Principles of Communications Lecture 11: Stochastic Processes (II)

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Outlines

- Terminology of Random Processes
- Correlation and Power Spectral Density
- Linear Systems and Random Processes
- Narrowband Noise

Bandlimited White Noise

- "White" ~ flat spectrum

$$psd = S(f) = \begin{cases} \frac{1}{2} N_0, & |f| \le B \\ 0, & \text{otherwise} \end{cases}.$$

$$R(\tau) = BN_0 \text{sinc}(2B\tau).$$

$$\bullet \text{ White Noise: } B \to \infty, \quad \Rightarrow \; R(\tau) \to \frac{N_0}{2}\delta(\tau).$$

$$\text{Therefore, the samples of } n(t) \text{ are uncorrelated.}$$

$$R_n(t_1, t_2) = m_n(t_1) m_n(t_2) = m_n \cdot m_n.$$

PAM with Random Delay

$$X(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT - \Delta).$$

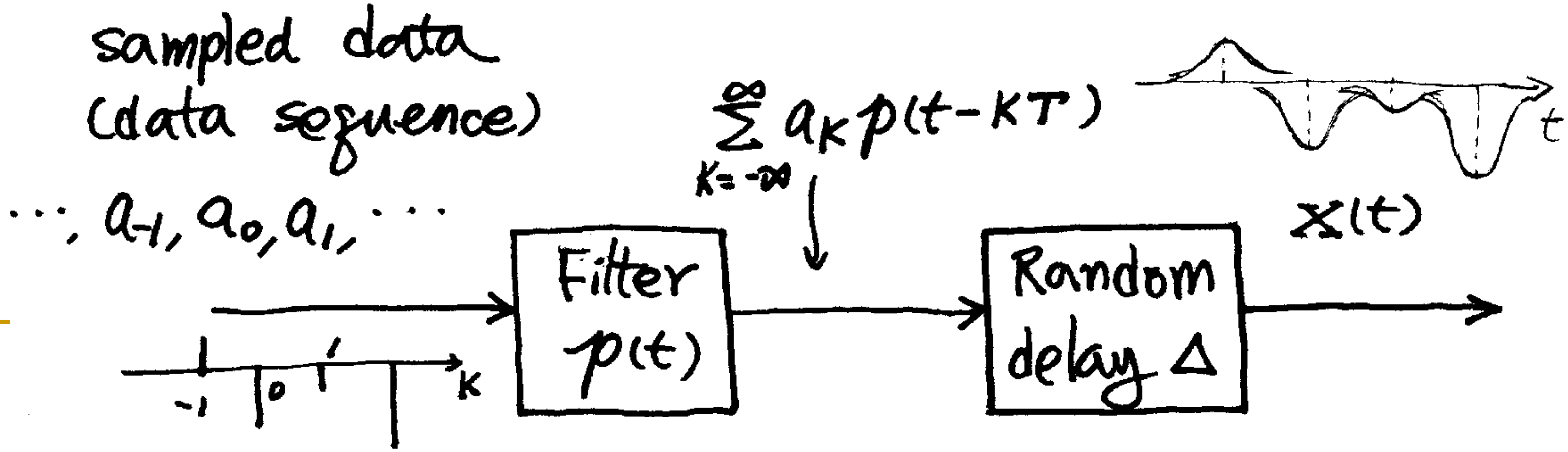
Assumptions:

a_K: WSS random process with $E[a_k] = 0$

$E[a_k a_{k+m}] = R_m$ a deterministic sequence of m.

Δ: random delay, uniformly distributed over $(-T/2, T/2)$.

Mean of $X(t)$: $E[X(t)] = \sum_{k=-\infty}^{\infty} E[a_k] E[p(t - kT - \Delta)] = 0.$



The autocorrelation function of *X(t)*:

$$R_X(\tau) = E[X(t)X(t+\tau)] = E\{\sum_{k=-\infty}^{\infty} a_k p(t-kT-\Delta) \sum_{i=-\infty}^{\infty} a_i p(t+\tau-iT-\Delta)\}$$

$$= E\{\sum_{k=-\infty}^{\infty} a_k p(t-kT-\Delta) \sum_{m=-\infty}^{\infty} a_{k+m} p(t+\tau-(k+m)T-\Delta)\}$$

$$= E\{\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k a_{k+m} p(t-kT-\Delta) p(t+\tau-(k+m)T-\Delta)\}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\{a_k a_{k+m}\} E\{p(t-kT-\Delta) p(t+\tau-(k+m)T-\Delta)\}$$

pdf of Δ

$$= \sum_{m=-\infty}^{\infty} R_m \sum_{k=-\infty}^{\infty} \int_{-T/2}^{T/2} p(t-kT-\Delta) p(t+\tau-(k+m)T-\Delta) \frac{1}{T} d\Delta$$

$$= \sum_{m=-\infty}^{\infty} R_m \sum_{k=-\infty}^{\infty} \int_{t-(k-\frac{1}{2})T}^{t-(k+\frac{1}{2})T} p(u) p(u+\tau-mT) \frac{-du}{T}.$$

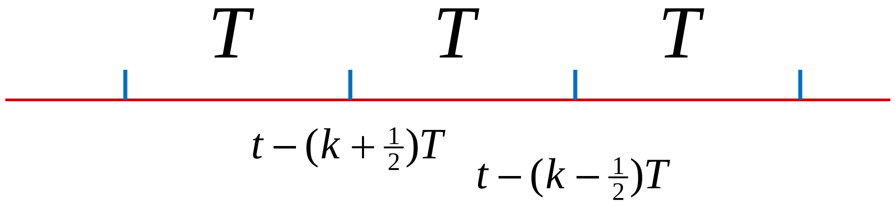
Let $u = t - kT - \Delta$

- (Continue on next page.)

If $\sum_{k=-\infty}^{\infty} \int_{t-(k-\frac{1}{2})T}^{t-(k+\frac{1}{2})T} p(u)p(u+\tau-mT)\frac{-du}{T}$ converges, it is independent of t. That is, the result is the same as $t=0$.

$$R_X(\tau) = \sum_{m=-\infty}^{\infty} R_m \frac{1}{T}\int_{-\infty}^{\infty} p(u)p(u+\tau-mT)du$$

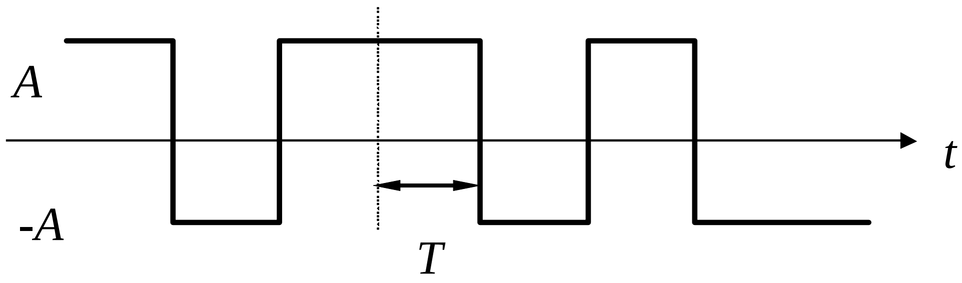
$$= \sum_{m=-\infty}^{\infty} R_m r(\tau - mT)$$



$$= \sum_{m=-\infty}^{\infty} R_m r(\tau + mT) \qquad (R_m = R_{-m})$$

$= R_m \otimes r(\tau)$. (discrete convolution)

where $r(\tau) \equiv \frac{1}{T}\int_{-T/2}^{T/2} p(u)p(u+\tau)du$ is the autocorrelation function of the pulse function $p(t)$.



- Case I : Binary random waveform (memoryless)

$$a_k = \pm A \quad \text{with } R_m = \begin{cases} A^2 & m = 0 \\ 0 & m \neq 0 \end{cases}. \qquad \text{(i.i.d. binary sequence)}$$

$$p(t) = \Pi(\frac{t}{T}). \qquad \text{(unit pulse)}$$

$$r(\tau) = \frac{1}{T}\int_{-\infty}^{\infty} \Pi(\frac{t}{T})\Pi(\frac{t+\tau}{T})dt = \frac{1}{T}\int_{-T/2}^{T/2} \Pi(\frac{t+\tau}{T})dt = \Lambda(\frac{\tau}{T}).$$

$$R_X(\tau) = R_m \otimes \Lambda(\frac{\tau}{T}) = A^2\delta(m) \otimes \Lambda(\frac{\tau}{T}) = A^2\Lambda(\frac{\tau}{T}).$$

$$\Rightarrow S_X(f) = A^2T \cdot \text{sinc}^2(fT).$$

- Case II: Correlated message (with memory)

$a_k = g_0 A_k + g_1 A_{k-1}$, where g_0 and g_1 are constants and A_k is a sequence of random variables that $A_k = \pm A$ and $E[A_k A_{k+m}] = \begin{cases} A^2, & m = 0 \\ 0, & m \neq 0 \end{cases} \equiv R_A(m)$.

$p(t) = \Pi(\frac{t}{T})$. (unit pulse)

$$R_m = E[a_k a_{k+m}] == E[(g_0 A_k + g_1 A_{k-1})(g_0 A_{k+m} + g_1 A_{k+m-1})]$$

$$= E[g_0^2 A_k A_{k+m} + g_1^2 A_{k-1} A_{k+m-1} + g_0 g_1 A_k A_{k+m-1} + g_0 g_1 A_{k-1} A_{k+m}]$$

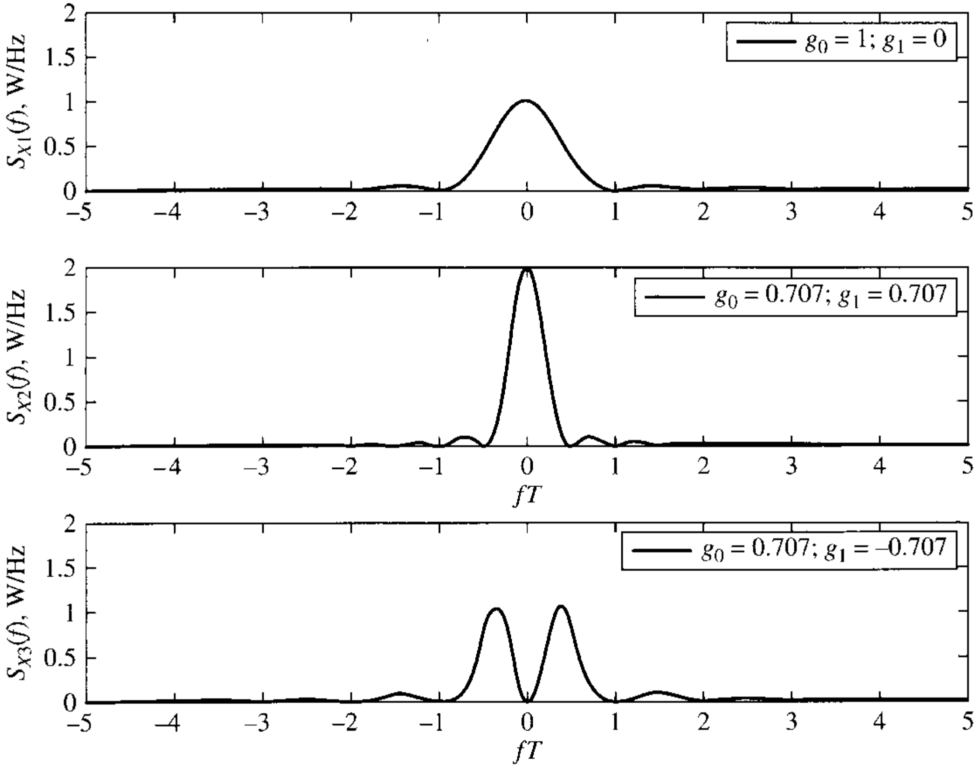
$$= g_0^2 R_A(m) + g_1^2 R_A(m) + g_0 g_1 R_A(m-1) + g_0 g_1 R_A(m+1)$$

$$\Rightarrow R_m = \begin{cases} (g_0^2 + g_1^2) \cdot A^2, & m = 0 \\ \mathrm{g}_0 g_1 A^2, & m = \pm 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$R_X(\tau) = R_m \otimes \Lambda(\frac{\tau}{T}) = A^2\{(g_0^2 + g_1^2)\Lambda(\frac{\tau}{T}) + g_0 g_1[\Lambda(\frac{\tau+T}{T}) + \Lambda(\frac{\tau-T}{T})].$$

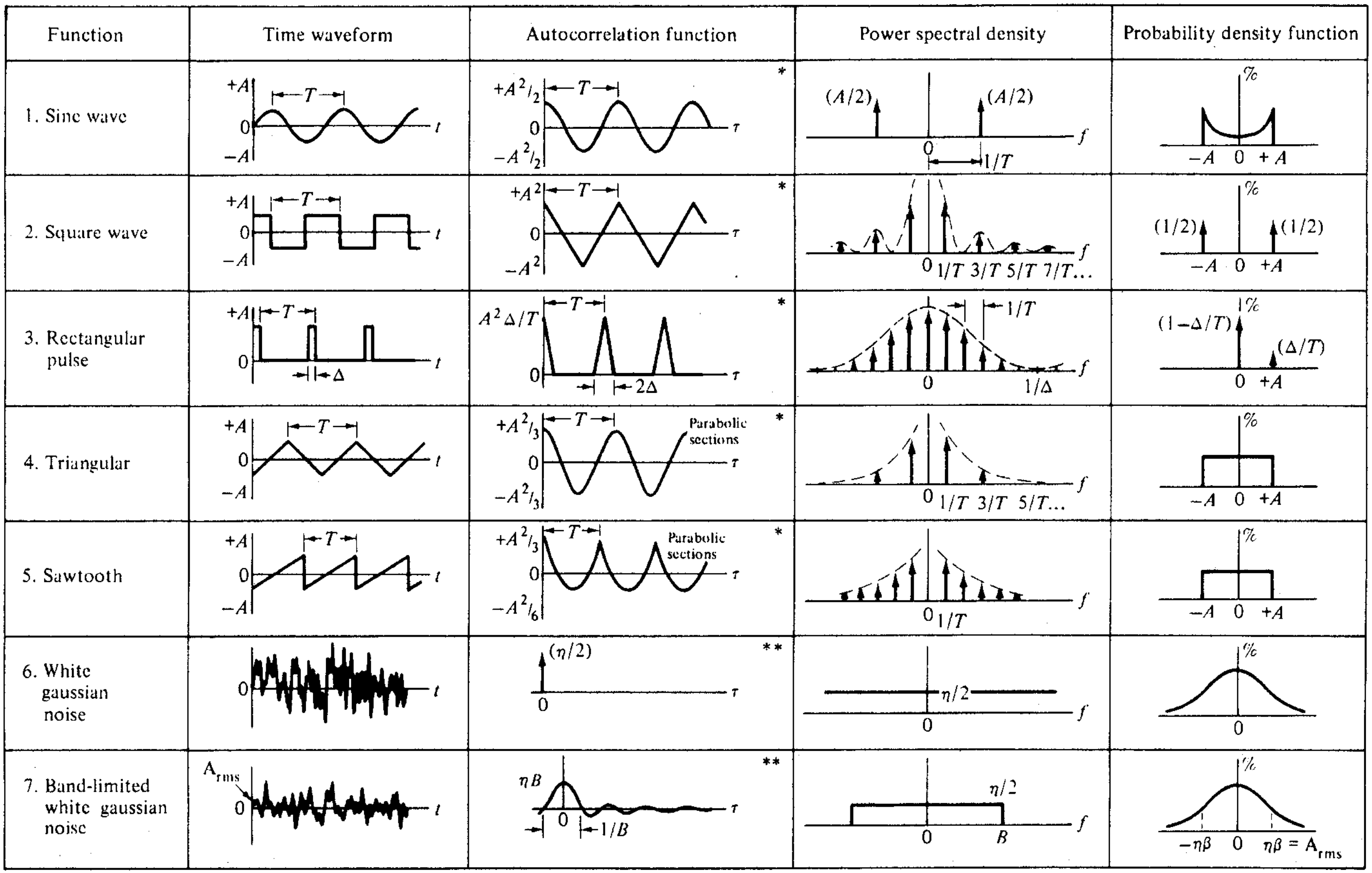
$$\Rightarrow S_X(f) = A^2 T \cdot \text{sinc}^2(fT)\{(g_0^2 + g_1^2 + g_0 g_1[e^{-j2\pi fT} + e^{j2\pi fT}]\}.$$

If $g_0 = g_1 = 1, \Rightarrow S_X(f) = 4A^2T \cdot \text{sinc}^2(fT)\cos^2(\pi fT).$



Compare different g_0 and g_1 values.

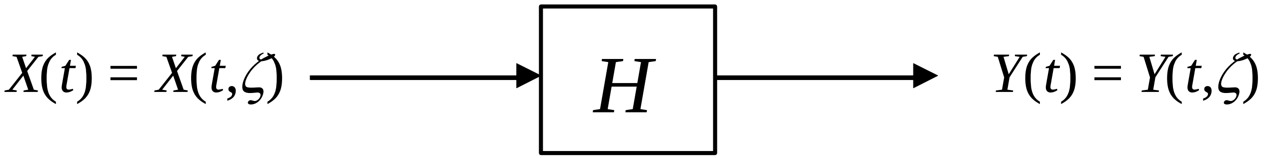
Figure 6.6
Power spectra of binary-valued waveforms. (a) Case in which there is no memory. (b) Case in which there is reinforcing memory between adjacent pulses.(c) Case where the memory between adjacent pulses is antipodal.



F.G. Stremler, *Intr. Commu. Systems*, p. 509, 3rd, ed, 1990.

Figure 8.19 Autocorrelation, spectra, and probability densities for some commonly used waveforms (* = time, ** = statistical).

Linear Systems and RP



- System without memory: a r.v.$\rightarrow$ a r.v.
- System with memory: correlated outputs
- Now, we study only the statistics between inputs and outputs, e.g., $m_y(t)$, $R_y(\tau)$, …
- Assume $X(t)$ stationary (or WSS at least) and $H(\cdot)$ is LTI.

- **Mean of Output** $Y(t)$:

$$m_y(t) = E_\varsigma[Y(t)] = E_\varsigma[\int_{-\infty}^{\infty} h(u)X(t-u)du]$$

$$= \int_{-\infty}^{\infty} h(u) \cdot E_\varsigma[X(t-u)]du = m_X \cdot H(0).$$

- **Cross-correlation Function**:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[X(t)\int_{-\infty}^{\infty} h(u)X(t+\tau-u)du]$$

$$= \int_{-\infty}^{\infty} h(u)E[X(t)X(t+\tau-u)]du = \int_{-\infty}^{\infty} h(u)R_X(\tau-u)du.$$

$$\Rightarrow R_{XY}(\tau) = h(\tau) * R_X(\tau).$$

$$\Rightarrow S_{XY}(f) = H(f)S_X(f).$$

- $R_{YX}(\tau) = R_{XY}(-\tau) = h(-\tau) * R_X(-\tau) = h(-\tau) * R_X(\tau).$

$$S_{YX}(f) = S_{XY}(-f) = H(-f)S_X(f) = H^*(f)S_X(f).$$

- **Autocorrelation Function of Output**:

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[Y(t)\{h(t) * X(t+\tau)\}]$$

$$= E[Y(t)\int_{-\infty}^{\infty} h(u)X(t+\tau-u)du] = \int_{-\infty}^{\infty} h(u)E[Y(t)X(t+\tau-u)]du$$

$$= \int_{-\infty}^{\infty} h(u)R_{YX}(\tau-u)du = h(\tau) * R_{YX}(\tau).$$

$$R_Y(\tau) = h(\tau) * R_{YX}(\tau) = h(\tau) * h(-\tau) * R_X(\tau).$$

$$S_Y(f) = H(f)H^*(f)S_X(f) = |H(f)|^2 S_X(f).$$

- Remarks: (1) If $X(t)$ is WSS and $h(t)$ is LTI (no initial condition),

  $\Rightarrow$ $Y(t)$ is WSS too.

(2) So far, we only consider 2nd order statistics (mean, correlation). In general, given the joint pdf of $X(t)$, it is very difficult to find the joint pdf of $Y(t)$. But if $X(t)$ is jointly Gaussian, then $Y(t)$ is also jointly Gaussian and thus is completely characterized by mean and correlation functions.

Gaussian Random Process

- Gaussian Random Process: $X(t)$ has joint Gaussian pdf (of all orders).
  - Special Case 1: **Stationary** Gaussian random process

    Mean = m_x; auto-correlation = $R_X(\tau)$.
  - Special Case 2: **White** Gaussian random process

    $R_X(t_1,t_2) = \delta(t_1-t_2) = \delta(\tau)$ and $S_X(f) = 1$ (constant).

- **White** → correlation in **time**
- **Gaussian** (or Lapacian,…) → pdf on **magnitude**

Response due to Gaussian Input

- Show that if $X(t)$ is stationary Gaussian, so is $Y(t)$.

Case I: $X(t)$ is white ($\Rightarrow$ independent random variables).

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} X(\tau) h(t-\tau) d\tau$$

$$= \lim_{\Delta\tau \to 0} \sum_{k=-\infty}^{\infty} X(k\Delta\tau) \cdot h(t - k\Delta\tau)\Delta\tau$$

$=$ weighted sum of Gaussian random variables.

$\therefore$ $Y(t)$ has a 1st-order Gaussian distribution. Similarly, the higher order joint pdf of $Y(t)$ is jointly Gaussian.

For example, $n = 2$, time $= t_1, t_2$

$f_Y(y_1, y_2) = f_Y(y_2 \mid y_1) f_Y(y_1)$ (would be jointly Gaussian).

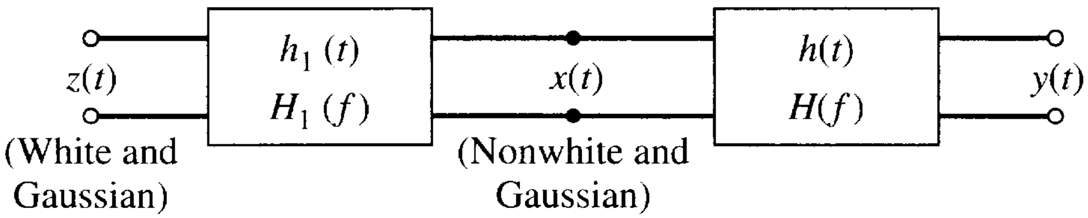


Figure 6.7
Cascade of two linear systems with Gaussian input.

- Case II: $x(t)$ is Gaussian but not white (more realistic case).

Claim: $x(t)$ is produced by passing a white Gaussian random process through $h_1(t)$.

Then, define $h_2(t) = h_1(t) * h(t)$, we have $y(t) = h_2(t) * Z(t)$.

$\Rightarrow$ Back to Case I. Therefore, $y(t)$ is stationary Gaussian.

- Note: We often use lower-case letters (x and y instead of X and Y) for denoting random processes. You need to judge by the context whether it is a deterministic signal or a random process.

Properties of Gaussian Processes

(1) $X(t)$ Gaussian, $H(\cdot)$ stable, linear $\Rightarrow$ $Y(t)$ Gaussian.

(2) $X(t)$ Gaussian and WSS $\Rightarrow$ $X(t)$ is SSS.

(3) Samples of a Gaussian process, $X(t_1)$, $X(t_2)$, …, are uncorrelated $\Rightarrow$ They are independent.

(4) Samples of a Gaussian process, $X(t_1)$, $X(t_2)$, …, have a joint Gaussian pdf specified completely by the set of means and auto-covariance function. $E[(X(t_j)-m_{X_j})(X(t_i)-m_{X_i})]$

- *Remarks*: Why do we use Gaussian model?
  - Easy to analyze.
  - *Central Limit Theorem*: Many "independent" events combined together become a Gaussian random variable (random process).

Ex: RC Filter with White Gaussian Input

Input: white Gaussian and zero-mean with $S_{n_i}(f) = \dfrac{N_0}{2}$.

Filter: $H(f) = \dfrac{1}{1 + j2\pi fRC} = \dfrac{1}{1 + j\, f/f_3}$,

where f_3=3 dB cutoff frequency $= \dfrac{1}{2\pi RC}$.

Output: $S_{n_0}(f) = S_{n_i}(f)\cdot\left|H(f)\right|^2 = \dfrac{N_0}{2}\dfrac{1}{1 + (f/f_3)^2}$.

$$R_{n_0}(\tau) = \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}}.$$

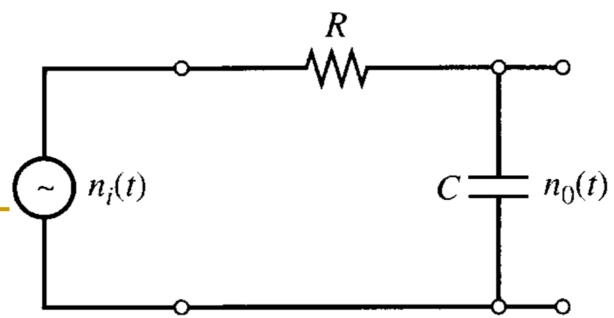


Figure 6.8
A lowpass RC filter with a white-noise input.

- Output power: $\overline{n_0^2(t)} = \sigma_{n_0}^2 = R_{n_0}(0) = \dfrac{N_0}{4RC}.$

  Another approach: $\overline{n_0^2(t)} = \int_{-\infty}^{\infty} S_{n_0}(f)df = \dfrac{N_0}{2\pi RC}\int_0^{\infty}\dfrac{dx}{1+x^2} = \dfrac{N_0}{4RC}.$

Mean: $\overline{n_0(t)} = 0 \cdot H(0) = 0.$

  Another approach: $\left(\overline{n_0(t)}\right)^2 = \lim_{|\tau|\to\infty} R_{n_0}(\tau) = 0.$

The first-order pdf: $f_{n_0}(y,t) = \dfrac{1}{\sqrt{\dfrac{\pi N_0}{2RC}}} e^{-\frac{2RCy^2}{N_0}}.$

- Note: $R(\tau)$ does not "completely" describes the behaviour of a random process.

Noise Equivalent Bandwidth

Q: What is the bandwidth of an ideal *fictitious* filter that has the same mid-band gain as *H*(*f*) and that passes the same noise power?

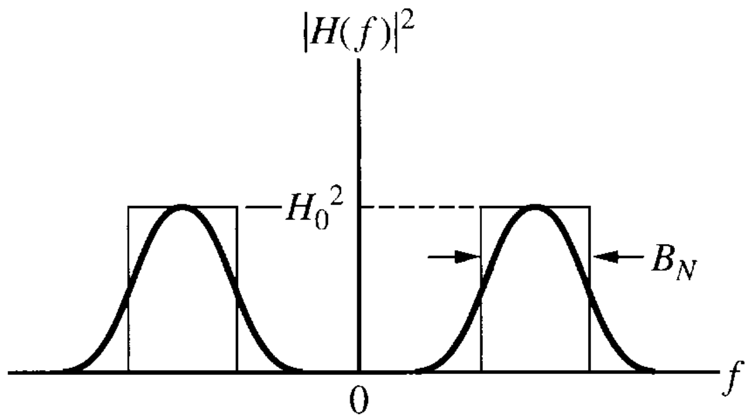


Figure 6.9
Comparison between $|H(f)|^2$ and an idealized approximation.

$$\text{Noise – equivalent bandwidth of } H(f): B_N \equiv \frac{1}{H_0^2}\int_0^{\infty}|H(f)|^2\,df,$$

where H_0 is the midband (maximum) gain.

***H*(*f*)**	**Ideal BPF**
Average Power	Average Power
$P_{n_0} = \int_{-\infty}^{\infty} \frac{N_0}{2} \lvert H(f) \rvert^2 df$ $= \int_{0}^{\infty} \lvert H(f) \rvert^2 df \cdot N_0$	$P_{n_0} = \frac{N_0}{2} B_N \cdot 2 \cdot H_0^2$ $= N_0 B_N H_0^2$

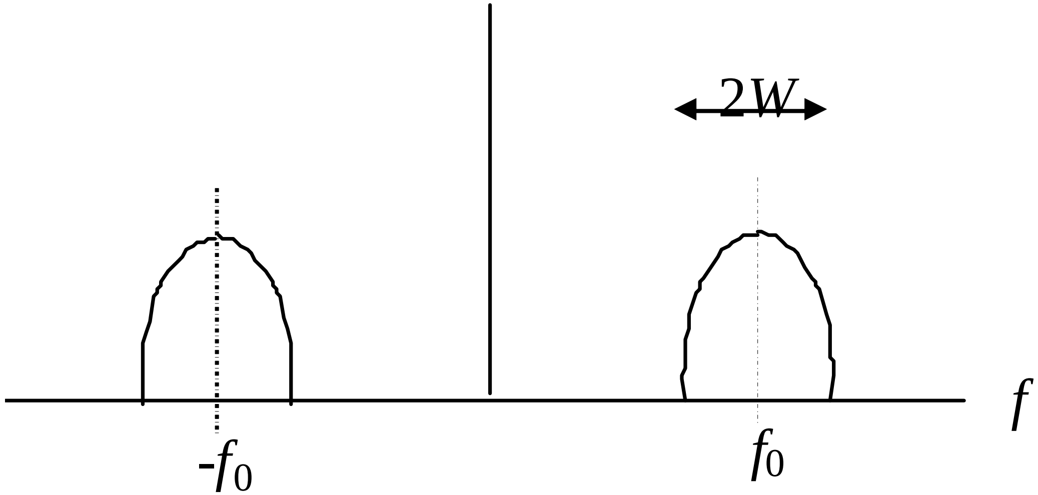
- If the target system is a lowpass filter with maximum gain at f=0, then we have the following relationships.

$$\because \begin{cases} \int_{-\infty}^{\infty} \lvert H(f) \rvert^2 df = \int_{-\infty}^{\infty} \lvert h(t) \rvert^2 dt & \text{Rayleigh's Theorem} \\ H_0 = \int_{-\infty}^{\infty} h(t) dt \end{cases}$$

$$\Rightarrow B_N = \frac{1}{H_0^2} \int_{0}^{\infty} \lvert H(f) \rvert^2 df = \frac{1}{2} \frac{\int_{-\infty}^{\infty} \lvert h(t) \rvert^2 dt}{\{\int_{-\infty}^{\infty} h(t) dt\}^2}$$

Narrowband Noise

- Usage: BP noise (signal)



$W << f_0$

- *Interpretation*: A baseband random process is shifted to a higher frequency.

$$n(t) = R(t)\cos 2\pi f_0 t, \quad R(t): \text{RP}; \cos(): \text{carrier}$$

However, a bandpass random signal can have a random phase:

$$n(t) = R(t)\cos(2\pi f_0 t + \phi(t)).$$

In general, if we allow a time-invariant phase bias θ, then

(**Envelope - phase** representation)

$$n(t) = R(t)\cos(\omega_0 t + \phi(t) + \theta)$$

or (**Quadrature - component** representation)

$$n(t) = n_c(t)\cos(\omega_0 t + \theta) - n_s(t)\sin(\omega_0 t + \theta)$$

with $R(t) = \sqrt{n_c^2(t) + n_s^2(t)}$ and $\phi(t) = \tan^{-1}(\dfrac{n_s(t)}{n_c(t)})$.

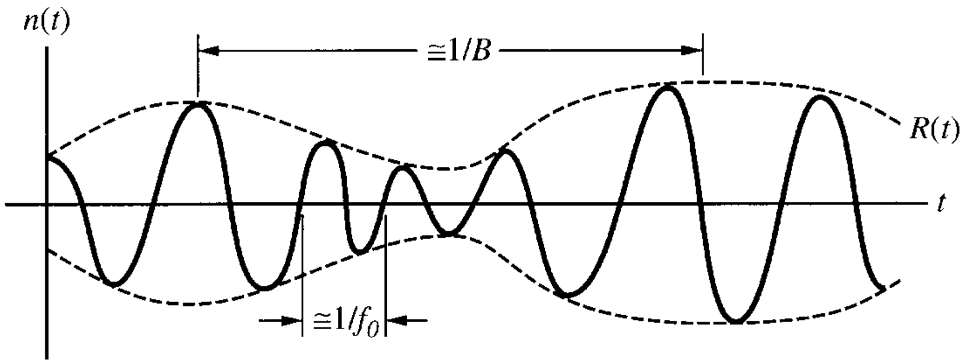


Figure 6.11
A typical narrowband noise waveform.

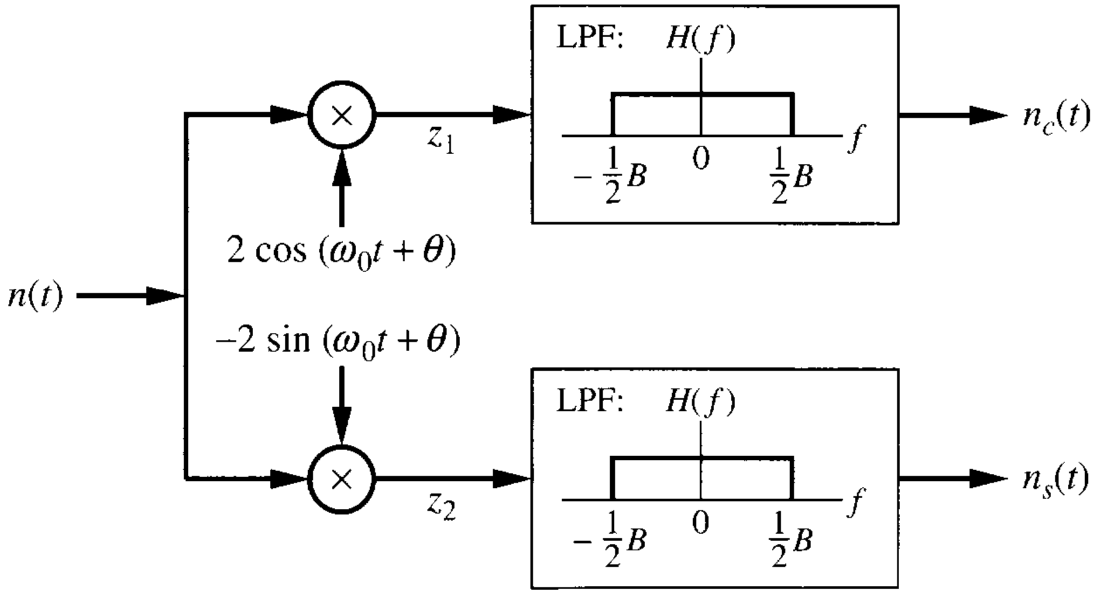


Figure 6.12
The operations involved in producing $n_c(t)$ and $n_s(t)$.

- How to produce n_c(t) and $n_s(t)$:

$$n(t) \approx n_c(t)\cos(\omega_0 t + \theta) - n_s(t)\sin(\omega_0 t + \theta)$$

($\approx$ in mean-square sense)

Remarks: Here, we assume (initial phase) θ is a random variable, independent of $n(t)$, uniformly distributed over $(0, 2\pi)$ or $(-\pi, \pi)$. If θ is not a random variable, $z_1(t)$ and $z_2(t)$ are not WSS. We cannot use LTI theory to predict the outputs of LPF's.

Quadrature-Component Representation

(1) $\overline{n(t)} = \overline{n_c(t)} = \overline{n_s(t)} = 0.$

Proof: (i) Let $X(t) \equiv n(t) - \overline{n(t)}.$

$$R_X(\tau) = E[X(t)X(t+\tau)] = \overline{(n(t)-\overline{n(t)})(n(t+\tau)-\overline{n(t+\tau)})}$$

$$= \overline{n(t)n(t+\tau)} + \overline{n(t)}\cdot\overline{n(t+\tau)} - \overline{n(t)}\cdot\overline{n(t+\tau)} - \overline{n(t)}\cdot\overline{n(t+\tau)}$$

$$= R_n(\tau) - (\overline{n(t)})^2. \qquad (\because \text{WSS})$$

$$S_X(f=0) = \int_{-\infty}^{\infty} R_X(\tau)d\tau = \int_{-\infty}^{\infty}[R_n(\tau) - (\overline{n(t)})^2]d\tau$$

$$= S_n(0) - \int_{-\infty}^{\infty}(\overline{n(t)})^2 d\tau = 0\text{-}\int_{-\infty}^{\infty}(\overline{n(t)})^2 d\tau \geq 0.$$

The only possibility is $(\overline{n(t)})^2 = 0.$

(ii) $E\{z_1(t)\} = 2\cdot\overline{n(t)}\cdot\overline{\cos(\omega_0 t+\theta)} = 0.$

$E\{n_c(t)\} = E\{z_1(t)\}\cdot H(0) = 0.$ Similarly, $E\{n_s(t)\} = 0.$

(2) $S_{n_c}(f) = S_{n_s}(f) = Lowpass\{S_n(f-f_0)+S_n(f+f_0)\}$

$$= \begin{cases} S_n(f-f_0)+S_n(f+f_0), & -W < f < W \\ 0, & \text{otherwise} \end{cases}.$$

$$S_{n_c n_s}(f) = j \cdot Lowpass\{S_n(f-f_0)-S_n(f+f_0)\}.$$

Proof: (i) $z_1(t) = 2n(t)\cos(\omega_0 t+\theta)$.

$$R_{Z_1}(\tau) = E\{z_1(t)z_1(t+\tau)\}$$

$$= E\{4n(t)n(t+\tau)\cos(\omega_0 t+\theta)\cos(\omega_0(t+\tau)+\theta)\}$$

$$= 2E\{n(t)n(t+\tau)\}\cos\omega_0\tau + 2E\{n(t)n(t+\tau)\cos(2\omega_0 t+\omega_0\tau+2\theta)\}$$

$$= 2R_n(\tau)\cos\omega_0\tau + 2E\{n(t)n(t+\tau)\}E\{\cos(2\omega_0 t+\omega_0\tau+2\theta)\}$$

$$= 2R_n(\tau)\cos\omega_0\tau.$$

Indept.

Thus, $S_{z_1}(f) = S_n(f) * [\delta(f-f_0)+\delta(f+f_0)]$

$$= S_n(f-f_0)+S_n(f+f_0).$$

(ii) n_c(t) is the low-pass portion of $z_1(t)$.

$\therefore \quad S_{n_c}(f) = Lowpass\{S_n(f-f_0)+S_n(f+f_0)\}.$

Similarly, $S_{n_s}(f) = Lowpass\{S_n(f-f_0)+S_n(f+f_0)\}.$

(Pf. conti.): (iii) $R_{z_1z_2}(\tau) = E\{z_1(t)z_2(t+\tau)\}$

$$= -E\{4n(t)n(t+\tau)\cos(\omega_0 t+\theta)\sin(\omega_0(t+\tau)+\theta)\}$$

$$= -2R_n(\tau)\sin\omega_0\tau.$$

Thus, $S_{z_1z_2}(f) = j[S_n(f-f_0) - S_n(f+f_0)].$

(iv) $R_{n_cn_s}(\tau) = E\{n_c(t)n_s(t+\tau)\}$

$$= E\{\int_{-\infty}^{\infty} h(u)Z_1(t-u)du \int_{-\infty}^{\infty} h(v)Z_2(t+\tau-u)dv\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)E\{Z_1(t-u)Z_2(t+\tau-v)\}dudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)R_{Z_1Z_2}(\tau+u-v)dudv$$

$$= h(-\tau)*h(\tau)*R_{Z_1Z_2}(\tau).$$

$$\Rightarrow S_{n_cn_s}(f) = H(f)H^*(f)S_{Z_1Z_2}(f)$$

$$= |H(f)|^2 S_{Z_1Z_2}(f) = j|H(f)|^2[S_n(f-f_0) - S_n(f+f_0)]$$

$$= j\cdot Lowpass\{S_n(f-f_0) - S_n(f+f_0)\}.$$

(3) $\overline{n^2(t)} = \overline{n_c^2(t)} = \overline{n_s^2(t)} \equiv N.$

Proof:

$$\overline{n_c^2(t)} = \int_{-\infty}^{\infty} S_{n_c}(f)df = \int_{-\infty}^{\infty} S_{n_s}(f)df = \overline{n_s^2(t)} = \int_{-\infty}^{\infty} S_n(f)df = \overline{n^2(t)}.$$

(4) If $S_n(f)$ is symmetric with respect to f_0, then $n_c(t_1)$ and $n_s(t_2)$ are uncorrelated for all t_1 and t_2.

Proof: $S_n(f)$ is symmetric with respect to f_0,

$$\Rightarrow \quad Lowpass\{S_n(f - f_0) - S_n(f + f_0)\} = 0.$$

$$\Rightarrow \quad R_{n_c n_s}(\tau) = 0 \qquad \forall \tau.$$

$$\Rightarrow \quad n_c(t) \text{ and } n_s(t+\tau) \text{ are uncorrelated.}$$

Remarks: If $S_n(f)$ is NOT symmetric with respect to f_0,

then $n_c(t_1)$ and $n_s(t_2)$ are uncorrelated at

Odd symmetry

(*i*) $t_1 = t_2$ $\quad (\because h(-\tau) * h(\tau) * 2R_n(\tau)\sin\omega_0\tau = 0)$

(*ii*) those t_1 and t_2 such that $R_{n_c n_s}(t_1 - t_2) = 0$

Even symmetry

(5) If $n(t)$ is Gaussian, then $n_c(t)$ and $n_s(t)$ are Gaussian.

Proof : $n_c(t)$ and $n_s(t)$ are weighted linear combination of $n(t)$.

(6) If $n_c(t)$ and $n_s(t)$ are independent, their joint pdf is

$$f(n_c, t; n_s, t+\tau) = \frac{e^{-(n_c^2 + n_s^2)/2N}}{2\pi N}, \qquad \forall \tau,$$

and for $R(t)$ and $\phi(t)$:

$$f(r, \phi) = \frac{r}{2\pi N} e^{-\frac{r^2}{2N}}, \qquad \forall r > 0, \quad |\phi| \le \pi.$$

Ex.: BP Signal

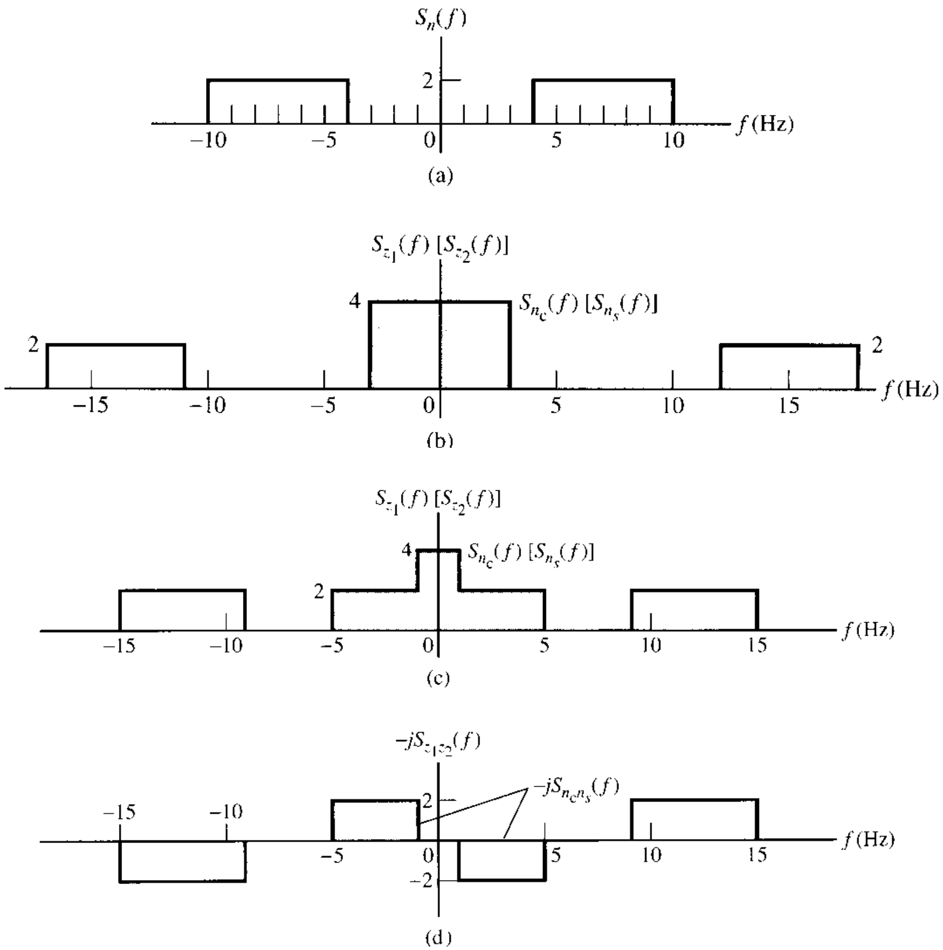


Figure 6.13
Spectra for Example 6.11. (a) Bandpass spectrum. (b) Lowpass spectra for $f_0 = 7$ Hz. (c) Lowpass spectra for $f_0 = 5$ Hz. (d) Cross-spectra for $f_0 = 5$ Hz.

(1) If $f_0 = 7$ Hz, $S_n(f)$ is symmetric with respect to f_0.

$\Rightarrow$ $n_c(t)$ and $n_s(t)$ are uncorrelated and

$$N = \int_{-\infty}^{\infty} S_n(f)df = 2 \cdot (2 \cdot 6) = 24.$$

(2) If $f_0 = 5$ Hz, $S_n(f)$ is NOT symmetric with respect to f_0.

$\Rightarrow$ $n_c(t)$ and $n_s(t)$ are correlated.

$$S_{n_c}(f) = S_{n_s}(t) = Lowpass\{S_n(f - f_0) + S_n(f + f_0)\} = (c).$$

$$S_{n_c n_s}(f) = j \cdot Lowpass\{S_n(f - f_0) - S_n(f + f_0)\} = (d).$$

$$R_{n_c n_s}(\tau) = 16 \cdot \text{sinc}(4\tau) \cdot \sin 6\pi\tau.$$

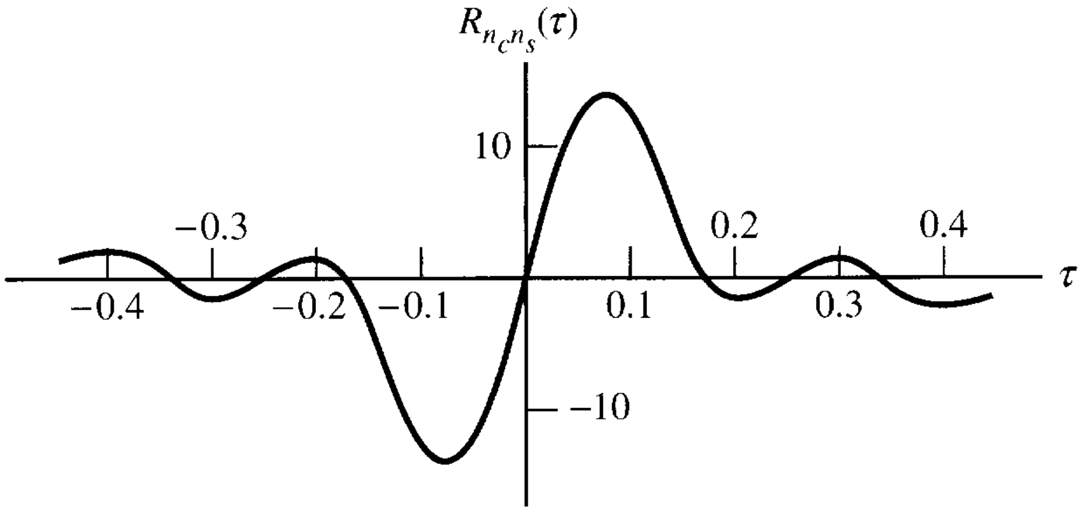


Figure 6.14
Cross-correlation function of $n_c(t)$ and $n_s(t)$ for Example 6.11.

Theorem: Given a WSS bandpass random process $n(t)$ with BW = B, then $n(t)$ can be represented by $n_c(t)\cos(\omega_0 t+\theta)-n_s(t)\sin(\omega_0 t+\theta)$ in mean-square sense; that is,

$$E\{[n(t)-(n_c(t)\cos(\omega_0 t+\theta)-n_s(t)\sin(\omega_0 t+\theta))]^2\}=0.$$

Note that θ is a random variable uniformly distributed over $(-\pi,\pi)$ and is independent of $n(t)$.

Proof: Let $\hat{n}(t)=n_c(t)\cos(\omega_0 t+\theta)-n_s(t)\sin(\omega_0 t+\theta)$.

Appendix C

Wish to show $E\{[n(t)-\hat{n}(t)]^2\}=0$. (converge in mean-square sense)

$$E\{[n(t)-\hat{n}(t)]^2\}=\overline{n^2(t)}-2\overline{n(t)\hat{n}(t)}+\overline{\hat{n}^2(t)}.$$

$$\begin{aligned}\overline{\hat{n}^2(t)}&=E\{[n_c(t)\cos(\omega_0 t+\theta)-n_s(t)\sin(\omega_0 t+\theta)]^2\}\\&=\overline{n_c^2(t)}\cdot\overline{\cos^2(\omega_0 t+\theta)}+\overline{n_s^2(t)}\cdot\overline{\sin^2(\omega_0 t+\theta)}\\&\quad-2\overline{n_c(t)n_s(t)}\cdot\overline{\cos(\omega_0 t+\theta)\sin(\omega_0 t+\theta)}\\&=\frac{1}{2}\overline{n_c^2(t)}+\frac{1}{2}\overline{n_s^2(t)}=\overline{n^2(t)}=N.\end{aligned}$$

$$\overline{n(t)\hat{n}(t)} = E\{n(t)[n_c(t)\cos(\omega_0 t+\theta) - n_s(t)\sin(\omega_0 t+\theta)]\}.$$

$$\because\quad n_c(t) = h(t) * [2n(t)\cos(\omega_0 t+\theta)] = \int_{-\infty}^{\infty} h(t-u)\cdot 2n(u)\cos(\omega_0 u+\theta)du.$$

$$n_s(t) = -\int_{-\infty}^{\infty} h(t-v)\cdot 2n(v)\sin(\omega_0 v+\theta)dv.$$

$$\begin{aligned}
\overline{n(t)\hat{n}(t)} &= E\{\int_{-\infty}^{\infty} h(t-u)\cdot n(t)\cdot 2n(u)\cos(\omega_0 u+\theta)\cos(\omega_0 t+\theta)du\} \\
&\quad + E\{\int_{-\infty}^{\infty} h(t-v)\cdot n(t)\cdot 2n(v)\sin(\omega_0 v+\theta)\sin(\omega_0 t+\theta)dv\} \\
&= E\{\int_{-\infty}^{\infty} h(t-u)\cdot n(t)\cdot 2n(u)[\cos(\omega_0(u-t)) + \cos(\omega_0(u+t)+2\theta)]du\} \\
&\quad + E\{\int_{-\infty}^{\infty} h(t-v)\cdot n(t)\cdot 2n(v)[\cos(\omega_0(v-t)) - \cos(\omega_0(v+t)+2\theta)]dv\} \\
&= \int_{-\infty}^{\infty} h(t-u)\cdot 1\cdot R_n(u-t)\cdot\cos(\omega_0(u-t))du \\
&\quad + \int_{-\infty}^{\infty} h(t-v)\cdot 1\cdot R_n(v-t)\cdot\cos(\omega_0(v-t))dv \\
&= 2\cdot\int_{-\infty}^{\infty} h(t-u)R_n(t-u)\cos(\omega_0(t-u))du \\
&= 2\cdot\int_{-\infty}^{\infty} h(\lambda)R_n(\lambda)\cos(\omega_0\lambda)d\lambda.
\end{aligned}$$

$$\overline{n(t)\hat{n}(t)} = 2 \cdot \int_{-\infty}^{\infty} h(\lambda) R_n(\lambda) \cos(\omega_0 \lambda) d\lambda.$$

$$\because \int_{-\infty}^{\infty} x(t) y(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df, \qquad \text{(Parseval's Theorem)}$$

and $h(\lambda)\cos w_0 \lambda \leftrightarrow \frac{1}{2} H(f - f_0) + \frac{1}{2} H(f + f_0)$, i.e.,

$$R_n(\lambda) \leftrightarrow S_n(f),$$

$$\overline{n(t)\hat{n}(t)} = \int_{-\infty}^{\infty} [H(f - f_0) + H(f + f_0)] \cdot S_n(f) df = \int_{-\infty}^{\infty} S_n(f) df = \overline{n^2(t)}.$$

$$\Rightarrow E\{[n(t) - \hat{n}(t)]^2\} = \overline{n^2(t)} - 2\overline{n^2(t)} + \overline{n^2(t)} = 0. \quad \text{Q.E.D.}$$