Principles of Communications
Lecture 10: Stochastic Processes

Chih-Wei Liu 劉志尉
National Chiao Tung University
cwliu@twins.ee.nctu.edu.tw

Outlines

- Terminology of Random Processes
- Correlation and Power Spectral Density
- Linear Systems and Random Processes
- Narrowband Noise

Random (Stochastic) Processes

- ***Informal Definition:*** The outcomes (events) of a chance (prob) experiment are mapped into *functions of time* (waveforms).

  Cf.: Random variables: outcomes are mapped to *numbers*.

  Note: Each waveform is called a **sample function**, or a **realization**. The totality of all sample functions is called an **ensemble**.
- Random Process ~ RP
- We next show the axiomatic-theory approach.

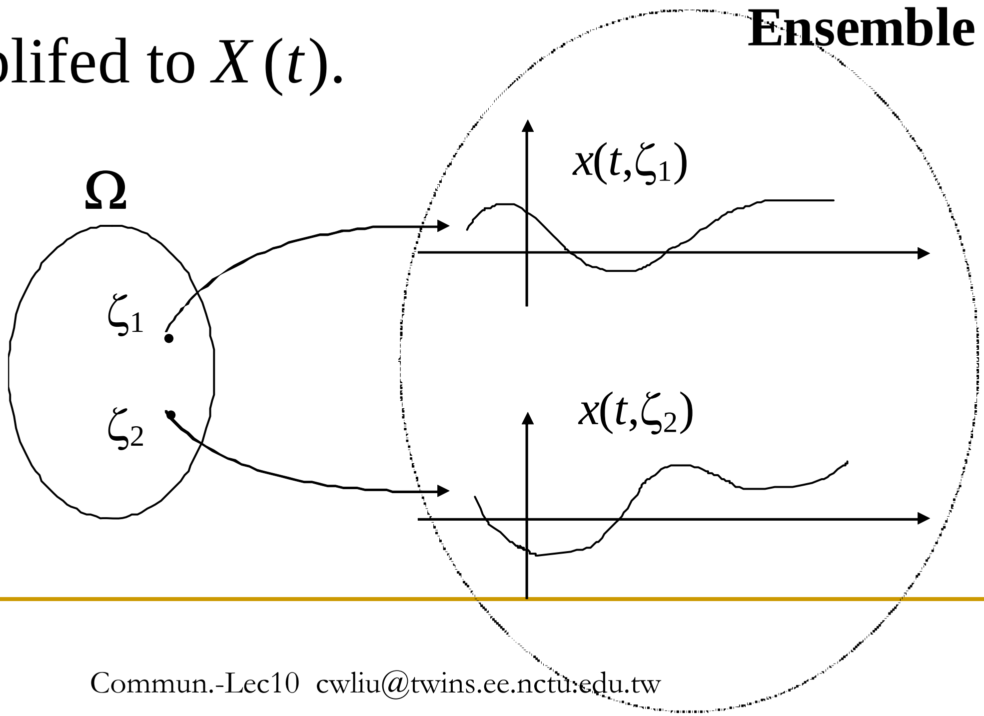
Formal Definition

$(\Omega, \Im, P)$: a probability space

Random Process: $X(t,\varsigma)$ is a mapping from Ω to an ensamble of sample functions and for each fixed t, $X(t,\varsigma)$ is a random variable.

Note: For each fixed ς, $X(t,\varsigma)$ is NOT random, it is an ordinary (deterministic) time function.

$X(t,\varsigma)$ is often simplifed to $X(t)$.



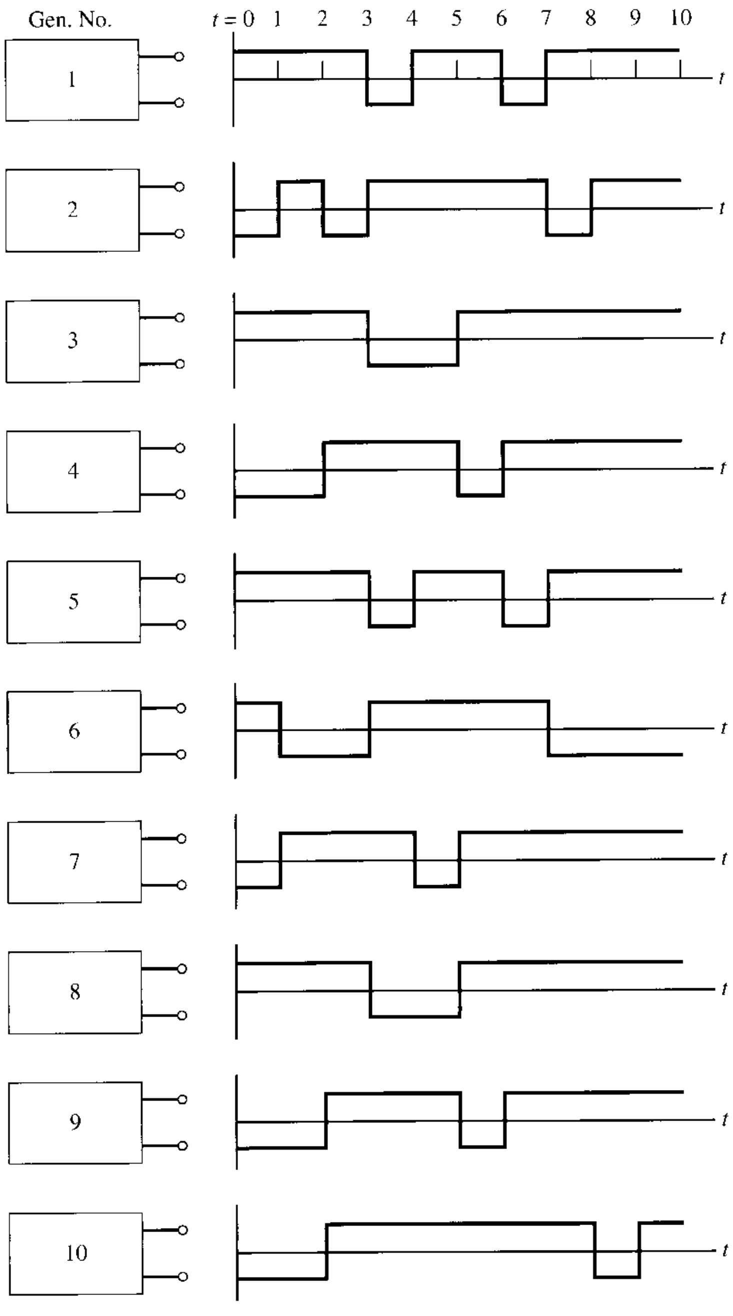


Figure 6.1
A statistically identical set of binary waveform generators with typical outputs.

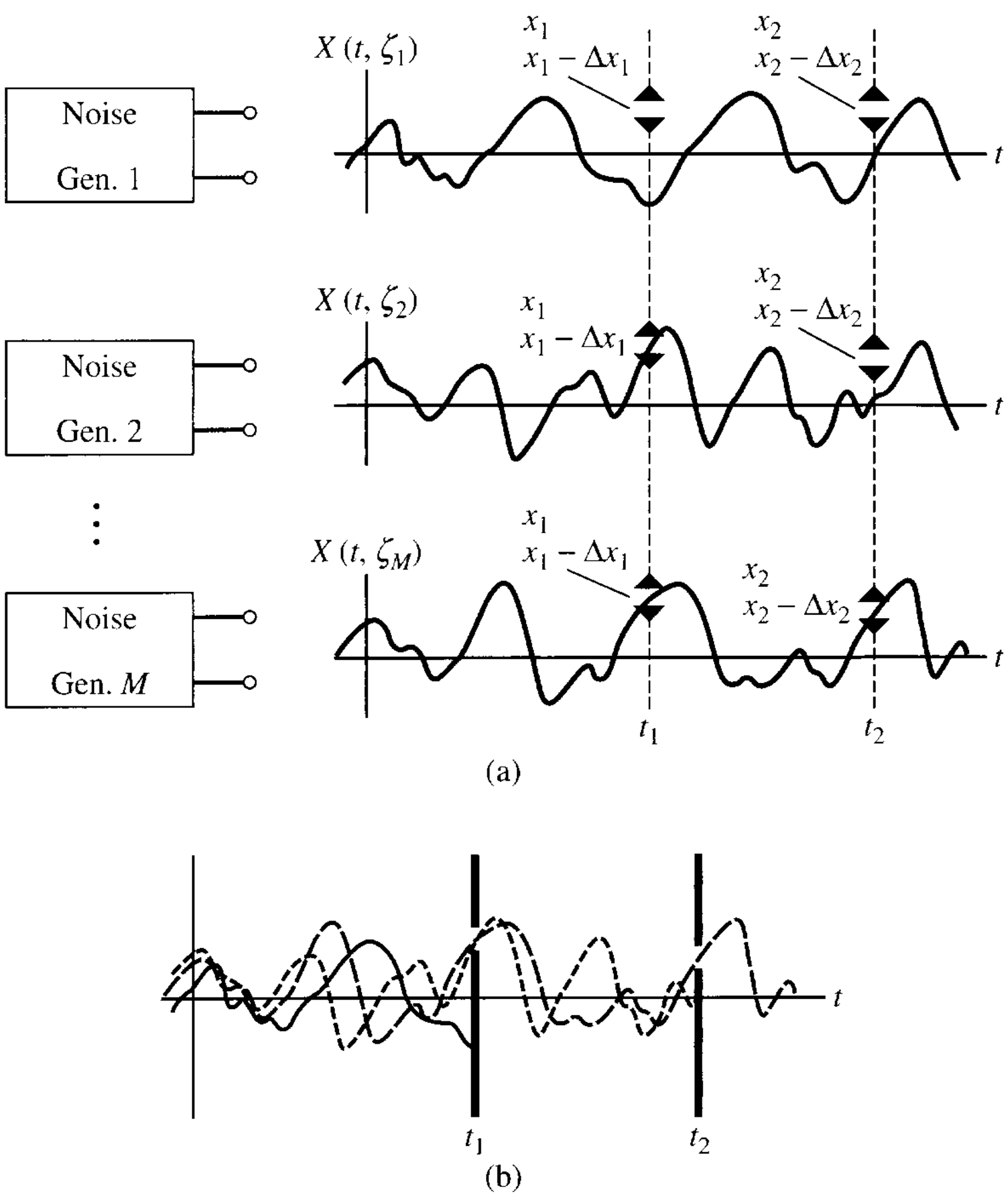


Figure 6.2
Typical sample functions of a random process and illustration of the relative-frequency interpretation of its joint pdf. (a) Ensemble of sample functions. (b) Superposition of the sample functions shown in (a).

Statistical Description

A random process is statistically specified if we know ALL its nth order joint pdf's for any $t_1,\ t_2,\ ..,\ t_n,$ for any n.

$$f(x_1,t_1;\ x_2,t_2;\ \cdots;\ x_n,t_n)dx_1dx_2..dx_n$$
$$= P(x_1-dx_1 < X_1 \le x_1;\ x_2-dx_2 < X_2 \le x_2;\cdots;$$
$$x_n-dx_n < X_n \le x_n).$$

Note: Strictly speaking, all the nth order joint pdf's are not sufficient to determine a random process completely. For example, we need to impose a continuity requirement on the sample functions.

Ensemble Average

- Statistics based on prob model -- Expectation

The result is a determinstic value or function (non-random)

Mean function: $m_x(t) \equiv \overline{X}(t) \equiv E\{X(t)\} = \int_{-\infty}^{\infty} \alpha f_X(\alpha;t)d\alpha.$

Variance function: $\sigma_x^2(t) \equiv E\{[X(t) - \overline{X}(t)]^2\}.$

(Auto)correlation function:

$$R_x(t_1,t_2) \equiv E\{X(t_1)X^*(t_2)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha_1\alpha_2 f_{x_1x_2}(\alpha_1,t_1;\alpha_2,t_2)d\alpha_1 d\alpha_2.$$

Time Average

Stochastic integral: integration over TIME

$\approx$ sum of r.v.'s = r.v. Let $\Delta t_i = t_i - t_{i-1}$

$$I \equiv \int_{T_1}^{T_2} X(t)dt \equiv \lim_{n\to\infty} \sum_{i=1}^{n} X(t_i)\Delta t_i, \quad T_1 \le t_0 \le t_1 \le \cdots \le t_n \le T_2.$$

Time average: $\langle X(t) \rangle \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt.$

$$\langle [X(t) - \langle X(t) \rangle]^2 \rangle \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (X(t) - \langle X(t) \rangle)^2 dt.$$

$$\langle X(t)X^*(t+\tau) \rangle \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)X^*(t+\tau)dt.$$

Stationary RP

- $(Strict\ sense)$**Stationary**: (pdf is shift-invariant)

A random process $X(t)$ is statistically stationary if $X(t)$ has the same nth-order pdf as $X(t+\Gamma)$, for $\forall\Gamma$ and $\forall n > 0$.

That is,

$$f(x_1,t_1;\ x_2,t_2;\cdots;\ x_n,t_n) = f(x_1,t_1+T;x_2,t_2+T;\cdots;x_n,t_n+T).$$

Remark: Consequently, the nth-order joint pdf depends only on the time differences $t_2-t_1,\ t_3-t_1,\ \cdots,\ t_n-t_1$.

Thus, $R_x(t_1,t_2) = R_x(0,t_2 - t_1) \equiv R_x(t_2 - t_1)$.

$$m_x(t) = \text{constant} = m_x(0).$$

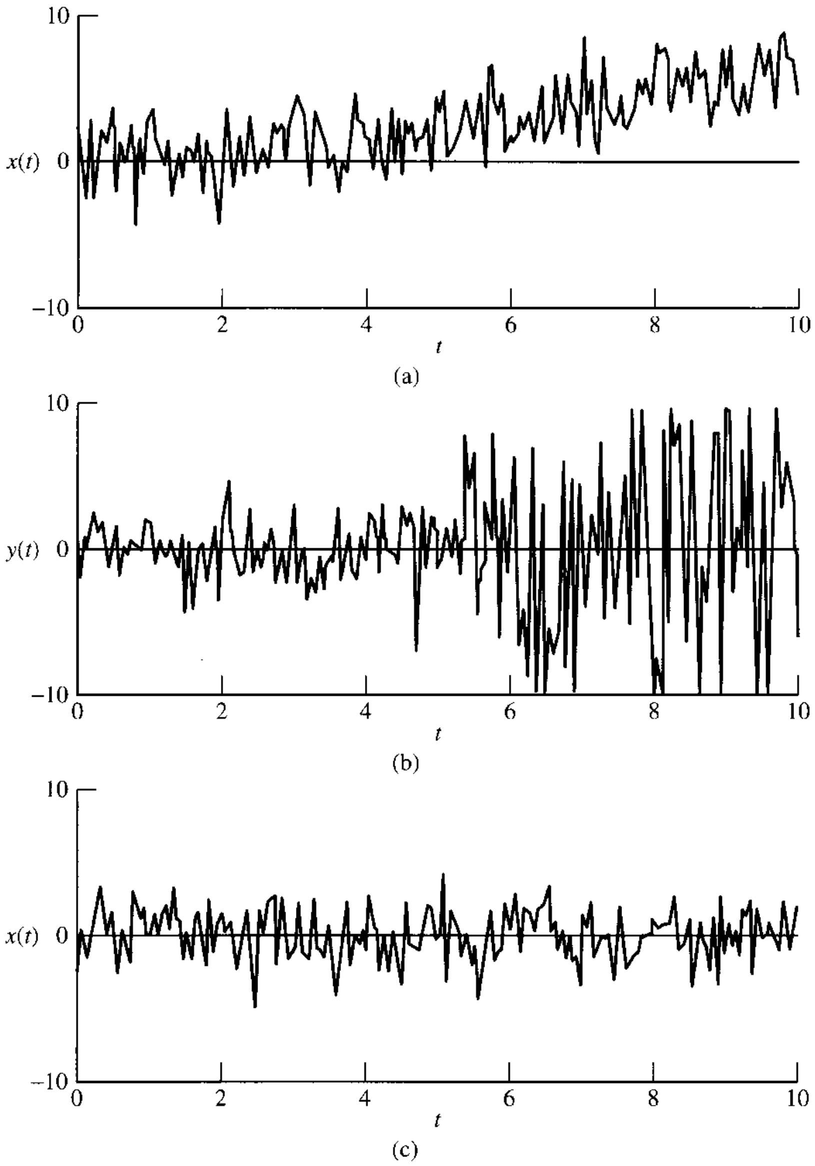


Figure 6.3
Sample functions of nonstationary processes contrasted with a sample function of a stationary process. (a) Time-varying mean. (b) Time-varying variance. (c) Stationary.

Wise-sense Stationary

- A random process $X(t)$ is **wide-sense stationary (WSS)** if $E[X(t)]$ = constant, and

  $E[X(t)X^*(t+\tau)] = R_X(\tau)$ function of τ only.

Remarks: (1) strict-sense stationary $\Rightarrow$ wide-sense stationary

(2) For Gaussian random processes, WSS = strict-sense stationary

(A Gaussian process is completely specified by only two parameters: <u>mean</u> and <u>covariance</u>.)

Ergodicity

- **Ergodicity**: "Ensemble average = Time average"
- 3 types: in mean, in correlation, "strict-sense"

• Definition: **Ergodic in the mean**:

A WSS random process $X(t)$ is ergodic in the mean if

$< X(t) > \; = \; m_x \; = \;$ constant.

Remark: $< X(t) >$ is the time average (a random variable).

A more rigorous definition:

$$M_T \equiv \frac{1}{2T}\int_{-T}^{T} X(t)dt \underset{T\to\infty}{=} m_X \quad \text{(in mean square sense)}$$

• A random variable Y_T converges to Y in m.s. sense,

if $E[|Y_T - Y|^2] \to 0, \quad$ as $T \to \infty$.

Therefore, ergodic in the mean $\Leftrightarrow \lim_{T\to\infty} \text{var}\{M_T\} = 0$.

Ergodicity (2)

- Definition: **Ergodic in correlation**:

A WSS random process $X(t)$ is ergodic in correlation if

$$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} X(t+\tau)X^*(t)dt\right] \underset{m.s.}{=} R_X(\tau), \forall \tau.$$

- Definition: **Ergodic**

A random process $X(t)$ is ergodic if all time and ensemble averages are interchangeable. (The nth order moments of both are identical for $\forall n$.)

Remark: Ergodicity $\Rightarrow$ strict-sense stationarity

Ex.: Ergodic RP

Ergodic process: $n(t) = A\cos(\omega_0 t + \Theta)$.

Θ: random variable $\quad f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & |\theta| \le \pi \\ 0, & \text{otherwise} \end{cases}$.

<Ensemble average> Mean: $\overline{n(t)} = E[A\cos(\omega_0 t + \Theta)]$

$$= \int_{-\infty}^{\infty} A\cos(\omega_0 t + \theta) f_\Theta(\theta) d\theta = \int_{-\pi}^{\pi} A\cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta = 0.$$

Variance: $\overline{n^2(t)} = \int_{-\pi}^{\pi} (A\cos(\omega_0 t + \theta))^2 \frac{1}{2\pi} d\theta = \frac{A^2}{2}$.

$$n(t) = A\cos(\omega_0 t + \Theta)....$$

<Time average>

$$\langle n(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(\omega_0 t + \theta)dt = 0, \quad \text{for any } \theta.$$

$$\langle n^2(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \cos^2(\omega_0 t + \theta)dt = \frac{A^2}{2}, \quad \text{for any } \theta.$$

$\because \langle n(t) \rangle = E[n(t)], \langle n^2(t) \rangle = E[n^2(t)] \Rightarrow n(t)$ is, at least, ergodic in power. It is in fact ergodic.

Moreover, it is ergodic in autocorrelation function.

Ensemble: $R_X(\tau) = E[X(t)X^*(t+\tau)]$

$$= E[A\cos(\omega_0 t + \Theta)A\cos(\omega_0 t + \omega_0\tau + \Theta)]$$

$$= \frac{A^2}{2}\{E[\cos(2\omega_0 t + 2\Theta + \omega_0\tau)] + E[\cos(\omega_0\tau)]\}$$

$$= \frac{A^2}{2}\{\cos(\omega_0\tau) + \int_{-\pi}^{\pi}\cos(2\omega_0 t + \omega_0\tau + 2\theta)\frac{1}{2\pi}d\theta\}$$

$$= \frac{A^2}{2}\cos(\omega_0\tau).$$

Time: $\langle X(t)X^*(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A^2\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)dt$

$$= \lim_{T\to\infty}\frac{A^2}{2T}\int_{-T}^{T}[\frac{1}{2}\cos(\omega_0\tau) + \frac{1}{2}\cos(2\omega_0 t + \omega_0\tau + 2\theta)]dt$$

$$= \frac{A^2}{2}\cos(\omega_0\tau) \quad = R_X(\tau).$$

Ex.: Non-stationary RP

- Nonstationary process (nonergodic) $n(t) = A\cos(\omega_0 t + \Theta)$.

Θ: a random variable with $f_\Theta(\theta) = \begin{cases} \frac{2}{\pi}, & |\theta| \le \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$.

Mean: $\overline{n(t)} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} A\cos(\omega_0 t + \theta)\frac{2}{\pi} d\theta = \frac{2A}{\pi}\sin(\omega_0 t + \theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{2\sqrt{2}A}{\pi}\cos\omega_0 t.$

Variance: $\overline{n^2(t)} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (A\cos(\omega_0 t + \theta))^2 \frac{2}{\pi} d\theta$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{A^2}{\pi}[1 + \cos(2\omega_0 t + 2\theta)]d\theta = \frac{A^2}{2} + \frac{A^2}{\pi}\cos 2\omega_0 t.$$

Random Telegraph Waveform

Definition: (i) At $t = t_0$, $\Pr\{X(t_0) = A\} = 0.5 = \Pr\{X(t_0) = -A\}$

(ii) The number of switching (K) in any time interval T

 Poisson distribution, i.e.,

$$P_T(K) = \frac{(\alpha T)^K}{K!} e^{-\alpha T}, \text{ for some } \alpha > 0.$$

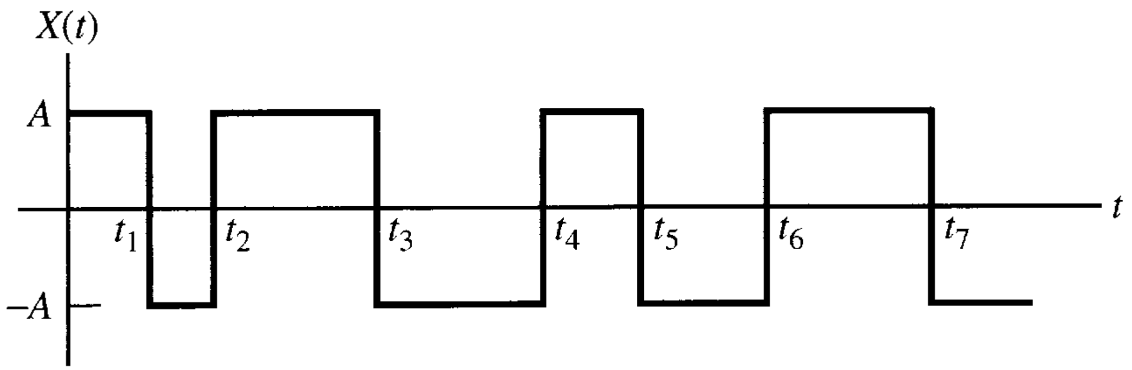


Figure 6.4
Sample function of a random telegraph waveform.

Mean of $X(t) = 0$. $\quad (\because X(t) = \pm A,$ equally likely)

Correlation function: $R_X(\tau) = E[X(t)X(t+\tau)]$

$$= A^2 \cdot P\{X(t) \text{ and } X(t+\tau) \text{ have the same sign}\}$$

$$-A^2 \cdot P\{X(t) \text{ and } X(t+\tau) \text{ have different signs}\}$$

$$= A^2 \cdot P\{\text{even number of switchings in } (t, t+\tau)\}$$

$$-A^2 \cdot P\{\text{odd number of switchings in } (t, t+\tau)\}$$

$$= A^2 \sum_{\substack{k=0,\\ k \text{ even}}}^{\infty} \frac{(\alpha\tau)^k}{k!} e^{-\alpha\tau} - A^2 \sum_{\substack{k=0\\ k \text{ odd}}}^{\infty} \frac{(\alpha\tau)^k}{k!} e^{-\alpha\tau}$$

$$\sum_{k=0}^{\infty} \frac{(-\alpha\tau)^k}{k!} = e^{-\alpha\tau}$$

$$= A^2 e^{-\alpha\tau} \sum_{k=0}^{\infty} \frac{(-\alpha\tau)^k}{k!} = A^2 e^{-2\alpha\tau}, \text{ for } \alpha\tau > 0 \quad (\text{ or } \tau > 0).$$

Similarly, for $\tau < 0$, $R_X(\tau) = A^2 e^{2\alpha\tau}$. $\quad \therefore R_X(\tau) = A^2 e^{-2\alpha|\tau|}$.

Interpretation of Moments

- For ergodic PR, the moments (statistics) have the *physical* meanings.

1. Mean $\overline{X(t)} = \langle X(t) \rangle$ is the dc component.
2. $\overline{X(t)}^2 = \langle X(t) \rangle^2$ is the dc power.
3. 2nd moment: $\overline{X^2(t)} = \langle X^2(t) \rangle$ is the total power.
4. Variance: $\sigma_X^2 = \overline{X(t)^2} - \overline{X(t)}^2 = \langle X^2(t) \rangle - \langle X(t) \rangle^2$

   is the power in the ac (time-varying) component.
5. The total power $\overline{X^2(t)} = \sigma_X^2 + \overline{X(t)}^2$ is the ac plus dc power.

Remark: The time averages can be measured in the laboratory.

Power Spectral Density (PSD)

For a WSS process $X(t)$, the PSD is defined as

$S_X(f) \equiv FT[R_X(\tau)]$ (Fourier transform)

- What is the realtionship between the PSD and the Fourier transform of a realization of the random process?

$$\text{Let } n_T(t,\zeta_i) = \begin{cases} n(t,\zeta_i), |t| < \dfrac{T}{2} \\ 0, \qquad \text{otherwise} \end{cases}.$$

$$N_T(f,\zeta_i) = FT\{n_T(t,\zeta_i)\} = \int_{-\frac{T}{2}}^{\frac{T}{2}} n(t,\zeta_i)e^{-j\omega t}dt.$$

Note: In general, $n(t,\zeta_i)$ has infinite energy. Hence, $FT[n(t,\zeta_i)]$ does not exist.

With finite T, for each ζ_i, energy spectral density of $n(t,\zeta_i) = |N_T(f,\zeta_i)|^2$.

Power spectral density of $n(t,\zeta_i) = \dfrac{|N_T(f,\zeta_i)|^2}{T}$.

- For all ζ_i, let $S_n(f) = \lim_{T\to\infty i} \frac{1}{T}\overline{\left|N_T(f,\zeta_i)\right|^2}$.
- The **Wiener - Khinchin theorem**:

$$S_n(f) = \lim_{T\to\infty i} \frac{1}{2T} E\{\left|FT[n_{2T}(t)]\right|^2\}.$$

- Proof: $\left|FT[n_{2T}(t)]\right|^2 = \int_{-T}^{T} n(t)e^{-j\omega t}dt \int_{-T}^{T} n^*(\sigma)e^{j\omega\sigma}d\sigma$

$$= \int_{-T}^{T}\int_{-T}^{T} n(t)n^*(\sigma)e^{-j\omega(t-\sigma)}dtd\sigma.$$

$$E\{\left|FT[n_{2T}(t)]\right|^2\} = \int_{-T}^{T}\int_{-T}^{T} E\{n(t)n^*(\sigma)\}e^{-j\omega(t-\sigma)}dtd\sigma$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_n(t-\sigma)e^{-j\omega(t-\sigma)}dtd\sigma.$$

Change of variables: $u = t - \sigma$ and $v = t$.

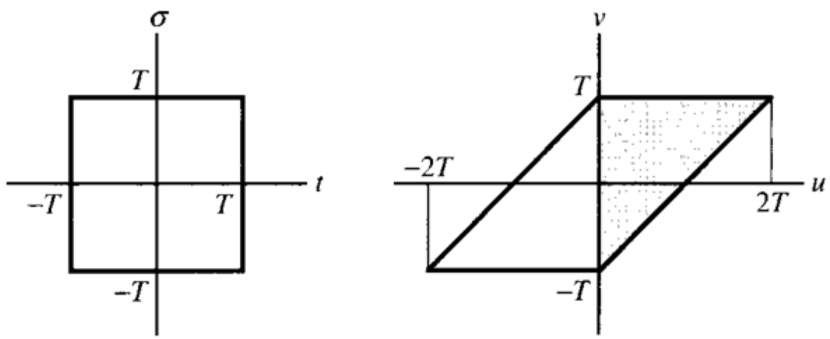


Figure 6.5
Regions of integration for (6.37).

For a given t value, the range of $u: \{t - T,\ t + T\}$.

For a given u value,

if $u > 0$, the range of $v: \{u - T,\ T\}$,

else ($u < 0$), the range of $v: \{-T,\ u + T\}$.

$$E\{|FT[n_{2T}(t)]|^2\} = \int_{-2T}^{0}\left(\int_{-T}^{u+T} R_n(u)e^{-j\omega u}dv\right)du$$

$$+ \int_{0}^{2T}\left(\int_{u-T}^{T} R_n(u)e^{-j\omega u}dv\right)du$$

$$= \int_{-2T}^{0} R_n(u)e^{-j\omega u}\left(\int_{-T}^{u+T} dv\right)du + \int_{0}^{2T} R_n(u)e^{-j\omega u}\left(\int_{u-T}^{T} dv\right)du$$

$$= \int_{-2T}^{0} (2T + u)R_n(u)e^{-j\omega u}du + \int_{0}^{2T} (2T - u)R_n(u)e^{-j\omega u}du$$

$$= 2T\int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right)R_n(u)e^{-j\omega u}du.$$

$$\lim_{T\to\infty} \frac{1}{2T} E\{|FT[n_{2T}(t)]|^2\} = \lim_{T\to\infty} \int_{-2T}^{2T} \left(1 - \frac{|u|}{2T}\right)R_n(u)e^{-j\omega u}du$$

$$= \int_{-\infty}^{\infty} R_n(u)e^{-j\omega u}du = FT\{R_n(u)\}.$$

Ex.: Random Phase Sinusoidal Wave

Sinusoidal wave with random phase:

$n(t) = A\cos(\omega_0 t + \Theta)$. Θ : uniform pdf $[-\pi, \pi]$

$$R_n(\tau) = \frac{A^2}{2}\cos\omega_0\tau.$$

$$S_n(f) = \frac{A^2}{4}[\delta(f + f_0) + \delta(f - f_0)].$$

Properties of $R(\tau)$

- $R(\tau)$: Autocorrelation function of a WSS random process

(1) $R(0) \geq \left| R(\tau) \right|, \forall \tau.$

Proof: $E\{[X(t) \pm X(t+\tau)]^2\} \geq 0$

$\Rightarrow \overline{X^2(t)} \pm 2\overline{X(t)X(t+\tau)} + \overline{X^2(t+\tau)} \geq 0$

$\Rightarrow 2R(0) \pm 2R(\tau) \geq 0$

$\Rightarrow -R(0) \leq R(\tau) \leq R(0).$

(2) $R(\tau)$ is even; $R(\tau) = R(-\tau)$ (if $x(t)$ is real).

Poof: From the definition (of WSS).

(3) $\lim_{|\tau|\to\infty} R(\tau) = (\overline{X(t)})^2$, if $X(t)$ does not contain a periodic component.

Proof: If $X(t)$ is not periodic, $X(t)$ and $X(t+\tau)$ are nearly independent when $\tau \to \infty$.

Thus, $\lim_{|\tau|\to\infty} \overline{X(t)X(t+\tau)} \approx (\overline{X(t)})(\overline{X(t+\tau)}) = (\overline{X(t)})^2$.

(4) If $X(t)$ is periodic with period T_0, then $R(\tau)$ is periodic with the same period. $[R(\tau) = R(T_0+\tau)]$

(5) $S(f) = FT[R(\tau)] \geq 0, \forall f$.

Proof: By Wiener-Khinchin theorem,

$$S(f) = \lim_{T\to\infty} \frac{1}{2T} E\{|FT[X_{2T}(t)\}|^2\} \geq 0.$$

Properties of $S(f)$

(1) $S(f) = FT[R(\tau)] \geq 0, \forall f.$

(2) $S(f)$ is real-valued. $[\because R(\tau)$ is conjugate symmeteric]

(3) If $X(t)$ is real, $S(f)$ is even. $[\because R(\tau)$ is real]

(4) $$\text{"}dc\ power\text{"} = \int_{-\infty}^{\infty} R(\tau) d\tau = S(0).$$

$$\text{"}total\ power\text{"} = \int_{-\infty}^{\infty} S(f) df = R(0).$$

Note: $S(f)$ is the average power at frequency f.

Crosscorrelation

- For two RPs: $X(t)$ and $Y(t)$
  - Definition: **Cross-correlation**:

    $R_{XY}(t_1, t_2) \equiv E\{X(t_1)Y^*(t_2)\}.$

    If $X(t)$ and $Y(t)$ are jointly WSS,

    $R_{XY}(\tau) \equiv E\{X(t)Y^*(t+\tau)\}.$

  - Definition: **Cross-power spectral density**:

    If $X(t)$ and $Y(t)$ are joint WSS, $S_{XY}(f) \equiv \Im\{R_{XY}(\tau)\}.$

- Definition: For two random processes $X(t)$ and $Y(t)$,

(1) **Uncorrelated**: if $R_{XY}(t_t, t_2) = m_X(t_1) m_Y^*(t_2), \ \forall t_1, t_2$.

(2) **Orthogonal**: if $R_{XY}(t_t, t_2) = 0, \ \forall t_1, t_2$.

(3) **Independent**: if $f_{XY}(x_1, y_1, t_1; x_2, y_2, t_2; \cdots; x_n, y_n, t_n)$

$$= f_X(x_1, t_1; x_2, t_2; \cdots; x_n, t_n) f_Y(y_1, t_1; y_2, t_2; \cdots; y_n, t_n).$$

<Remarks>

(1) Independent $\Rightarrow$ Uncorrelated.

(2) Uncorrelated $\Rightarrow$ $(X(t) - m_{X(t)})$ and $(Y(t) - m_{Y(t)})$ are orthogonal.

(3) {Uncorrelated and either $m_{X(t)} = 0$ or $m_{Y(t)} = 0$} $\Rightarrow$ orthogonal.

(4) Uncorrelated and Gaussian $\Rightarrow$ Independent.