Principles of Communications
Lecture 1: Signals and Systems

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Outlines

- Signal Models & Classifications
- Signal Space & Orthogonal Basis
- Fourier Series &Transform
- Signals & Linear Systems
- Correlation & Power Spectral Density
- Sampling Theory
- DFT & FFT

Signal Models and Classifications

- **Physical world → Models → Math description**
- What is a signal?
- Usually we think of one-dimensional signals (waveforms); can our schemes be extended to higher dimensions?
- How about representing something uncertain, say, a noise?
- Random variables/processes – mathematical models for random signals

Deterministic vs. Random

- **Deterministic signals:** Completely specified functions of time.
  - Predictable, no uncertainty,
  - e.g. , $x(t) = A\cos(\omega_0 t)$ with A and ω_0 are fixed.
- **Random signals (stochastic signals):** Take on *random* values at any given time instant and are characterized by pdf (probability density function).
  - Not completely predictable, with uncertainty,
  - e.g. $x(n)$ = dice value at the n-th toss.

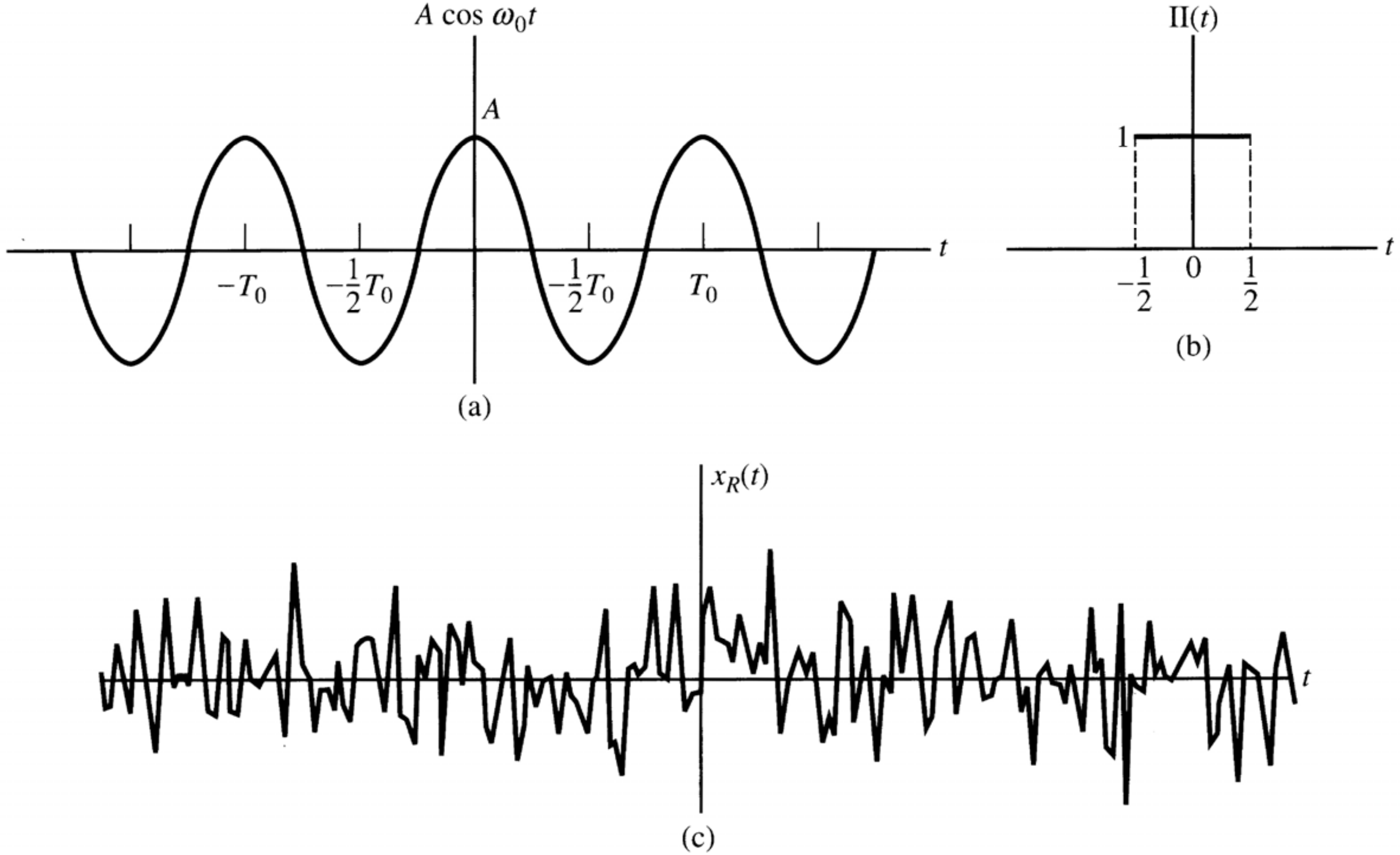

Figure 2.1
Examples of various types of signals. (a) Deterministic (sinusoidal) signal. (b) Unit rectangular pulse signal. (c) Random signal.

Periodic vs. Aperiodic Signals

- **Periodic signals**: A signal $x(t)$ is periodic iff (if and only if) there exists a constant T_0 such that

$$x(t+T_0)=x(t), \quad \forall t$$

- The smallest T_0 is called fundamental period or simply period.

- **Aperiodic signals**: Cannot find a finite T_0 such that

$$x(t+T_0)=x(t), \quad \forall t$$

Phasor Signals

- **Phasor**: A complex sinusoidal function:

$$\tilde{x}(t) = Ae^{j(\omega_0 t+\theta)} = Ae^{j\theta}e^{j\omega_0 t}$$

$\tilde{x}(t)$: *rotation phasor*; $Ae^{j\theta}$: *phasor*

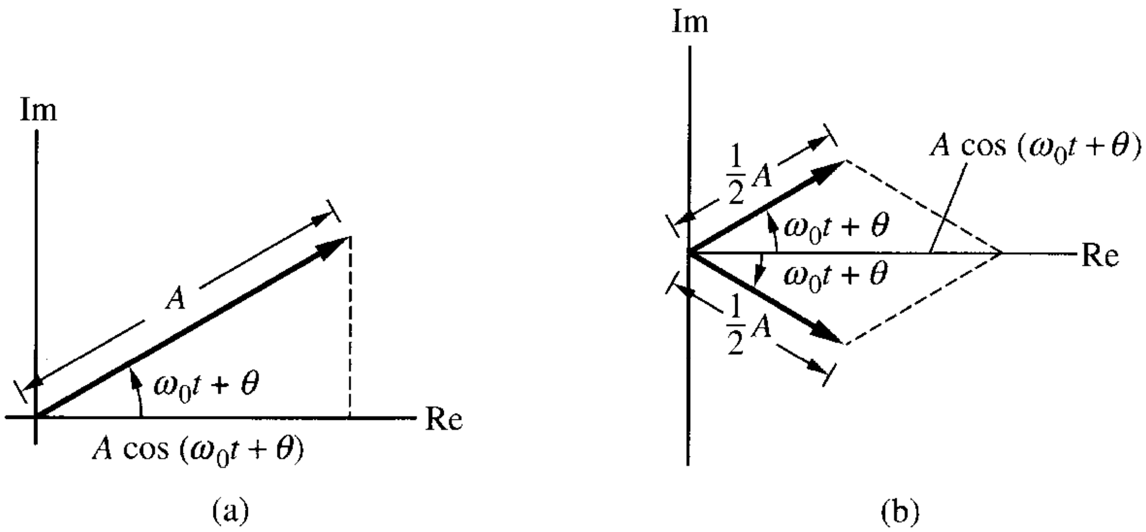

Figure 2.2
Two ways of relating a phasor signal to a sinusoidal signal. (a) Projection of a rotating phasor onto the real axis. (b) Addition of complex conjugate rotating phasors.

Why Phasor?

- Why complex number?
  - Easy mathematical analysis (exp function)
  - Two degree of freedoms at a single frequency (I,Q) → modulation
- Information is contained in A and θ.
- The related real sinusoidal function

$$x(t) = \mathrm{Re}\,\tilde{x}(t) = A\cos(\omega_0 t + \theta).$$

by projection …

- Express $x(t)$ in terms of rotating phasors:

$$A\cos(\omega_0 t + \theta) = \tfrac{1}{2}\tilde{x}(t) + \tfrac{1}{2}\tilde{x}^*(t).$$

mathematical representation …

Frequency Domain Representations

- **Single-sided**: Complex exponential

$$\tilde{x}(t) = Ae^{j\theta} \cdot e^{j\omega_0 t}.$$

- **Double-sided**: Real-value sinusoidal

$$x(t) = A\cos(\omega_0 t + \theta).$$

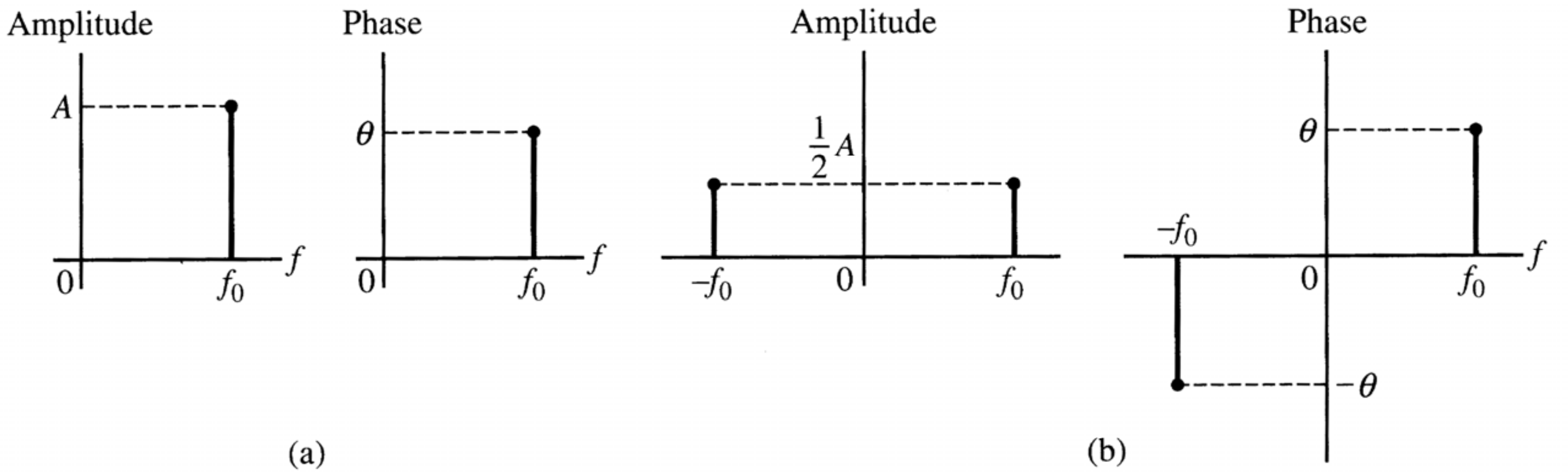

Figure 2.3
Amplitude and phase spectra for the signal $A\cos(\omega_0 t + \theta)$. (a) Single sided. (b) Double sided.

Singularity Functions

- **Unit impulse function:** $\delta(t)$

  -- Not an ordinary function. It is a *generalized function*, defined by its associated operation (just an *operator*)

- Defined by

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

  Note: $x(t)$ should be continuous, …

- It acts like a sampling device working only on "one point"

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = \int_{0_-}^{0_+} x(t)\delta(t)dt = x(0)$$

Impulse Function

- $\delta(t)$ is approximated by a *narrow* pulse with unit area

$$\text{(a)}\ \delta(t) = \begin{cases} \lim_{\varepsilon \to 0} \dfrac{1}{2\varepsilon}, & |t| < \varepsilon \\ 0, & \text{otherwise (or elsewhere)} \end{cases}$$

$$\text{(b)}\ \delta(t) = \lim_{\varepsilon \to 0} \varepsilon \left(\frac{1}{\pi t} \sin \frac{\pi t}{\varepsilon} \right)^2$$

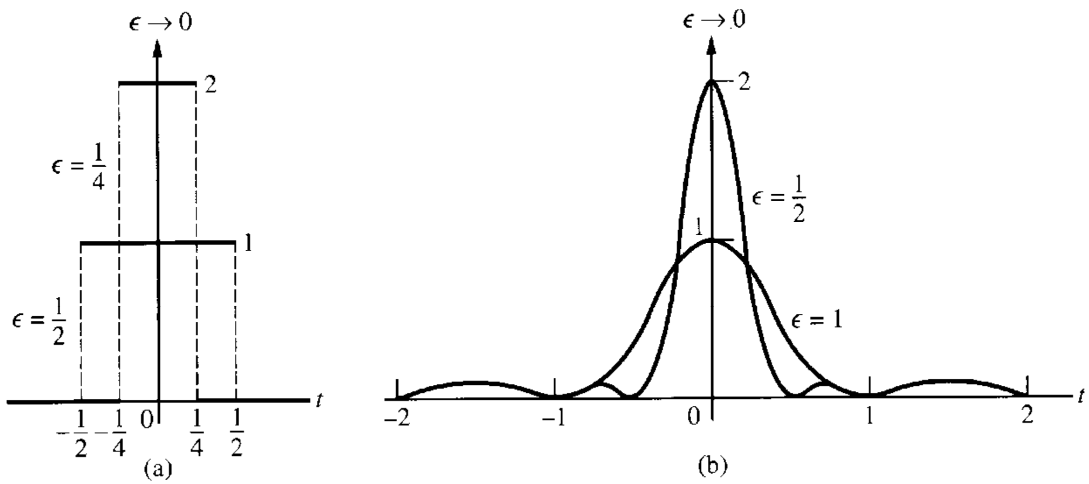

Figure 2.4
Two representations for the unit impulse function in the limit as $\epsilon \to 0$. (a) $(1/2\epsilon)\Pi(t/2\epsilon)$. (b) $\epsilon[(1/\pi t)\sin(\pi t/\epsilon)]^2$.

Cor

Properties of Impulse Function

- Use $\delta(t)$ *only* by its properties (Z&T, pp.21~22)
  - Shifting: $$x(t_0) = \int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt$$
  - Time-scaling: $$\delta(at) = \frac{1}{|a|}\delta(t)$$
  - Symmetry: $$\delta(t) = \delta(-t)$$
- **Unit step function**:

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)d\lambda; \quad \text{or} \quad \delta(t) = \frac{du(t)}{dt}$$

Energy and Power Signals

- Energy: $$E \equiv \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$
- Power: $$P \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$
- **Energy signals**: iff $0 < E < \infty \;\; (P = 0)$
- **Power signals**: iff $0 < P < \infty \;\; (E = \infty)$

Note: If $x(t)$ is periodic, we only need to check (calculate) its power in one period. (Often, it has infinite energy.)

Examples

$$x_1(t) = Ae^{-\alpha t}u(t)$$

$$x_2(t) = Au(t)$$

$$x_3(t) = A\cos(\omega_0 t + \theta)$$

Concluding Remarks

- Periodic signals and random signals are often power signals.
- Deterministic and aperiodic signals are often energy signals.
- The energy and power classifications of signals are mutually exclusive (cannot be both at the same time). But a signal can be neither energy nor power signal. E.g.

$$x_4(t) = t^{-1/4}u(t)$$

Signal Space & Orthogonal Basis

- Waveform $\leftrightarrow$ vector (in geometry, linear algebra)
- The consequence of linearity:

  N-dimensional basis vectors: $\underline{b_1}, \underline{b_2}, \cdots, \underline{b_N}$
- Degree of freedom and independence: For example, in geometry, any 2-D vector $\underline{x}$ can be decomposed into components along two *orthogona*l basis vectors, (or expanded by these two vectors) $\underline{x} = x_1 \underline{b_1} + x_2 \underline{b_2}$
- Meaning of "linear" in linear algebra:

$$\underline{x} + \underline{y} = (x_1 + y_1)\underline{b_1} + (x_2 + y_2)\underline{b_2}$$

- A *general* function (waveform) can also be expanded by a set of *basis* functions

  $x(t) = \sum_{n=1}^{N} X_n \varphi_n(t)$, where N can be ∞.

- Define the **inner product of functions** as

$$< x(t), y(t) >= \int_{-\infty}^{\infty} x(t) y^*(t) dt.$$

and the basis is orthogonal, then

$$\int_{-\infty}^{\infty} \phi_n(t)\phi_m^*(t)dt \equiv \delta(n-m) \equiv \begin{cases} 1, & n = m \\ 0, & o.w. \end{cases}$$

$$X_m = \int_{-\infty}^{\infty} x(t)\phi_m^*(t)dt.$$

Basis Functions

- *Example*: cosine waves

Q1: How to construct a "good" set of basis functions? (What conditions? What purposes? …)

Q2: Can "any" function (waveform) be represented by this set of functions?

Q3: How to compute X_i?

Fourier Series

- If $x(t)$ is periodic with period T_0
- **Fourier Series**:
  - Synthesis: $$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$
  - Analysis: $$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-j2\pi n f_0 t} dt$$

(Notice the integral bounds)

- Decompose a periodic signal into *countable* (frequency) components (sin(), cos())

$$\hat{x}(t) = X_0 + \sum_{n=1}^{\infty}\left[X_n e^{j2\pi n f_0 t} + X_{-n} e^{j2\pi(-n) f_0 t}\right]$$

FS of Real Functions

- If $x(t)$ is *real*, X_n is conjugate symmetric.

$$\hat{x}(t) = X_0 + \sum_{n=1}^{\infty} \left[X_n e^{j2\pi n f_0 t} + X_{-n} e^{j2\pi(-n) f_0 t} \right]$$

$$X_{-n} = \left| X_{-n} \right| e^{j\angle X_{-n}} = \left| X_n \right| e^{-j\angle X_{-n}} = X_n^{\ *}$$

$$\hat{x}(t) = X_0 + \sum_{n=1}^{\infty} \left| X_n \right| \left(e^{j(2\pi n f_0 t + \angle Xn)} + e^{-j(2\pi n f_0 t + \angle Xn)} \right)$$

$$\hat{x}(t) = X_0 + \sum_{n=1}^{\infty} 2 \left| X_n \right| \cos(2\pi n f_0 t + \angle X_n)$$

- Or, use both cosine and sine:

$$\hat{x}(t) = X_0 + \sum_{n=1}^{\infty} 2|X_n|\left[\cos(\angle X_n)\cos(2\pi n f_0 t) - \sin(\angle X_n)\sin(2\pi n f_0 t)\right]$$

$$= X_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

with $a_n = 2|X_n|\cos(\angle X_n)$ $\quad b_n = -2|X_n|\sin(\angle X_n)$

Or (rewrite):

$$a_n = 2|X_n|\cos(\angle X_n) = 2\,\mathrm{Re}\{X_n\} = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(2\pi n f_0 t)\,dt$$

$$b_n = -\frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(2\pi n f_0 t)\,dt$$

$$\hat{x}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

DC or average component of $x(t)$ + (n=1) the fundamental harmonic of $x(t)$ + (n=2) the second harmonic of $x(t)$ + …

DC and AC Coefficients

- DC coefficient
$$X_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-j2\pi(0)f_0 t}dt = \int_{t_0}^{t_0+T_0} x(t)dt$$
$$= \text{average of } x(t)$$

- AC coefficients
$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)[\cos(2\pi n f_0 t) - j\sin(2\pi n f_0 t)]dt$$
$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(2\pi n f_0 t)dt - j\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(2\pi n f_0 t)dt$$
  - If $x(t)$ is even and real, that is $x(t) = x(-t)$, the second term is zero. Hence X_n is purely real and even
  - If $x(t)$ is odd and real, that is $x(t) = -x(-t)$, the first term is zero. Hence X_n is purely imaginary and odd

FS Properties

- ***Linearity*** If $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$

  then $Ax(t)+By(t) \leftrightarrow Aa_k + Bb_k$
- ***Time Reversal***

  If $x(t) \leftrightarrow a_k$ then $x(-t) \leftrightarrow a_{-k}$
- ***Time Shifting*** $x(t-t_0) \overset{F}{\longleftrightarrow} e^{-j2\pi nf_0t_0} a_k$
- ***Time Scaling***

  $$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X_{\frac{n}{a}}$$

  but the fundamental frequency changes
- ***Multiplication*** $x(t)y(t) \leftrightarrow a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$
- ***Conjugation and Conjugate Symmetry***

  $x(t) \leftrightarrow a_k$ and $x^*(t) \leftrightarrow a^*_{-k}$

  If $x(t)$ is real $\Rightarrow a_{-k} = a_k^*$

Example: FS

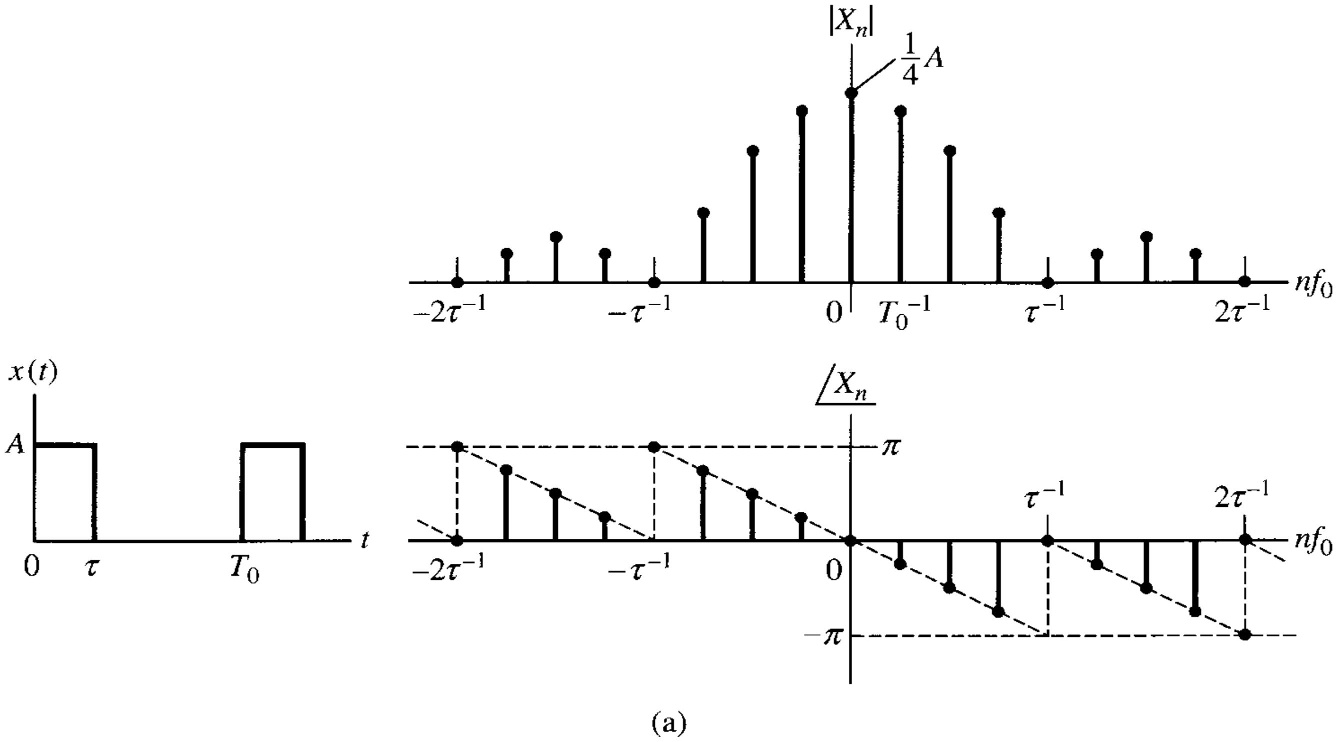

Figure 2.7
Spectra for a periodic pulse train signal. (a) $\tau = \frac{1}{4}T_0$. (b) $\tau = \frac{1}{8}T_0$; T_0 same as in (a). (c) $\tau = \frac{1}{8}T_0$; τ same as in (a).

Example: Time Scaling

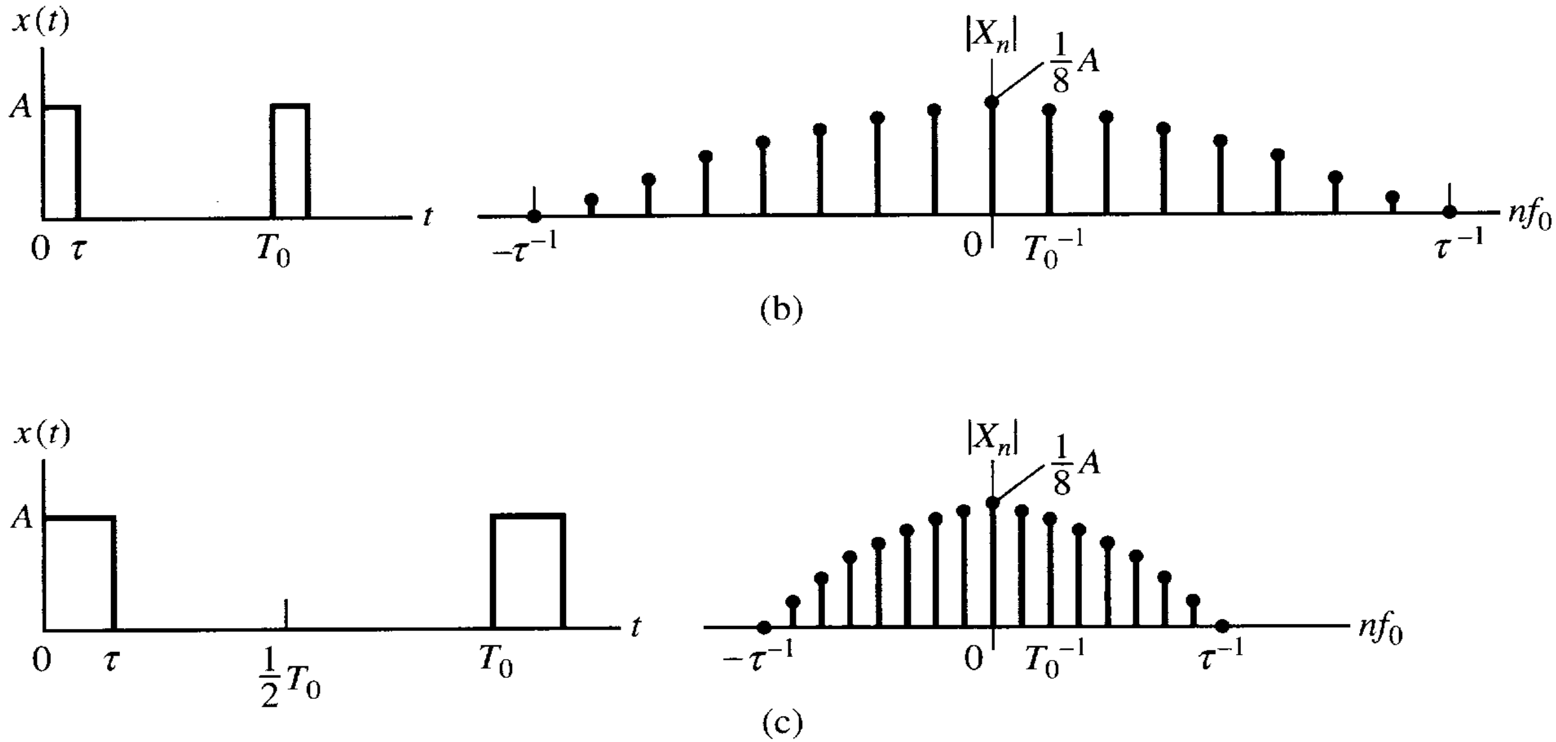

Figure 2.7
Continued.

Parseval's Theorem

Power in time domain = Power in frequency domain

$$P_x = \frac{1}{T_o}\int_{t_1}^{t_1+T_0} |x(t)|^2\, dt \qquad P_x = \frac{1}{T_0}\left[\sum_{n=-\infty}^{\infty} T_o |X_n|^2\right] = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Table 2.1 Fourier Series for Several Periodic Signals

Signal (one period)	Coefficients for exponential Fourier series
1. Asymmetrical pulse train; period $= T_0$:	
$x(t) = A\Pi\left(\frac{t-t_0}{\tau}\right), \tau < T_0$	$X_n = \frac{A\tau}{T_0}\text{sinc}(nf_0\tau)e^{-j2\pi nf_0 t_0}$
$x(t) = x(t+T_0)$, all t	$n = 0, \pm 1, \pm 2, \ldots$
2. Half-rectified sine wave; period $= T_0 = 2\pi/\omega_0$:	
$x(t) = \begin{cases} A\sin(\omega_0 t), & 0 \le t \le T_0/2 \\ 0, & -T_0/2 \le t \le 0 \end{cases}$	$X_n = \begin{cases} \frac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \cdots \\ 0, & n = \pm 3, \pm 5, \cdots \\ -\frac{1}{4}jnA, & n = \pm 1 \end{cases}$
$x(t) = x(t+T_0)$ all t	
3. Full-rectified sine wave; period $= T_0 = \pi/\omega_0$:	
$x(t) = A\lvert\sin(\omega_0 t)\rvert$	$X_n = \frac{2A}{\pi(1-4n^2)}, \quad n = 0, \pm 1, \pm 2, \ldots$
4. Triangular wave:	
$x(t) = \begin{cases} -\frac{4A}{T_0}t + A, & 0 \le t \le T_0/2 \\ \frac{4A}{T_0}t + A, & -T_0/2 \le t \le 0 \end{cases}$	$X_n = \begin{cases} \frac{4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$
$x(t) = x(t+T_0)$, all t	

Remarks

- Fourier found that the sinusoids are good orthonormal basis functions to expand a periodic function
- The Fourier series is derived from the good orthonormal basis functions for a periodic function, defined over a period interval (t_0, t_0+T_0)
- How about the aperiodic signal?
  - We consider the aperiodic energy signal $x(t)$, that is $x(t)$ is integrable in the interval (-∞,∞)
  - Note that aperiodic signals are mostly finite duration
  - We may interpret the aperiodc fnction as a special case of periodic function with infinite period

$$x(t) = \lim_{T_0 \to \infty} \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} x(\lambda) e^{-j2\pi f_0 \lambda} d\lambda \right] e^{j2\pi n f_0 t}, \quad |t| < \frac{T_0}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} d\lambda \right] e^{j2\pi f t} df$$

Fourier Transform Definition

- Then

$$x(t) = \lim_{df \to 0} \sum_{n=-\infty}^{\infty} X_n e^{j2\pi ndft} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f\lambda} d\lambda \right) e^{j2\pi ft} df$$

$$\equiv \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$\equiv \text{Fourier Transform of } x(t)$$

$$= \text{frequency response of } x(t)$$

Fourier Transform for Aperiodic Signals

- Aperiodic signals may be viewed as having periods that are “infinitely” long.
- Summation is replaced by integration.
- Decompose an aperioidc signal into *uncountable* (infinite) frequency components.
- Similar mathematical form, and similar interpretation.
- To be discussed: FT of impulses (samples)
- Why sinusoidal functions? (a) eigenfunctions of linear system; (b) orthogonal & complete basis

Energy Spectral Density

- For periodic signal, we have power spectral density $|X_n|^2$
- For aperiodic energy signal, we have the similar energy spectral density $\boxed{G(f) \equiv |X(f)|^2}$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x^*(t)x(t)dt$$

$$= \int_{-\infty}^{\infty} x^*(t)\left(\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df\right)dt = \int_{-\infty}^{\infty} X(f)\int_{-\infty}^{\infty} x^*(t)e^{j2\pi ft}dtdf$$

$$= \int_{-\infty}^{\infty} X(f)X^*(f)df = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

By Parseval's theorem

Fourier Series

$x(t)$: **periodic**, with period $T_0 = \dfrac{1}{f_0}$

$\omega_0 = 2\pi f_0$

Synthesis:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

X_n : Fourier coefficient (spectra coefficient)

Analysis:

$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jn\omega_0 t}dt$$

Fourier Transform

$x(t)$: **aperiodic**, $\omega = 2\pi f$

Synthesis:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Analysis:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$\equiv Fourier\ \ Transform\ \ of\ \ x(t)$$

$$\equiv frequency\ \ response\ \ of\ \ x(t)$$

Frequency components:

1.Has a *fundamental* freq. and many harmonics.

n=1, fundamental

n=2, second harmonic

n=3, third harmonic

2. (Discrete) *line* spectra

$$X_n e^{jn\omega_0 t} = |X_n| e^{j\angle X_n} e^{jn\omega_0 t}$$

$$-\infty < n < \infty$$

$$|X_n|\text{: amplitude}$$

$$\angle X_n\text{ : phase}$$

Power Spectral Density: $|X_n|^2$

and (by Parseval's equality)

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Frequency components:

1.No fundamental freq. and contain all possible freq.

2.Continuous spectra (density)

$$X(f)e^{j2\pi ft} = |X(f)| e^{j\angle X(f)} e^{j2\pi ft}$$

$$-\infty < f < \infty$$

$$|X(f)|\text{: amplitude}$$

$$\angle X(f)\text{ : phase}$$

Energy Spectral Density:

$$G(f) \equiv |X(f)|^2$$

and

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Conditions of Existence

- Does *any* periodic function have FS?

$$e_N(t) \equiv [x(t) - \sum_{n=-N}^{N} X_n e^{jn\omega_0 t}]$$

Would $e_N(t) \to 0, \forall t$ as $N \to \infty$?

a) **square integrable** condition (for the power signal): $\int_{T_0} |x(t)|^2 \, dt < \infty$

$$\int_{T_0} [e_N(t)]^2 dt \to 0 \text{ as } N \to \infty$$

but not necessarily $|e_N(t)| \to 0, \forall t$

b) **Dirichlet's conditions**:

i) finite no. of finite discontinuities;

ii) finite no. of finite max & min.;

iii) absolute integrable:

$$\int_{T_0} |x(t)| \, dt < \infty$$

- Dirichlet's condition implies convergence almost everywhere, except at some discontinuities.

- Does *any* aperiodic function have FT?

$$e_T(t) \equiv [x(t) - \int_{-T}^{T} X(f) e^{j2\pi ft} df]$$

Would $e_T(t) \to 0, \forall t$ as $T \to \infty$?

a) **square integrable** condition (for the energy signal): $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$

$$\int_{-\infty}^{\infty} [e_T(t)]^2 dt \to 0 \text{ as } T \to \infty$$

but not necessarily $|e_T(t)| \to 0, \forall t$

b) **Dirichlet's conditions**:

i) finite no. of finite discontinuities;

ii) finite no. of finite max & min.;

iii) absolute integrable:

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

- Dirichlet's condition implies convergence almost everywhere, except at some discontinuities.

Symmetry Properties

- **Real-valued** *x*(*t*) and Its Fourier Function

  For real periodic *x*(*t*), $X_{-n} = X_n^*$

  For real aperiodic *x*(*t*), $X(f) = X^*(-f)$

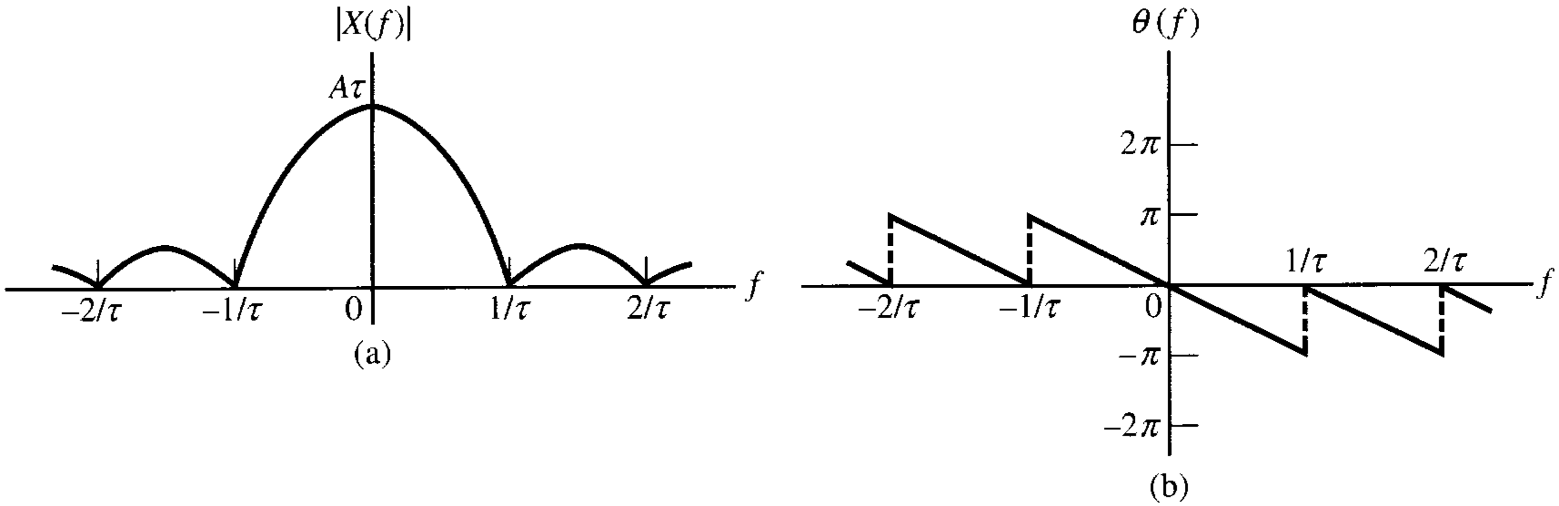

Figure 2.8
Amplitude and phase spectra for a pulse signal. (a) Amplitude spectrum. (b) Phase spectrum ($t_0 = \frac{1}{2}\tau$ is assumed).

FT of Singular Functions

- $\delta(t)$ is not an energy signal (hence doesn't satisfy Dirichlet condition).

  However, its FT can be obtained by "generalization" (formal definition). $\Im[\delta(t)] = \Im\left[\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)\Pi\left(\frac{t}{\tau}\right)\right] = \lim_{\tau\to 0}\text{sinc}(f\tau) = 1$

$$\delta(t)\xrightarrow{FT}1, \qquad 1\xrightarrow{FT}\delta(f)$$

- *Ex*: The FT of $\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$?

$$A\delta(t-t_0)\xrightarrow{FT}Ae^{-j2\pi t_0 f},\quad Ae^{j\pi f_0 t}\xrightarrow{FT}A\delta(f-f_0)$$

FT of Periodic Signals

- Periodic signals are not energy signals (don't satisfy Dirichlet's conditions). But we are doing it anyway (justified by advanced math.)…
- Map FS to FT:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \Rightarrow X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

- Ex-1: $\cos 2\pi f_0 t$
- Ex-2: $\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ (A pulse train! What good are they for?)

FT of Periodic Signals (2)

- Let FT of an energy signal *p*(*t*) be

$$\Im\{p(t)\} = P(f)$$

- Aperiodic signal *x*(*t*) is generated by duplicating *p*(*t*) at every interval T_s. Then

$$x(t) = [\sum_{n=-\infty}^{\infty} \delta(t - nT_s)] * p(t) = \sum_{n=-\infty}^{\infty} p(t - nT_s)$$

- From convolution theorem,

$$X(f) = \Im\{[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)]\} \times P(f)$$

$$= f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \times P(f) = \sum_{n=-\infty}^{\infty} f_s P(nf_s)\delta(f - nf_s)$$

FT of Periodic Signals (3)

Since $$\sum_{m=-\infty}^{\infty} p(t-mT_s) \leftrightarrow \sum_{n=-\infty}^{\infty} f_s P(nf_s)\delta(f-nf_0)$$

- Take inverse FT of the eqn.

$$\Im^{-1}\{X(f)\} = x(t) = \sum_{n=-\infty}^{\infty} p(t-nT_s) = \Im^{-1}\{\sum_{n=-\infty}^{\infty} f_s P(nf_s)\delta(f-nf_s)\}$$

$$= \sum_{n=-\infty}^{\infty} f_s P(nf_s)\Im^{-1}\{\delta(f-nf_s)\} = \sum_{n=-\infty}^{\infty} f_s P(nf_s)e^{j2\pi nf_s t}$$

$$\Rightarrow \boxed{\sum_{n=-\infty}^{\infty} p(t-nT_s) = \sum_{n=-\infty}^{\infty} f_s P(nf_s)e^{j2\pi nf_s t}}$$ Poisson sum formula

The sample values $P(nf_s)$ of $P(f)=\Im\{p(t)\}$ are the Fourier series coefficients of $T_s\sum p(t-mT_s)$

Examples of FT

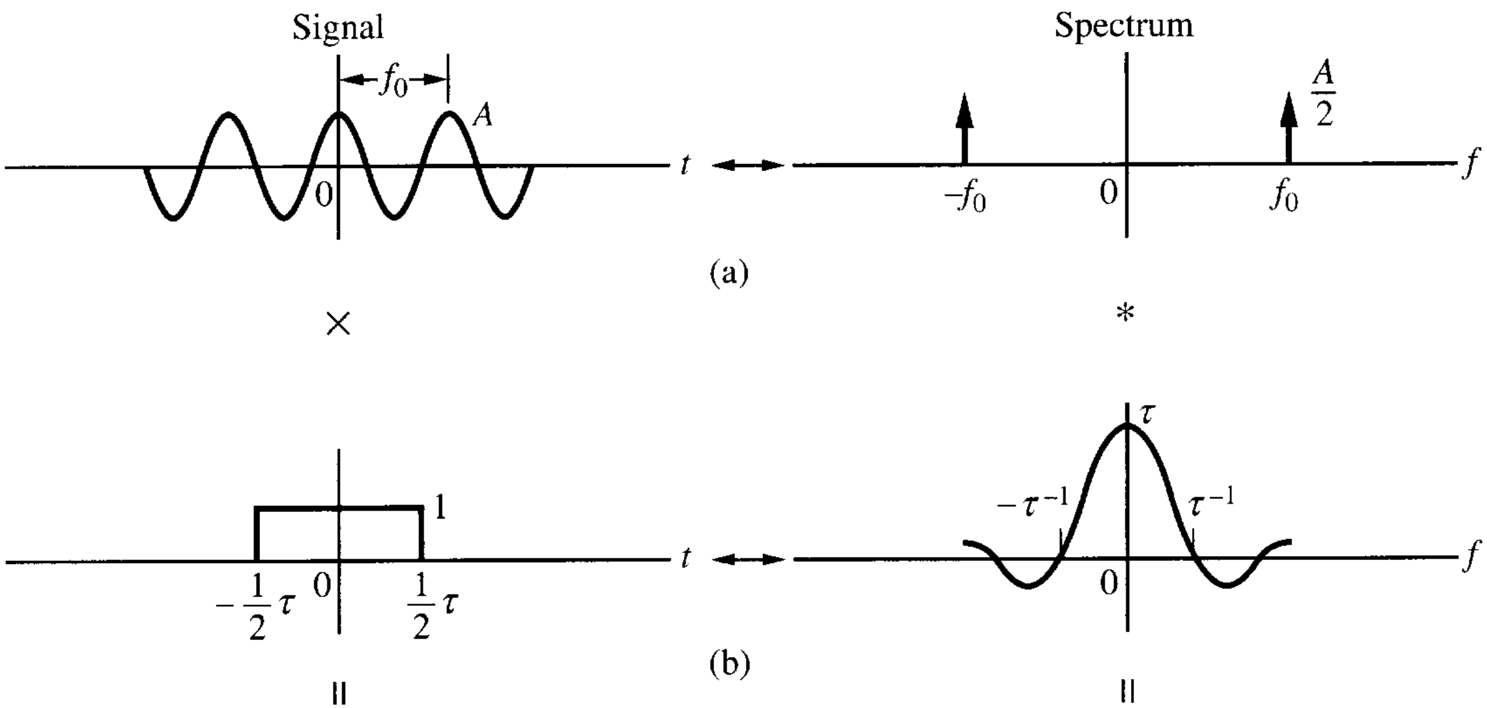

Figure 2.11
(a)–(c) Application of the multiplication theorem. (c)–(e) Application of the convolution theorem.
Note: × denotes multiplication; ∗ denotes convolution, ↔ denotes transform pairs.

Ex of FT (Periodic Signals)

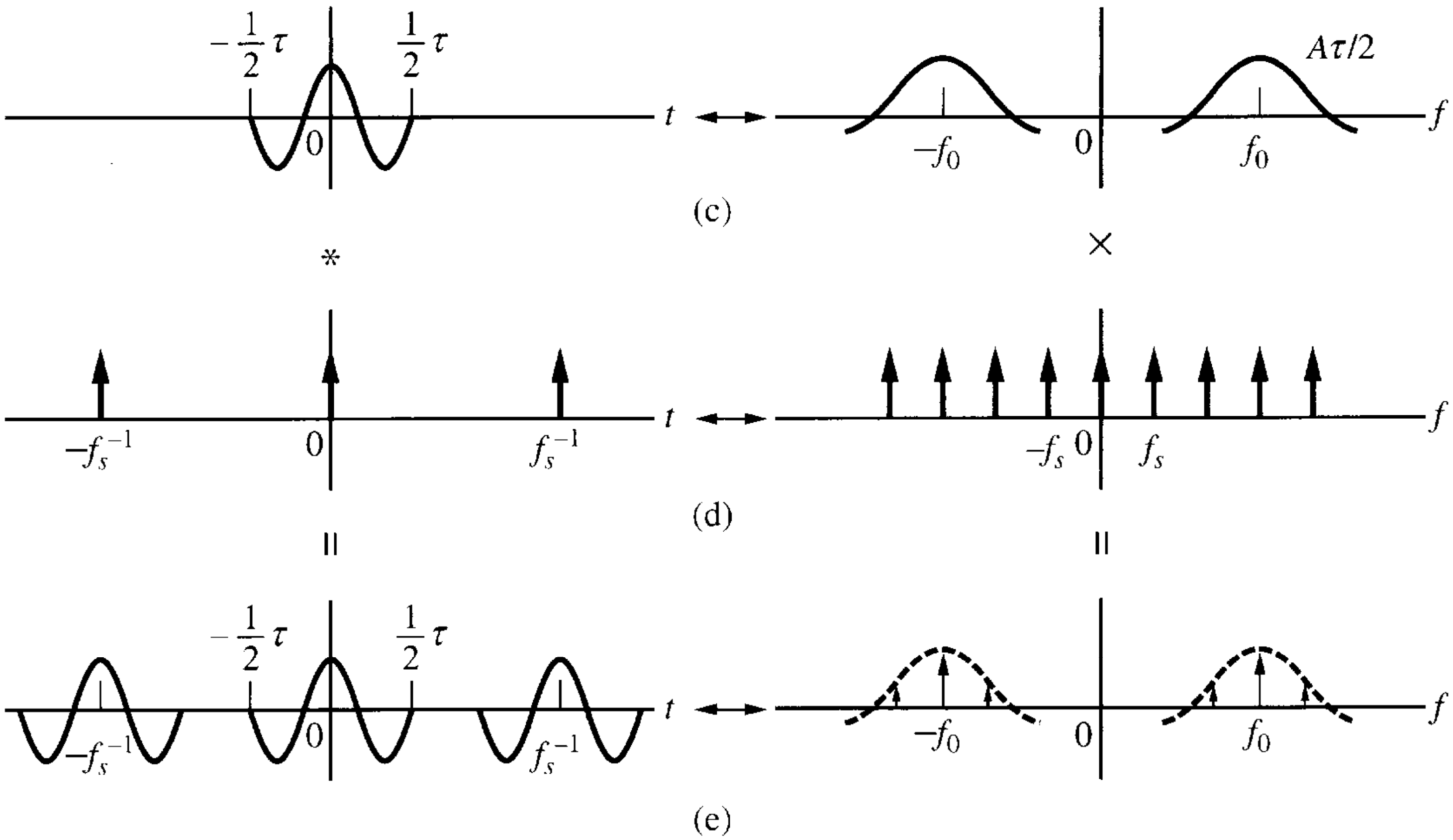

Figure 2.11
Continued.

FT Properties

Name	Time domain operation (signals assumed real)	Frequency domain operation
Superposition	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
Time delay	$x(t-t_0)$	$X(f)\exp(-j2\pi t_0 f)$
Scale change	$x(at)$	$\|a\|^{-1}X\left(\frac{f}{a}\right)$
Time reversal	$x(-t)$	$X(-f) = X^*(f)$
Duality	$X(t)$	$x(-f)$
Frequency translation	$x(t)\exp(j2\pi f_0 t)$	$X(f-f_0)$
Modulation	$x(t)\cos(2\pi f_0 t)$	$\frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$
Convolution*	$x_1(t)*x_2(t)$	$X_1(f)X_2(f)$
Multiplication	$x_1(t)x_2(t)$	$X_1(f)*X_2(f)$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda$	$\frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$

$^*x_1(t)*x_2(t) \triangleq \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda.$

FT Pairs

Signal	Fourier transform
$\Pi(t/\tau) = \begin{cases} 1, & \|t\| \le \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$	$\tau\,\text{sinc}(f\tau) = \tau\frac{\sin(\pi f\tau)}{\pi f\tau}$
$2W\,\text{sinc}(2Wt)$	$\Pi\left(\frac{f}{2W}\right)$
$\Lambda(t/\tau) = \begin{cases} 1 - \frac{\|t\|}{\tau}, & \|t\| \le \tau \\ 0, & \text{otherwise} \end{cases}$	$\tau\,\text{sinc}^2(f\tau)$
$W\,\text{sinc}^2(Wt)$	$\Lambda\left(\frac{f}{W}\right)$
$\exp(-\alpha t)u(t), \quad \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)}$
$t\exp(-\alpha t)u(t), \quad \alpha > 0$	$\frac{1}{(\alpha + j2\pi f)^2}$
$\exp(-\alpha\|t\|), \quad \alpha > 0$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
$\exp\left[-\pi\left(\frac{t}{\tau}\right)^2\right]$	$\tau\exp[-\pi(\tau f)^2]$
$\delta(t)$	1
1	$\delta(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
$u(t)$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$
$\frac{1}{(\pi t)}$	$-j\,\text{sgn}f;\ \text{sgn}f = \begin{cases} 1, & f > 0 \\ -1, & f < 0 \end{cases}$
$\sum_{m=-\infty}^{\infty}\delta(t - mT_s)$	$f_s\sum_{n=-\infty}^{\infty}\delta(f - nf_s);\quad f_s = \frac{1}{T_s}$