HW5

Problem 5.10

a. As $x \to \infty$, the cdf approaches 1. Therefore $B = 1$. Assuming continuity of the cdf, $F_x(12) = 1$ also, which says $A \times 12^4 = 1$ or $A = 4.8225 \times 10^{-5}$.

b. The pdf is given by

$$\begin{aligned} f_X(x) &= \frac{dF_X(x)}{dx} = 4 \times 4.8225 \times 10^{-5} x^3 u(x) u(12-x) \\ &= 1.929 \times 10^{-4} x^3 u(x) u(12-x) \end{aligned}$$

The graph of this pdf is 0 for $t < 0$, the cubic $1.929 \times 10^{-4} x^3$ for $0 \le x \le 12$, and 0 for $t > 12$.

c. The desired probability is

$$P(X > 5) = 1 - F_X(5) = 1 - 4.8225 \times 10^{-5} \times 5^4 = 1 - 0.0303 = 0.9699$$

d. The desired probability is

$$\begin{aligned} P(4 \le X < 6) &= F_X(6) - F_X(4) \\ &= 4.8225 \times 10^{-5} \times 6^4 - 4.8225 \times 10^{-5} \times 4^4 \\ &= 4.8225 \times 10^{-5} \left(6^4 - 4^4\right) \\ &= 0.0502 \end{aligned}$$

Problem 5.13

a. $\int \int f_{XY}(x,\ y)\, dxdy = 1$ gives $C = 1/32$;

b. $f_{XY}(1, 1.5) = \frac{1+1.5}{32} = 0.0781$

c. $f_{XY}(x, 3) = 0$ because $y = 3$ is in the domain where f_{XY} is zero.

d. By integration, $f_Y(y) = \frac{(1+2y)}{8}$, $0 \le y \le 2$, so the desired conditional pdf is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{32}(1+xy)}{\frac{1}{8}(1+2y)} = \frac{(1+xy)}{4(1+2y)},\ 0 \le x \le 4,\ 0 \le y \le 2$$

Substitute $y = 1$ to get the asked for result:

$$f_{X|Y}(x|3) = \frac{1+x}{4(1+2)} = \frac{1+x}{12},\ 0 \le x \le 4$$

Problem 5.17
First note that

$$P(Y=0)=P(X\le 0)=1/2$$

For $y>0$, transformation of variables gives

$$f_Y(y)=f_X(x)\left|\frac{dg^{-1}(y)}{dy}\right|_{x=g^{-1}(y)}$$

Since $y=g(x)=ax$, we have $g^{-1}(y)=y/a$. Therefore

$$f_Y(y)=\frac{\exp\left(-\frac{y^2}{2a^2\sigma^2}\right)}{\sqrt{2\pi a^2\sigma^2}},\ y>0$$

For $y=0$, we need to add $0.5\delta(y)$ to reflect the fact that Y takes on the value 0 with probability 0.5. Hence, for all y, the result is

$$f_Y(y)=\frac{1}{2}\delta(y)+\frac{\exp\left(-\frac{y^2}{2a^2\sigma^2}\right)}{\sqrt{2\pi a^2\sigma^2}}u(y)$$

where $u(y)$ is the unit step function.

Problem 5.22

$$\begin{aligned}E[X] &= \int_{-\infty}^{\infty}x\left\{\frac{1}{2}\delta(x-5)+\frac{1}{8}[u(x-4)-u(x-8)]\right\}dx\\ &= \frac{1}{2}\int_{-\infty}^{\infty}x\delta(x-5)\,dx+\frac{1}{8}\int_4^8 x dx=\frac{1}{2}(5)+\frac{1}{8}\frac{8^2-4^2}{2}=\frac{11}{2}\end{aligned}$$

$$\begin{aligned}E\left[X^2\right] &= \int_{-\infty}^{\infty}x^2\left\{\frac{1}{2}\delta(x-5)+\frac{1}{8}[u(x-4)-u(x-8)]\right\}dx\\ &= \frac{1}{2}\int_{-\infty}^{\infty}x^2\delta(x-5)\,dx+\frac{1}{8}\int_4^8 x^2dx=\frac{1}{2}(25)+\frac{1}{8}\frac{8^3-4^3}{3}=\frac{187}{6}\end{aligned}$$

$$\alpha_X^2=E\left[X^2\right]-E^2[X]=\frac{187}{6}-\frac{121}{4}=\frac{11}{12}$$

Problem 5.29

a. The characteristic function is

$$M_X(jv)=\frac{a}{a-jv}$$

b. The mean and mean-square values are

$$E[X]=-j\left.\frac{\partial M_X(jv)}{\partial v}\right|_{v=0}=1/a$$

and

$$E\left[X^2\right]=(-j)^2\left.\frac{\partial^2 M_X(jv)}{\partial v^2}\right|_{v=0}=\frac{2}{a^2}$$

respectively.

d. $\mathrm{var}[X]=E\left[X^2\right]-E^2[X]=1/a^2$.

Problem 5.33

a. The probability of exactly one error in 10^5 digits, by the binomial distribution, is

$$\begin{aligned} P\left(1 \text{ error in } 10^5\right) &= \binom{10^5}{1}\left(1-10^{-5}\right)^{99,999} 10^{-5} \\ &= 10^5 \times 0.3679 \times 10^{-5} = 0.3679 \end{aligned}$$

By the Poisson approximation, it is

$$P\left(1 \text{ error in } 10^5\right) = \exp\left(-1\right) = 0.3679$$

b. The Poisson approximation gives

$$P\left(2 \text{ errors in } 10^5\right) = \frac{10^5 \times 10^{-5}}{2!}\exp\left(-10^5 \times 10^{-5}\right) = 0.18395$$

c. Use the Poisson approximation:

$$\begin{aligned} P(\text{more than 5 errors in } 10^5) &= 1 - P\left(0,\ 1,\ 2,\ 3,\ 4, \text{ or 5 errors}\right) \\ &= 1 - \sum_{k=0}^{5}\frac{(np)^k}{k!}\exp\left(-np\right) \\ &= 1 - \exp\left(-1\right)\left(1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}\right) \\ &= 1 - 0.9994 = 0.0006 \end{aligned}$$

Problem 5.40

The following relationships involving the Q-function will prove useful:

$$\begin{aligned} \int_a^b \frac{\exp\left(-u^2/2\right)}{\sqrt{2\pi}}du &= Q\left(a\right) - Q\left(b\right),\ b > a; \\ Q\left(x\right) &= 1 - Q\left(|x|\right),\ x < 0;\quad Q\left(0\right) = 0.5 \end{aligned}$$

a. The probability is

$$\begin{aligned} P\left(|X| \le 15\right) &= \int_{-15}^{15}\frac{e^{-(x-10)^2/50}}{\sqrt{50\pi}}dx \\ &= \int_{-5}^{1}\frac{\exp\left(-u^2/2\right)}{\sqrt{2\pi}}du;\quad u = \frac{x-10}{\sqrt{25}} \\ &= Q\left(-5\right) - Q\left(1\right) = 1 - Q\left(5\right) - Q\left(1\right) \\ &= 0.8413 \end{aligned}$$

b. This probability is

$$\begin{aligned} P\left(10 < X \le 20\right) &= \int_{10}^{20}\frac{e^{-(x-10)^2/50}}{\sqrt{50\pi}}dx \\ &= \int_{0}^{2}\frac{\exp\left(-u^2/2\right)}{\sqrt{2\pi}}du;\quad u = \frac{x-10}{\sqrt{25}} \\ &= Q\left(0\right) - Q\left(2\right) = 0.5 - Q\left(2\right) \\ &= 0.4772 \end{aligned}$$

c. The desired probability is

$$
\begin{aligned}
P(5 < X \leq 25) &= \int_5^{25} \frac{e^{-(x-10)^2/50}}{\sqrt{50\pi}} dx \\
&= \int_{-1}^{3} \frac{\exp(-u^2/2)}{\sqrt{2\pi}} du; \quad u = \frac{x-10}{\sqrt{25}} \\
&= Q(-1) - Q(3) = 1 - Q(1) - Q(3) \\
&= 0.8400
\end{aligned}
$$

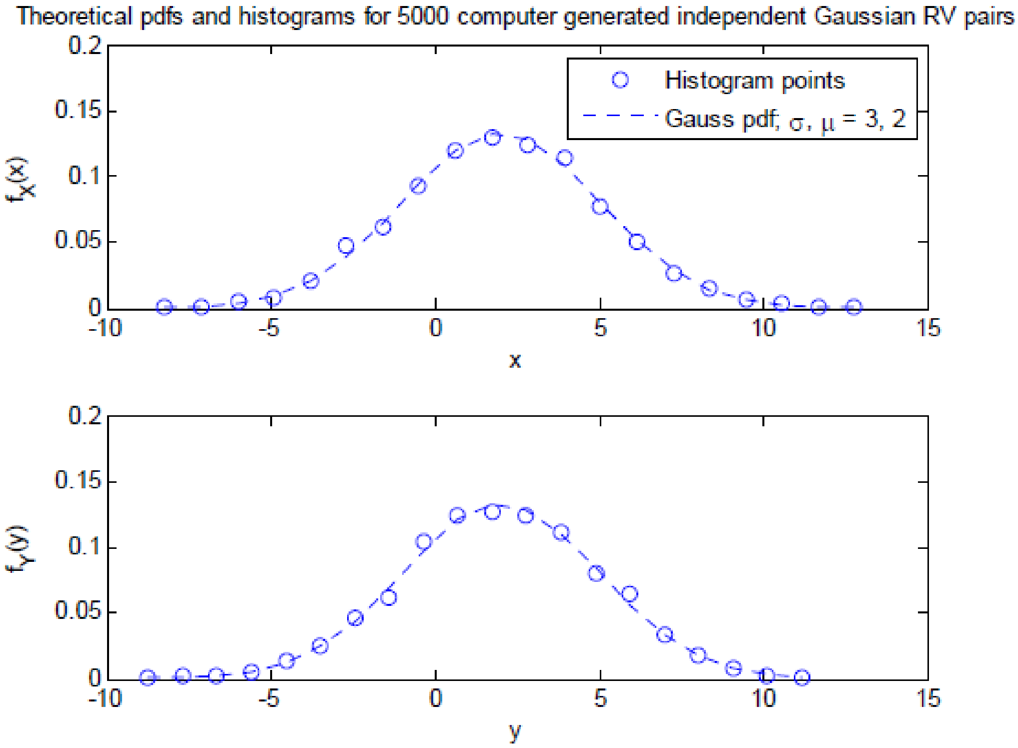
Computer Exercise 5.2

```
% ce5_2.m: Generate pairs of Gaussian random variables from Rayleigh and uniform RVs
%
clf
m = input('Enter the mean of the Gaussian random variables => ');
sigma = input('Enter the standard deviation of the Gaussian random variables => ');

N = input('Enter number of Gaussian random variable pairs to be generated => ');
U = rand(1,N);
V = rand(1,N);
R = sqrt(-2*log(V));
X = sigma*R.*cos(2*pi*U)+m;
Y = sigma*R.*sin(2*pi*U)+m;
disp(' ')
disp('Covarance matrix of X and Y vectors:')
disp(cov(X,Y))
disp(' ')
[MX, X_bin] = hist(X, 20);
norm_MX = MX/(N*(X_bin(2)-X_bin(1)));
[MY, Y_bin] = hist(Y, 20);
norm_MY = MY/(N*(Y_bin(2)-Y_bin(1)));
gauss_pdf_X = exp(-(X_bin - m).^2/(2*sigma^2))/sqrt(2*pi*sigma^2);
gauss_pdf_Y = exp(-(Y_bin - m).^2/(2*sigma^2))/sqrt(2*pi*sigma^2);
subplot(2,1,1), plot(X_bin, norm_MX, 'o')
hold
subplot(2,1,1), plot(X_bin, gauss_pdf_X, '–'), xlabel('x'), ylabel('f_X(x)'),...
    title(['Theoretical pdfs and histograms for ',num2str(N),' ...
        computer generated independent Gaussian RV pairs'])
    legend(['Histogram points'],['Gauss pdf; \sigma, \mu = ', num2str(sigma),', ', num2str(m)])
subplot(2,1,2), plot(Y_bin, norm_MY, 'o')
hold
subplot(2,1,2), plot(Y_bin, gauss_pdf_Y, '–'), xlabel('y'), ylabel('f_Y(y)')
```

A typical run follows with a plot given in Fig. 5.3.

```
>> ce5_2
Enter the mean of the Gaussian random variables => 2
Enter the standard deviation of the Gaussian random variables => 3
Enter number of Gaussian random variable pairs to be generated => 5000
Covarance matrix of X and Y vectors:
9.0928    0.2543
0.2543    9.0777
Current plot held
```


Theoretical pdfs and histograms for 5000 computer generated independent Gaussian RV pairs
Histogram points
Gauss pdf; σ, μ = 3, 2
$f_X(x)$
x
$f_Y(y)$
y
0.2
0.15
0.1
0.05
0
-10
-5
0
5
10
15