$

Odd



Even



Uniform Sampling Thm for Bandpass Signals

- If the signal spectrum has a BW W (Hz) and upper frequency f_u, then a rate f_s at which the signal can be sampled is $2f_u/m$, where m is the largest integer not exceeding f_u/W. All higher sampling rates are not necessarily usable unless they exceed $2f_u$. (p.81)
- *Keys*: 1) BP signal; 2) BW= W Hz ; 3) $X(f)$<= f_u ;

  4) Lower limit f_s: $2f_u/m$, where $m \le f_u/W$

  5) High (lowpass Nyquist) f_s: $\geq 2f_u$

  6) Rates in between *may* (may not) work

  7) Additional details

Example of BP Signal Sampling

- W =1.2; f_u =3 → m =2; f_s =2x3/2=3

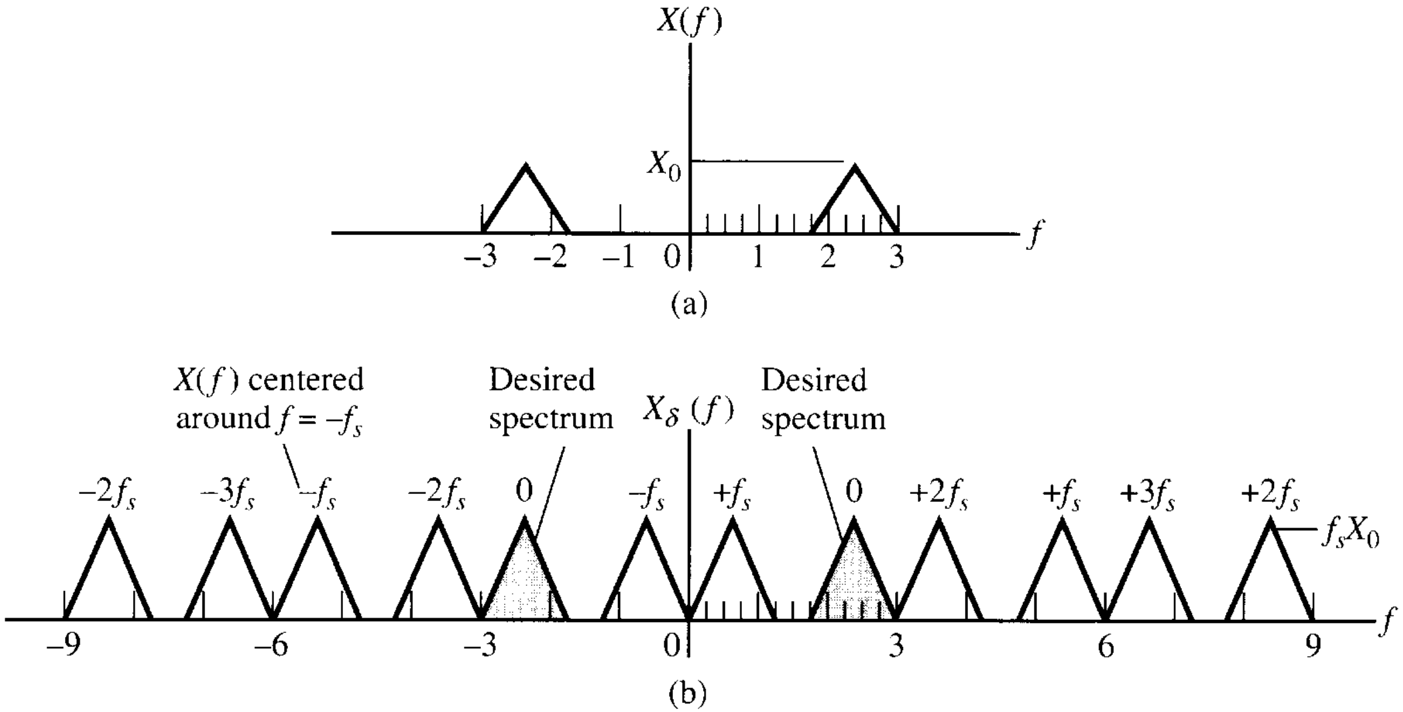


Figure 2.29
Signal spectra for bandpass sampling. (a) Assumed bandpass signal spectrum. (b) Spectrum of the sampled signal.

HW 2.69

- W =1(B); f_u =3 → m =3; f_s =2x3/2=2, okay!
- f_s =3, okay! f_s =4, okay!
- f_s >=6, okay!
- f_s =2.5, NOT! f_s =5, NOT!

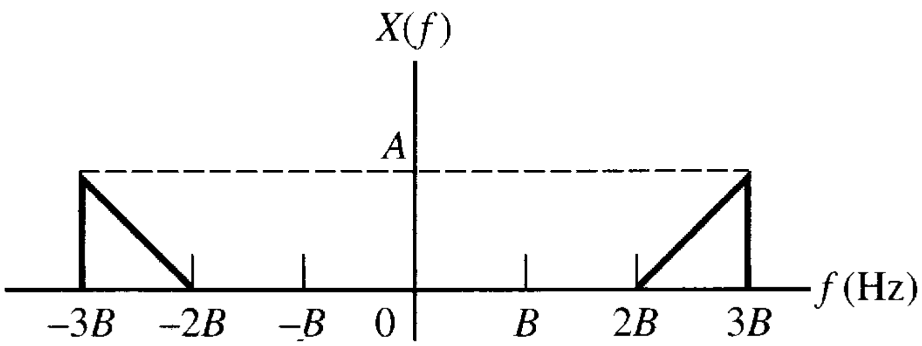
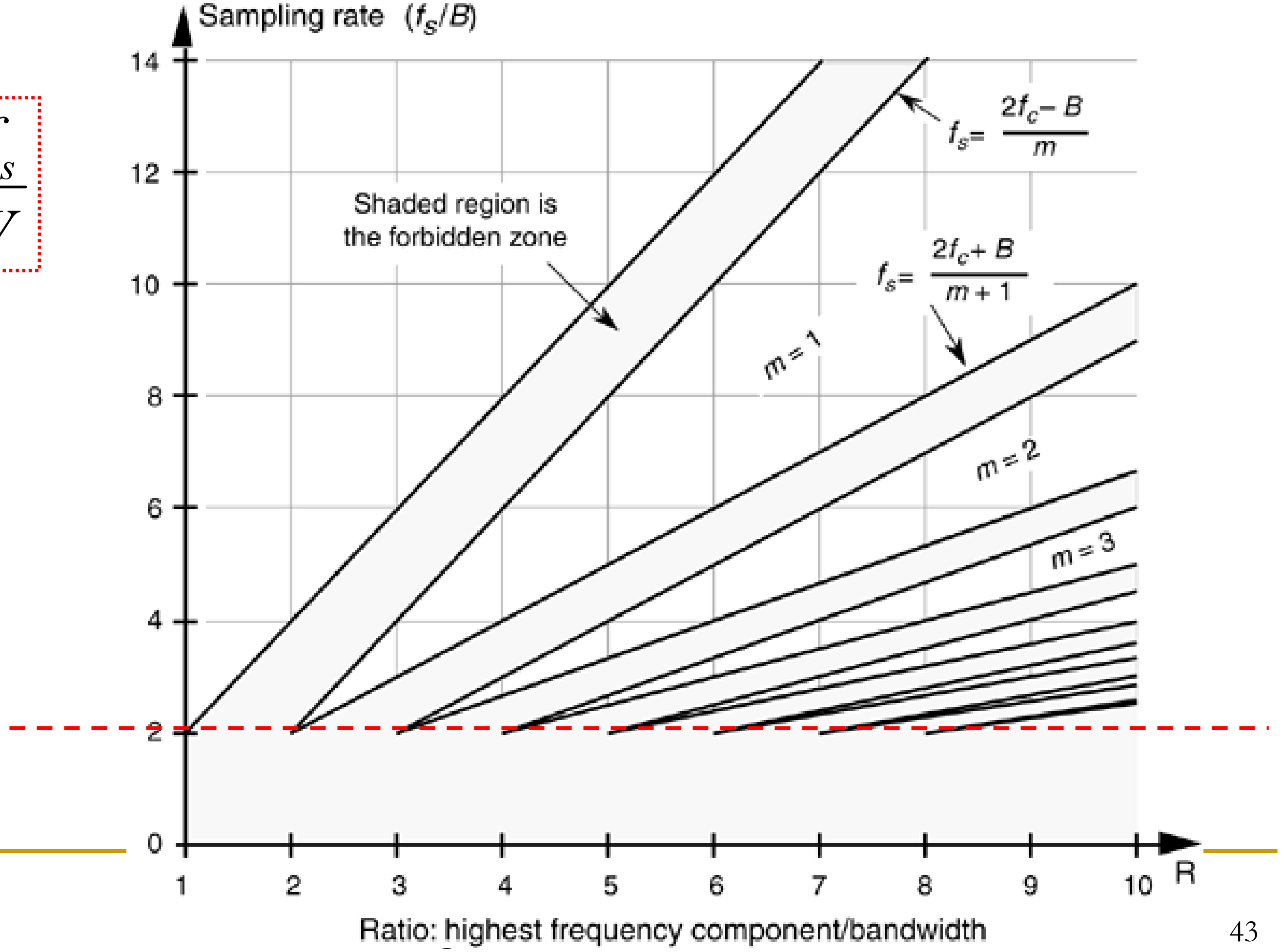


Figure 2.43

BP Sampling

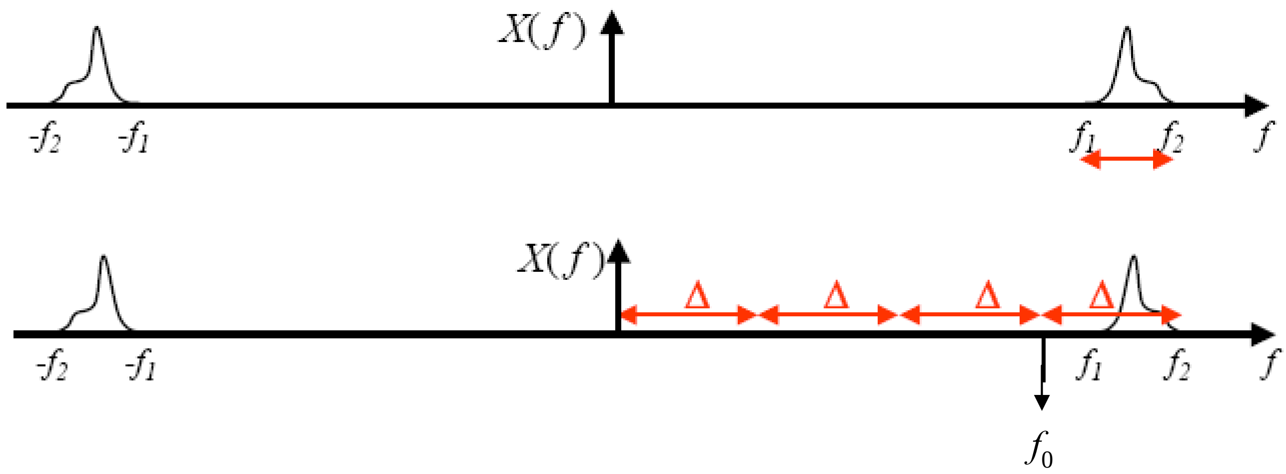
- R.G. Lyons, *Understanding Digital Signal Processing*, 2004, p.30~

$$R = \frac{f_u}{W}; \quad \frac{f_s}{W}$$



BP Sampling: General Case

- The general case: $f_u \neq L \times \text{W}$, L an integer
- Choose an frequency interval $[f_0, f_2]$ such that
  - $[f_0, f_2] \supseteq [f_1, f_2]$
  - $\Delta f = f_2 - f_0$ satisfies $f_2 = L \times \Delta f$, L an integer
- BP-Sample the signal at $f_s = 2\ \Delta f$



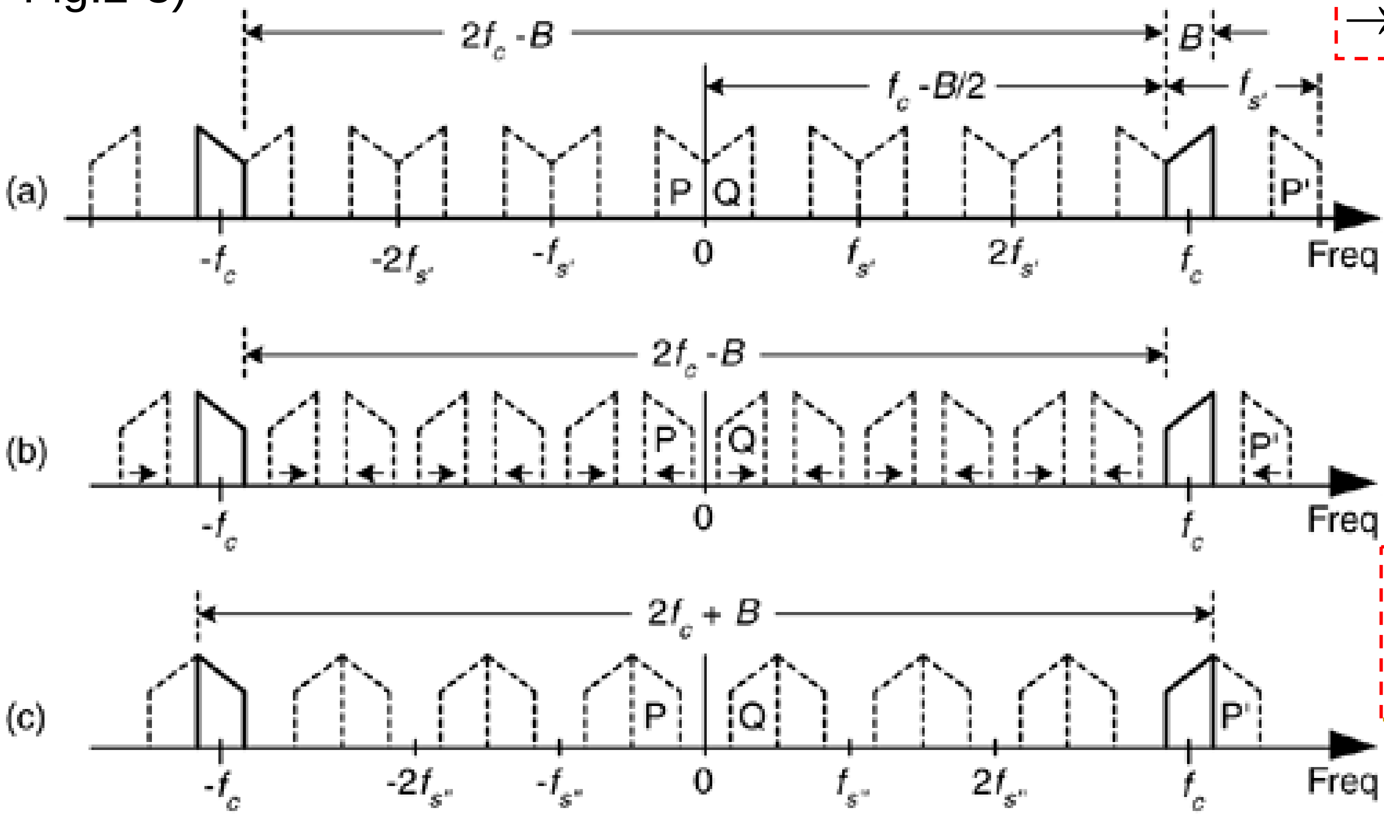
Derivation

- ***Key*: The replicas (in freq) due to sampling cannot overlap with "the original".**
- General: Insert replicas between $(-f_u + W)$ and $(f_u - W)$. Assume the number of replicas is m.
- Rule 1: *Higher limit* (of a given m) $mf_s^H = 2f_u - 2W$
  – Find the f_s that touches $(f_u - W)$ from left.
- Rule 2: *Lower limit* (of a given m) $(m+1)f_s^L = 2f_u$
  – Find the f_s that touches f_u from right.
- Rule 3: $f_s \geq 2W$ ➔ $\dfrac{2f_u - 2W}{m} \geq f_s \geq \dfrac{2f_u}{m+1}$

Higher & Lower Limits

- Insert m replicas → the f_s that lower SB just touches (f_u -W). [Note: B=W; f_c = (f_u –W/2), (f_c –B/2)= (f_u –W)]
- Reduce f_s ← the f_s that higher SB just touches (f_u)

(Lyons, Fig.2-8)



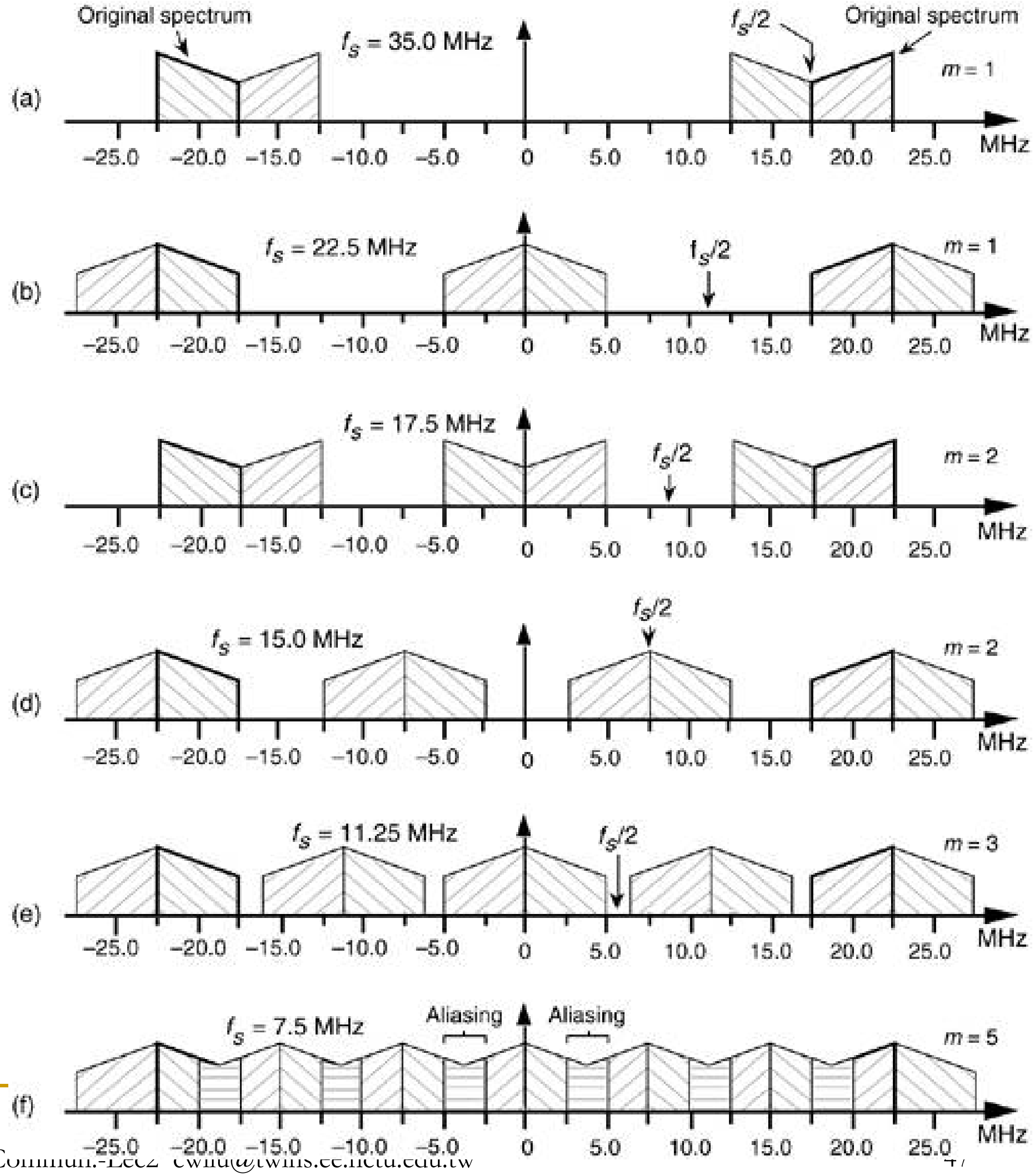
$$mf_s^H - f_u + W = f_u - W$$
$$\rightarrow mf_s^H = 2f_u - 2W$$

$$(m+1)f_s^L - f_u = f_u$$
$$\rightarrow (m+1)f_s^L = 2f_u$$

Example

(Lyons, Fig.2-9)
17.5 MHz ~22.5 M;
***W*=5**

m	$(2f_u - 2W)/m$	$2f_u/(m+1)$	Lowest sampling
1	35.0 MHz	22.5	22.5
2	17.5	15	15
3	11.66	11.25	11.25
4	8.75	9	--
5	7.0	7.5	--



Lowest Sampling Rate

$$f_s \geq \frac{2f_u}{m+1}$$

$$= \left[\frac{2(f_u / W)}{m+1}\right] W$$

Let $R = f_u / W$

Also, $f_s \geq 2W$

$\rightarrow$ largest integer m such that $\dfrac{2R}{m+1} \geq 2$

Let $m' = m+1 \rightarrow \dfrac{R}{m'} \geq 1$ or $R \geq m'$ $\Rightarrow f_s = \dfrac{2f_u}{m'} \geq 2W$

DFT & FT

- You can view DFT as a totally new definition for a different set of signals. Or, you can try to connect it to the Fourier Transform.
- DFT (Discrete FT) is defined on finite duration sequences.
- One approach: Finite-duration sequence

  → FT → (continuous) FT representation

  → sampling → DFT coefficients

DFT

- Definition:

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi nk}{N}}, \qquad n = 0,1,\cdots,N-1$$

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi nk}{N}}, \qquad k = 0,1,\cdots,N-1$$

- DFT may be view as an *N*-point to *N*-point *linear transformation* (matrix operation)
- z-transform, FT, DFT

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

$$H(k) = H(e^{j\omega})\big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} h(n)e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} h(n)W_N^{nk}$$

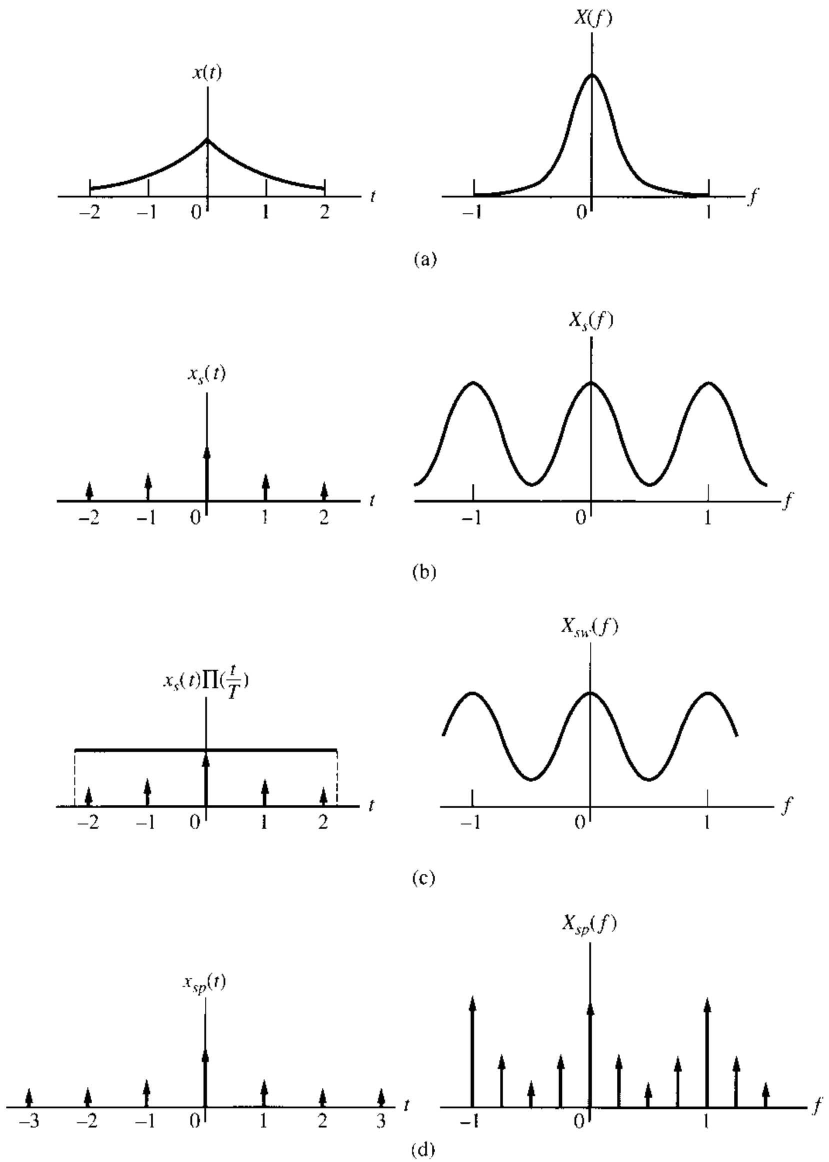


Figure 2.32
Signals and spectra illustrating the computation of the DFT. (a) Signal to be sampled and its spectrum ($\tau = 1$ s). (b) Sampled signal and its spectrum ($f_s = 1$ Hz). (c) Windowed, sampled signal and its spectrum $T \geq 4$ s). (d) Sampled signal spectrum and corresponding periodic repetition of the sampled, windowed signal.

FFT

- **Fast Fourier Transform** (FFT) is not a new transform; it is simply a fast way to compute DFT.
- *Application*: Convolution by FFT

  (Need to partition signal into segments and bridge the "gaps" between segments)

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) W_N^{nk} = \frac{1}{N}\sum_{k=0}^{N-1} H(k) X(k) W_N^{nk}$$